On the Deque and Rique Numbers of Complete and Complete Bipartite Graphs

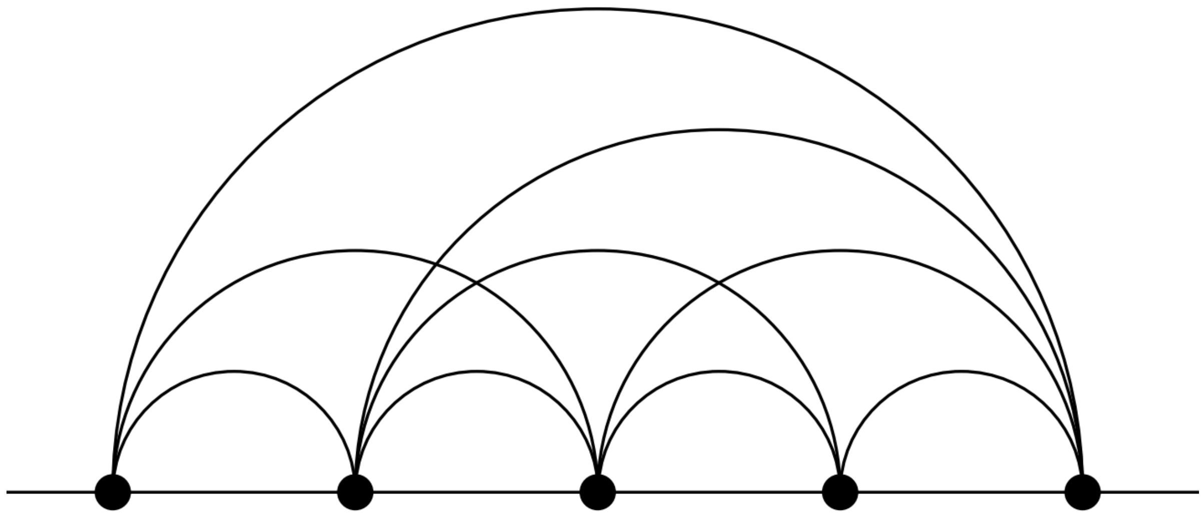
Michael A. Bekos, Michael Kaufmann, Maria Eleni Pavlidi, Xenia Rieger

