On the Budgeted Hausdorff Distance Problem

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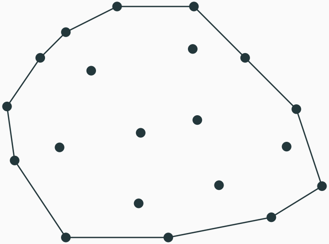
Convex Hull Approximation



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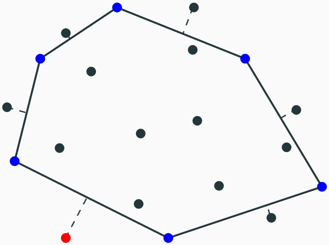

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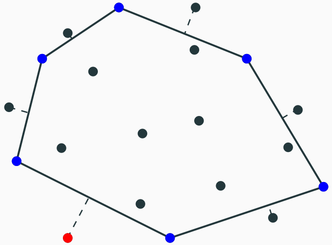

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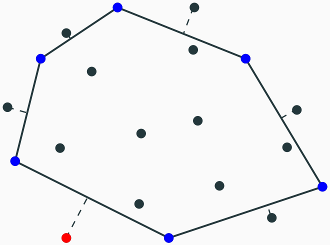

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- $\mathcal{C}(P)$ might contain many points, thus natural to consider finding a smaller subset $Q \subseteq P$ such that $\mathcal{C}(Q) \approx \mathcal{C}(P)$.
- We will measure similarity using the Hausdorff distance between $\mathcal{C}(Q)$ and $\mathcal{C}(P)$, which we denote $\mathrm{D}_H(Q, P)$.
- Since $Q \subseteq P$, and $\mathcal{C}(Q), \mathcal{C}(P)$ convex, one can argue $\mathrm{D}_H(Q, P) = \max_{p \in P} ||p - \mathcal{C}(Q)||$.

Problem Variants

MinCardin

Given a set $P \subset \mathbb{R}^2$ of n points, and a value $\tau > 0$, find the smallest cardinality subset $Q \subseteq P$ such that $\mathrm{D}_H(Q, P) \leq \tau$.

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MinDist

Given a set $P \subset \mathbb{R}^2$ of n points, and an integer k, find the subset $Q \subseteq P$ that minimizes $\mathrm{D}_H(Q, P)$ subject to $|Q| \leq k$.

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MinCardin and MinDist are polynomial time solvable and are dual. A solution to one can be used to search for the solution to the other. Naively the solutions and searching are at least cubic time.

Prior Work

- [KR21] For points in convex position, $O(n \log^2(n))$ time for MinCardin and $O(cn \log^3(n))$ w.h.p. for MinDist. For arbitrary position, by reducing to unweighted APSP, $O(n^{2.5302})$ time for both.

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Many other prior works on approximating the convex hull, though those works generally do not compute the optimal approximation, or give cubic time algorithms for Hausdorff or other related measures.

Decision Procedure

High level idea: solve MinDist, by searching using decider for MinCardin.

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Theorem ([AH23])

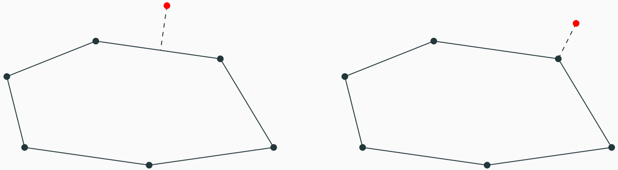
Given an instance P, k of MinDist and a value τ. Let $\tau^\star = \tau^\star(P, k)$.

There is a procedure **decider**(P, k, τ)*, that in $O(nk \log n)$ time returns*

1. $\tau = \tau^\star$,
2. $\tau < \tau^\star$, *or*
3. $\tau > \tau^\star$ *and a set Q where $|Q| \leq k$ and $\mathrm{D}_H(Q, P) \leq \tau$*

Note the result in [AH23] is actually for MinCardin, but with some massaging it yields the above decider.