Linear Layouts

Introduction and Example

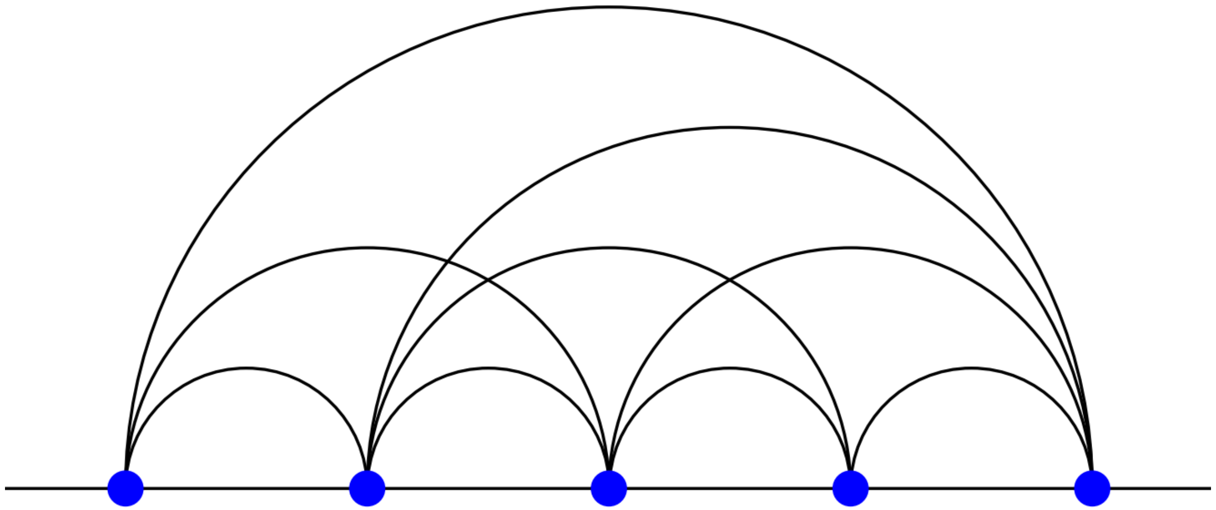
Linear Layouts: Introduction

- Undirected graph $G(V, E)$



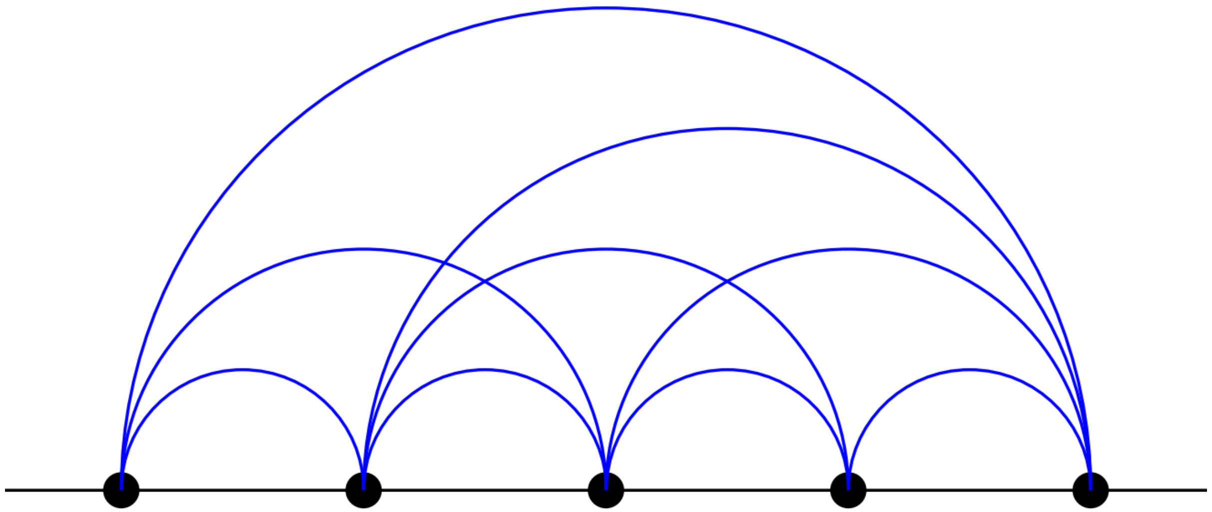
Linear Layouts: Introduction

- Undirected graph $G(V, E)$
- Linear order of V



Linear Layouts: Introduction

- Undirected graph $G(V, E)$
- Linear order of V
- Partition of E into pages



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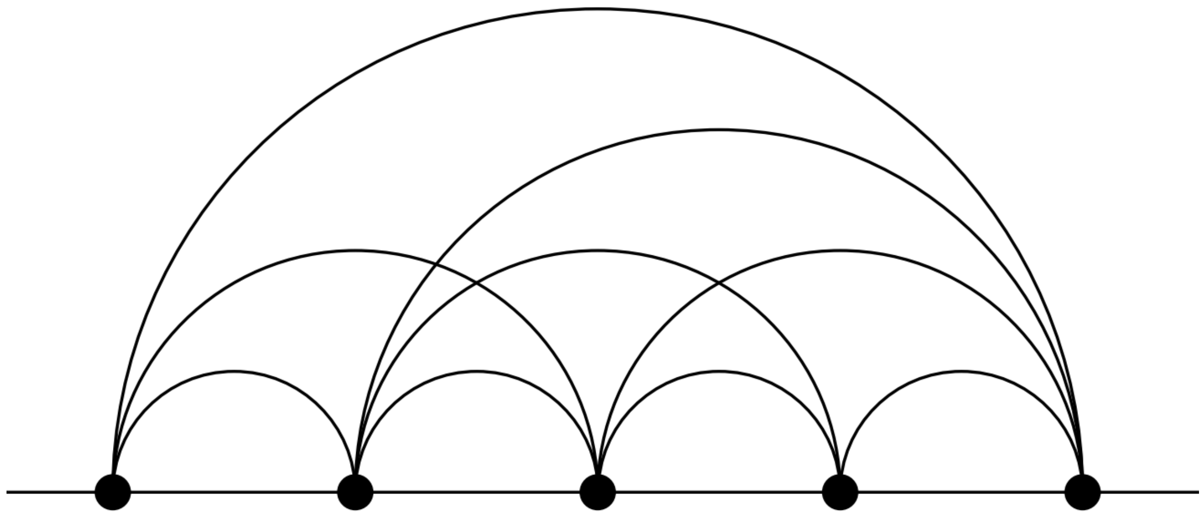
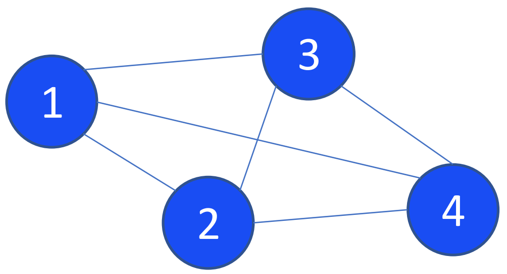
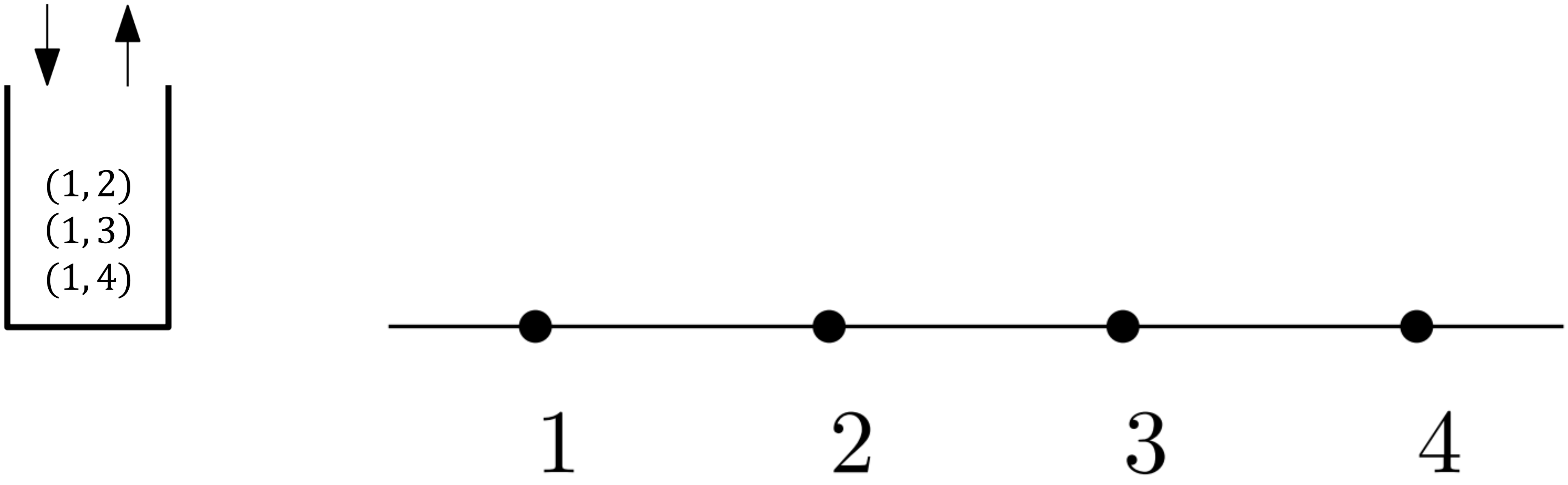


- page assignment depends on data structure D

Linear Layouts: Example



- Edges of each page P are processed by a data structure D
- Originally: Stack



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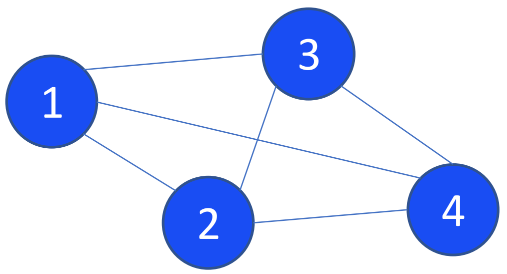

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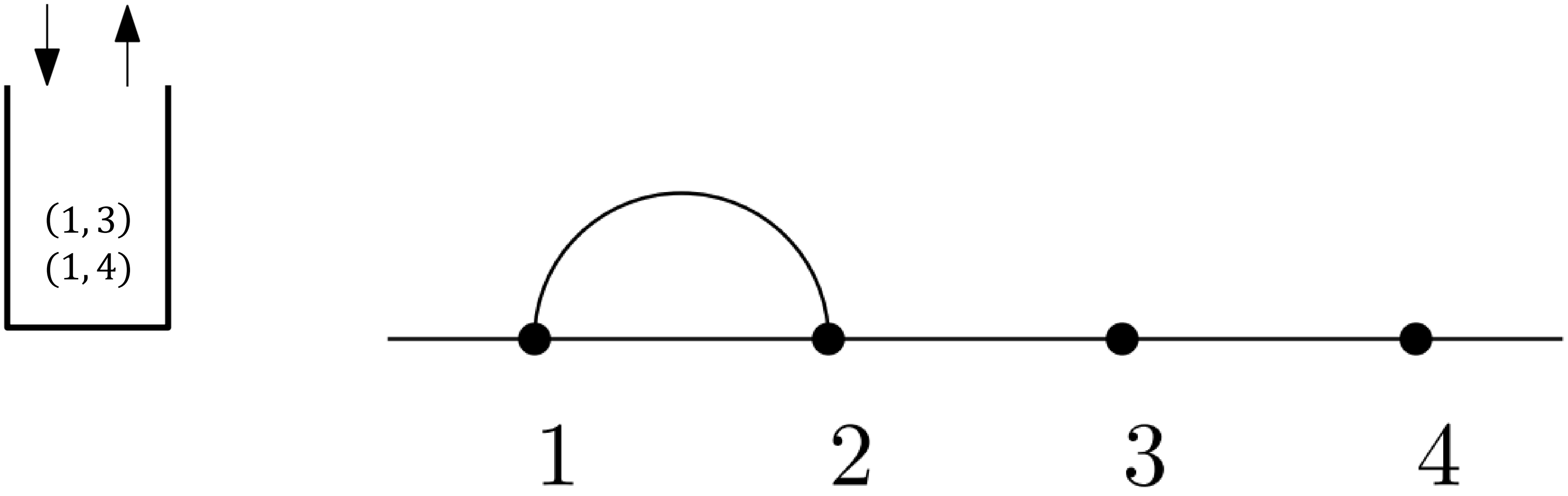

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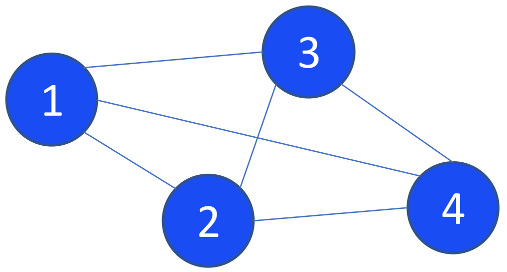

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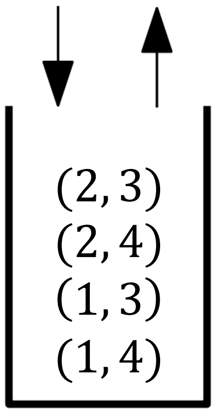

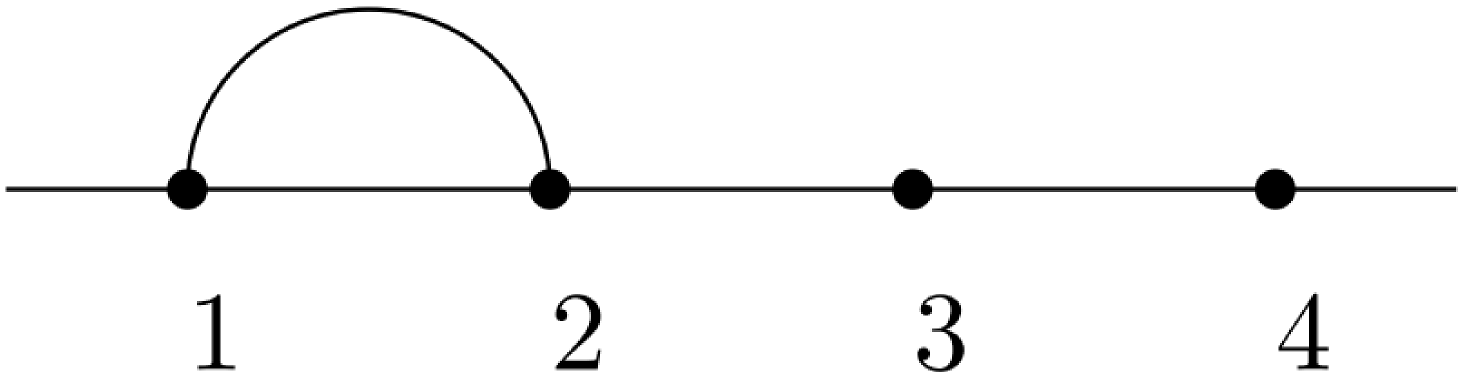

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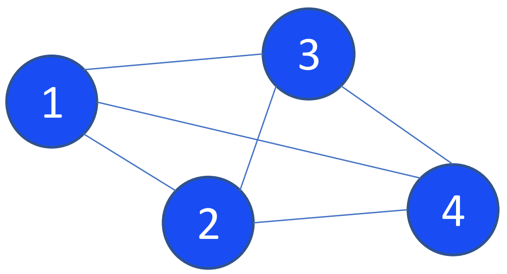

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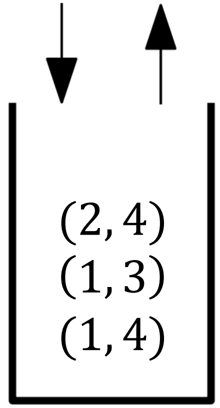