Canonical Set of Values



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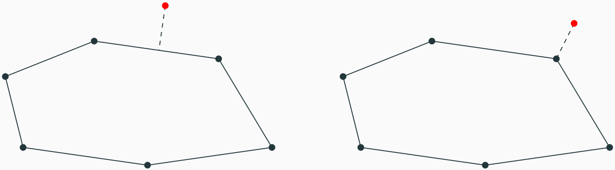

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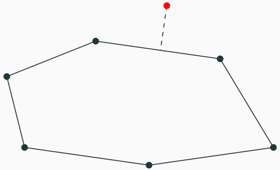

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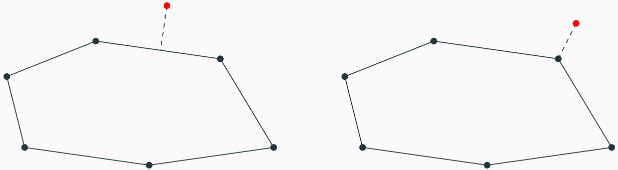

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- Let $P_{a,b}$ be subset of P left of $\ell_{a,b}$. Can argue if $\tau^\star$ is the distance to $\ell_{a,b}$, then p is point from $P_{a,b}$ furthest from $\ell_{a,b}$.

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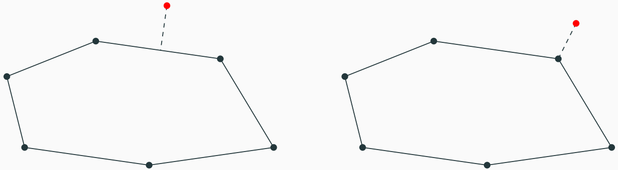

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- Thus $\tau^\star \in \mathcal{V} \cup \mathcal{L}$, where

$$\mathcal{V} = \left\{ ||x - y|| \;\middle|\; x, y \in P \right\} \quad \text{and} \quad \mathcal{L} = \left\{ \max_{p \in P_{a,b}} \mathrm{d}(p, \ell_{a,b}) \;\middle|\; a, b \in P \right\}.$$

Searching

- Let **extremal**$(\ell_{a,b}) = \max_{p \in P_{a,b}} \mathrm{d}(p, \ell_{a,b})$. If one precomputes $\mathcal{C}(P)$, **extremal**$(\ell_{a,b})$ takes $O(\log n)$ time by binary searching.

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- Otherwise, returns value $r, R \in \mathcal{V}$ such that $\tau^\star \in (r, R)$.
- Takes $O((n^{4/3} + nk \log n) \log n)$ time w.h.p.

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- The problem is we do not have direct access to the set $(r', R') \cap \mathcal{L}$.

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- This open interval doesn't contain β and so has fewer values from $\mathcal{L}$.
- Thus in total Stage 3 runs for $O(\sqrt{n})$ rounds, each of which costs $O(nk \log n)$ time as this is the time for **decider**

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- Stage III: $O(n^{3/2} k \log n)$
- Total Time: $O(n^{3/2}(k + \log n) \log n)$
- Optimizing the $\mathcal{L}$ sample size can improve the time to: $O\big(n^{3/2}\sqrt{k} \log^{3/2} n + kn \log^2 n\big)$

Total Time

- Stage I: $O((n^{4/3} + nk \log n) \log n)$
- Stage II: $O((n^{3/2} \log n + nk \log n) \log n)$
- Stage III: $O(n^{3/2} k \log n)$
- Total Time: $O(n^{3/2}(k + \log n) \log n)$
- Optimizing the $\mathcal{L}$ sample size can improve the time to: $O\big(n^{3/2}\sqrt{k} \log^{3/2} n + kn \log^2 n\big)$

Main Result

The MinDist problem can be solved in $O\big(n^{3/2}\sqrt{k} \log^{3/2} n + kn \log^2 n\big)$ time with high probability.

Thank You

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