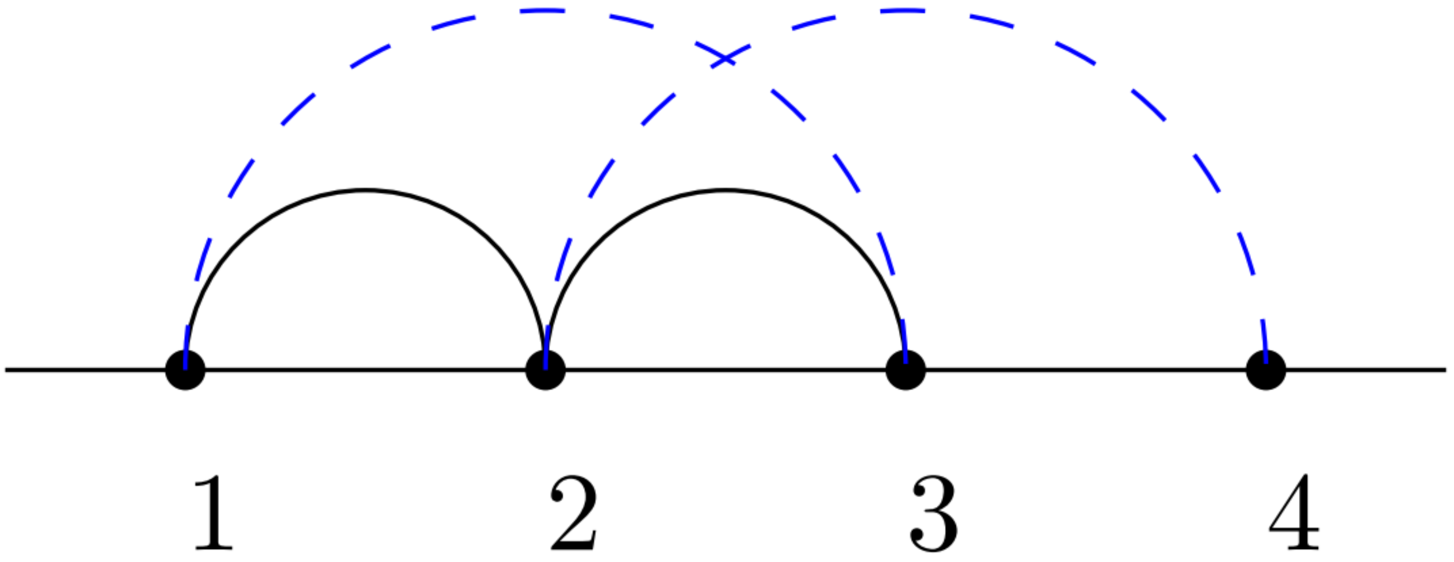

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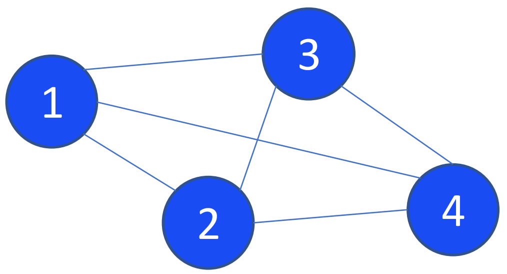

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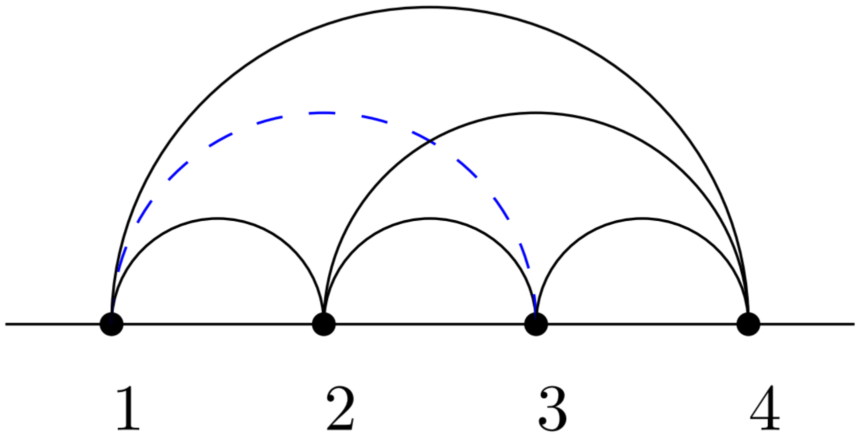


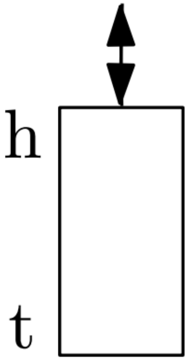
- Goal: D-Number of Graphs

Data Structures

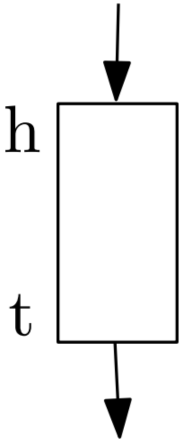
Introduction and Research

Data structures: Introduction

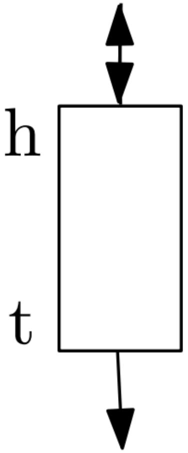
Stack



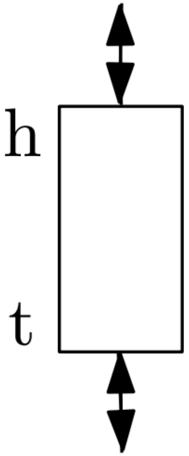
Queue



Rique

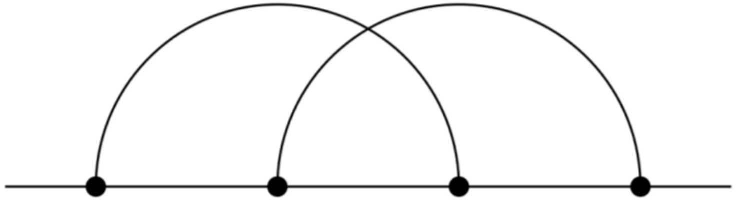


Deque

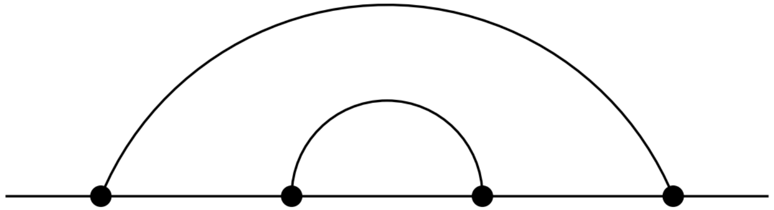


Data structures: Forbidden Patterns

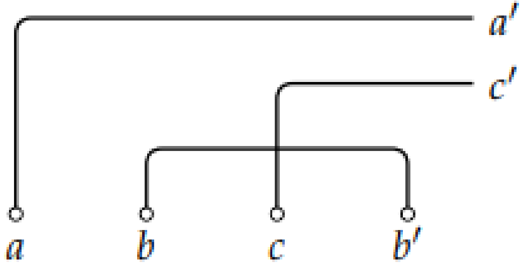
Stack



Queue



Rique



Bekos et al. (2022)

Deque

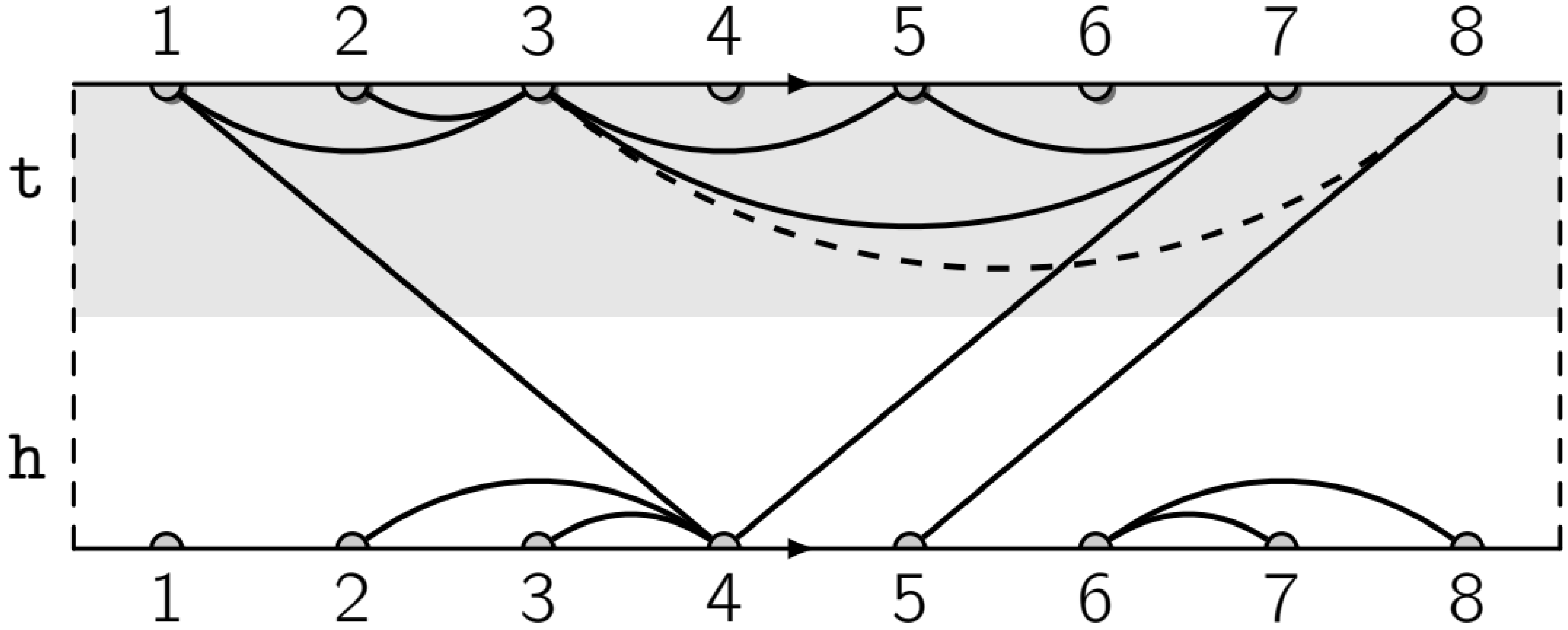
Spanning subgraphs of planar graphs with a Hamiltonian path

Auer et al. (2018)

Contribution

Computation of Deque Layouts

Fundamental Polygon Representation



Auer et al. (2018)

Computation of Linear Layouts: Variables

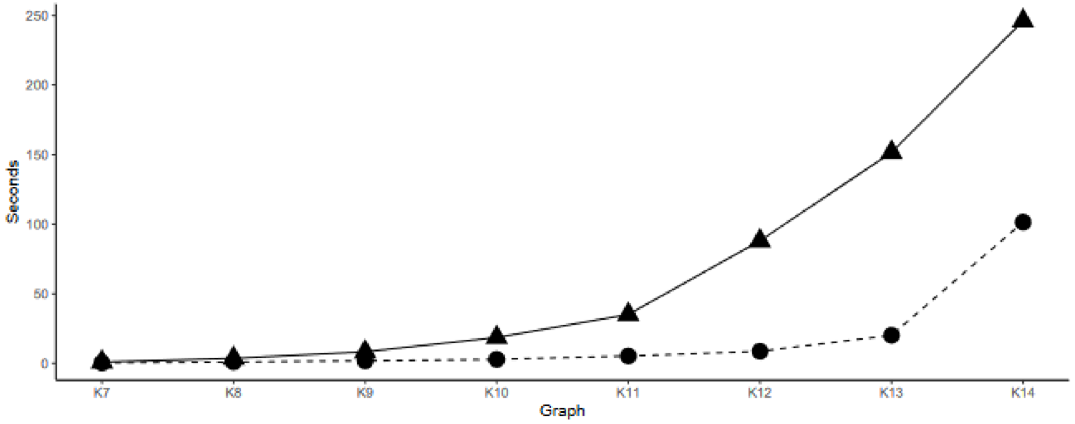
Original Variables

$$(i)\ \sigma(u,v) \Leftrightarrow u \prec v$$
$$(ii) \phi_p(e) \Leftrightarrow e\ is\ assigned\ to\ page\ p$$
$$(iii)\ \chi(e,e) \Leftrightarrow e\ and\ e'\ are\ assigned\ to\ the\ same\ page$$

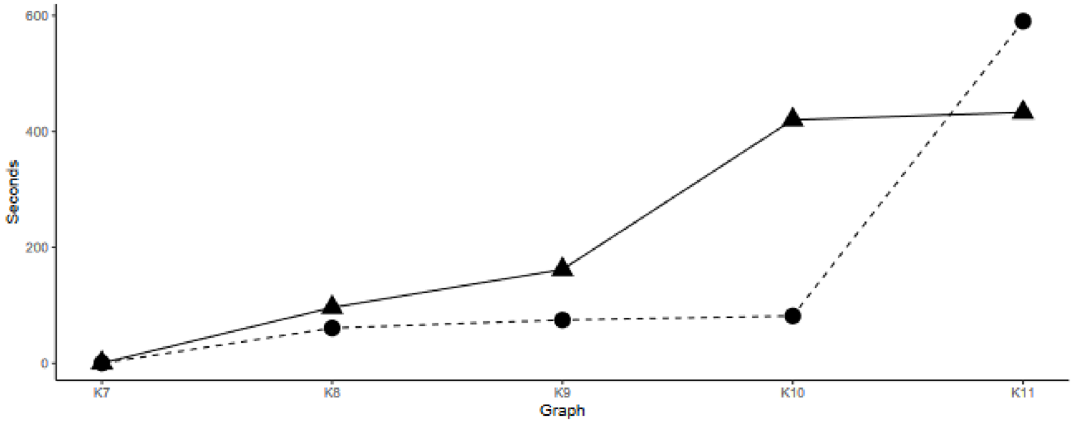
Addition for Deque-Support

$$\tau_p(e,x) with\ x \in \{hh, tt, th, ht\}$$
$$\Leftrightarrow the\ type\ of\ edge\ e\ at\ page\ \rho\ is\ x$$

Computation of Linear Layouts: Runtime



(a)



(b)

Comparison of runtimes of the computation of Rique layouts.

Runtimes of
(a) satisfiable instances of K_7 to K_{14} (Rique-Number)
(b) unsatisfiable instances of K_7 to K_{11} (Rique-Number – 1).

▲ Forbidden Pattern

● Deque-Like

Complete Graphs

$$DEQ(K_n) = \left\lceil \frac{n}{4} \right\rceil$$

Theorem 2

$$RIQ(K_n) \leq \left\lfloor \frac{n - 1}{3} \right\rfloor$$

Theorem 4

Deque-Number of Complete Graphs

Edge count: A graph with n vertices admitting a deque layout with k pages has at most $2kn - 5k + n - 1$ edges.

Each deque page is a planar graph $\Rightarrow$ $3n - 6$ edges per page

Spine edges $\Rightarrow$ $2n - 5$ non-spine edges per page

$\Rightarrow (2n - 5)k + n - 1$

Deque-Number of Complete Graphs

Lower Bound: Edge count

$$2kn - 5k + n - 1 \geq \frac{n^2 - n}{2}$$

$$k(2n - 5) \geq \frac{n^2 - n - 2n + 2}{2}$$

$$k \geq \frac{n^2 - 3n + 2}{4n - 10} \; for \; n \geq 3$$

Lemma: $\left\lceil \frac{n^2 - 3n + 2}{4n - 10} \right\rceil = \left\lceil \frac{n}{4} \right\rceil$

Upper Bound

$$S(K_n) = \left\lceil \frac{n}{2} \right\rceil$$

Bernhart, P. C. Kainen (1979)

$$\Rightarrow DEQ(K_n) \leq \left\lceil \frac{n}{4} \right\rceil$$

Rique-Number of Complete Graphs

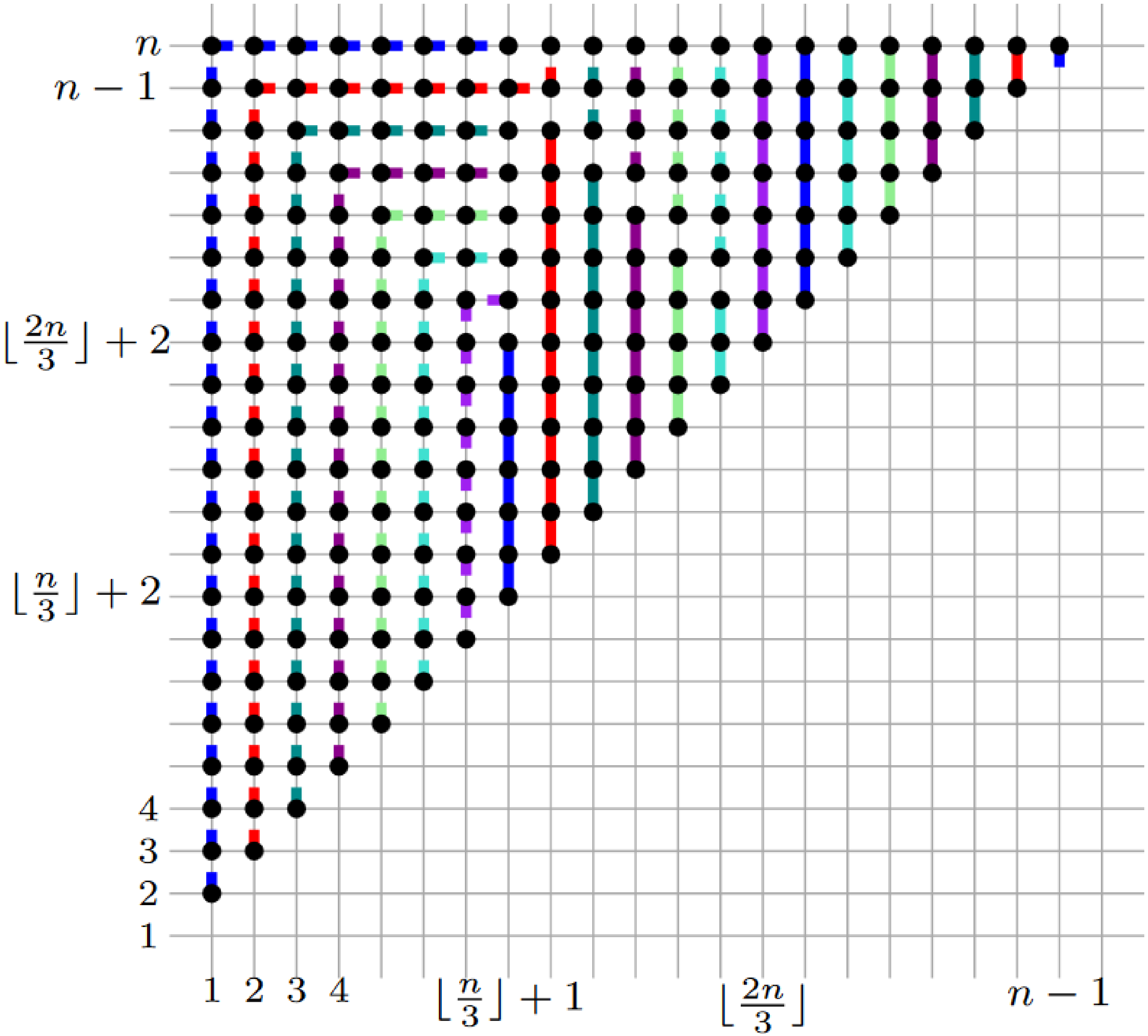
Known upper bound

$$RIQ(K_n) \leq \left\lceil \frac{n}{3} \right\rceil$$

Bekos et al. (2022)

Improved upper bound

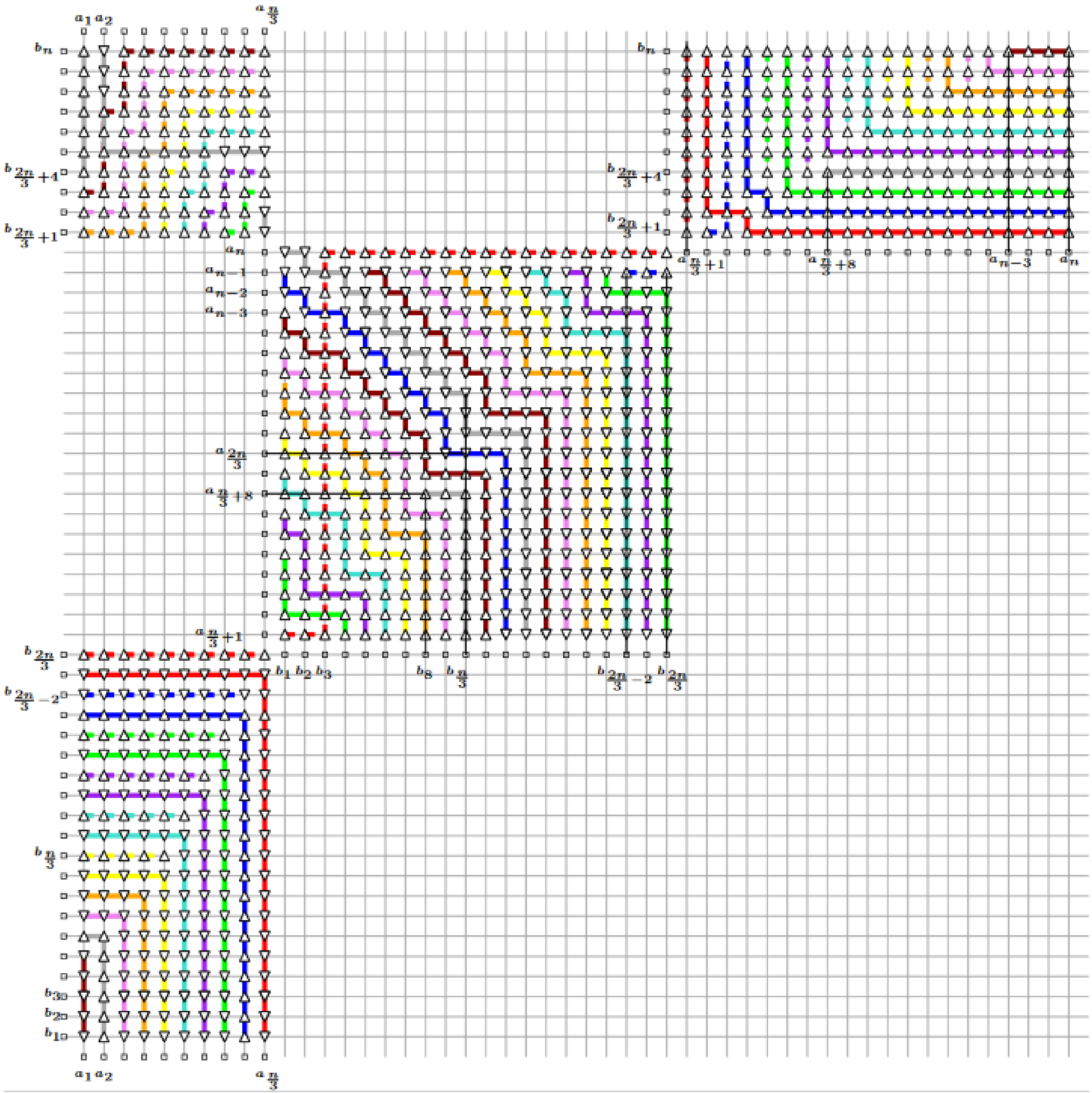
$$RIQ(K_n) \leq \left\lfloor \frac{n-1}{3} \right\rfloor$$


n
$n-1$
$\lfloor \frac{2n}{3} \rfloor + 2$
$\lfloor \frac{n}{3} \rfloor + 2$
4
3
2
1
1
2
3
4
$\lfloor \frac{n}{3} \rfloor + 1$
$\lfloor \frac{2n}{3} \rfloor$
$n-1$

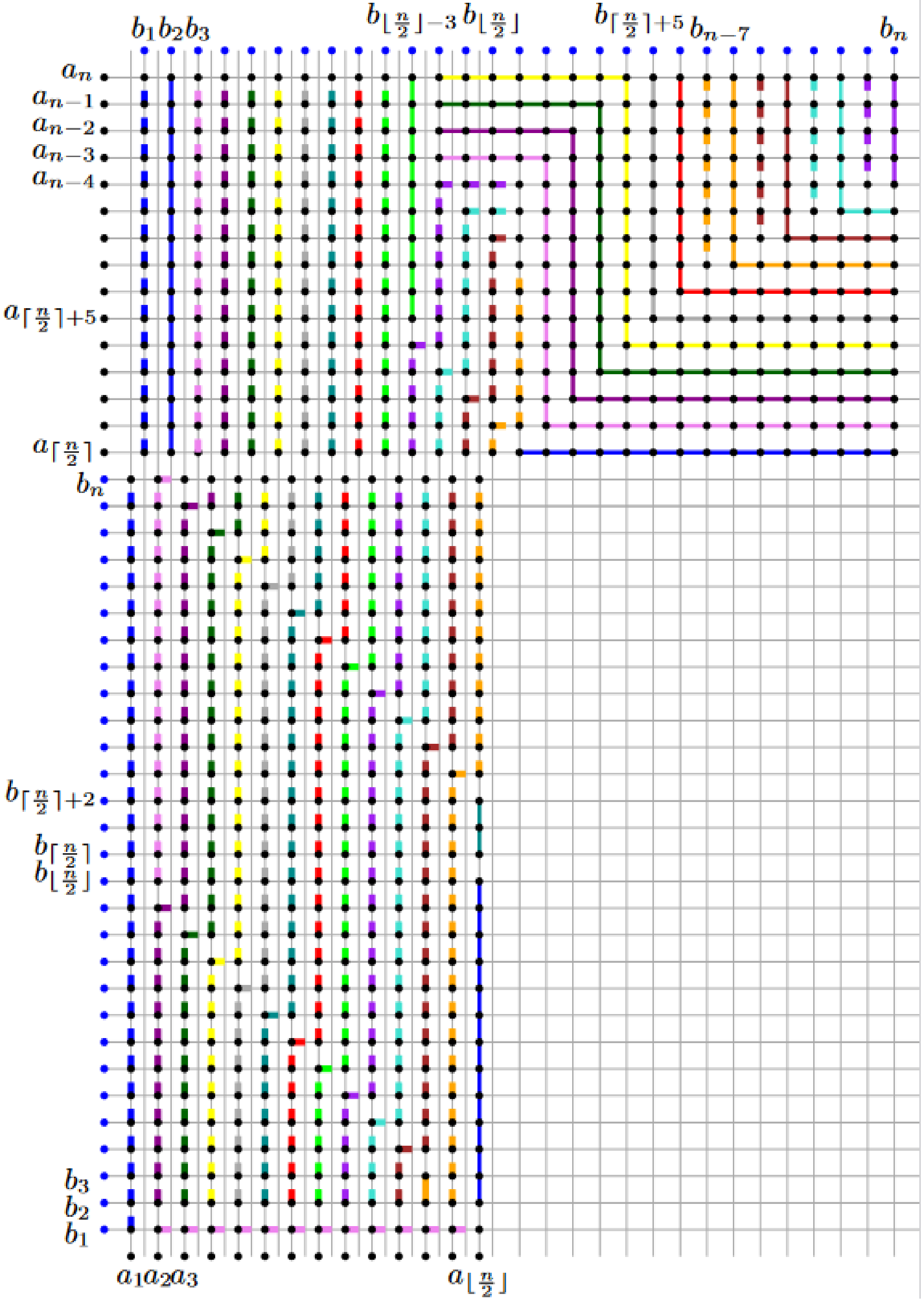
Complete Bipartite Graphs

$DEQ(K_{n,n}) \leq \left\lceil \frac{n}{3} \right\rceil$

$RIQ(K_{n,n}) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ - 1



a_1 a_2
$a_{\frac{n}{3}}$
b_n
$b_{\frac{2n}{3}+4}$
$b_{\frac{2n}{3}+1}$
a_n
a_{n-1}
a_{n-2}
a_{n-3}
$a_{\frac{2n}{3}}$
$a_{\frac{n}{3}+8}$
$a_{\frac{n}{3}+1}$
$a_{\frac{n}{3}+8}$
a_{n-3}
$b_{\frac{2n}{3}}$
$b_{\frac{2n}{3}-2}$
$b_{\frac{n}{3}}$
b_3
b_2
b_1
b_8


$b_1 b_2 b_3$
$b_{\lfloor \frac{n}{2} \rfloor -3}$ $b_{\lfloor \frac{n}{2} \rfloor}$
$b_{\lceil \frac{n}{2} \rceil +5}$ b_{n-7}
b_n
a_n
a_{n-1}
a_{n-2}
a_{n-3}
a_{n-4}
$a_{\lceil \frac{n}{2} \rceil +5}$
$a_{\lceil \frac{n}{2} \rceil}$
b_n
$b_{\lceil \frac{n}{2} \rceil +2}$
$b_{\lceil \frac{n}{2} \rceil}$
$b_{\lfloor \frac{n}{2} \rfloor}$
b_3
b_2
b_1
$a_1 a_2 a_3$
$a_{\lfloor \frac{n}{2} \rfloor}$

Open Problems

Deque-Number of Complete Bipartite Graphs: Computational Results

Graph	$K_{4,4}$	$K_{5,5}$	$K_{6,6}$	$K_{7,7}$	$K_{8,8}$	$K_{9,9}$
Deque-Number	2	2	2	3	3	3
Stack-Number	4	4	5	6	6	7

? Are Deque Layouts more powerful than Double Stack Layouts for Complete Bipartite Graphs?

Open Problems

- Rique-Number for Planar Graphs
- Exact Deque-Number of Complete Bipartite Graphs and exact Rique-Numbers
- Study of Monotone Deque
- Comparison of the Data Structures and study of other Graph Classes, e.g., k-trees or bipartite graphs
- Improve computation

Questions