Overlapping of Lattice Unfolding for Cuboids

CCCG 2023

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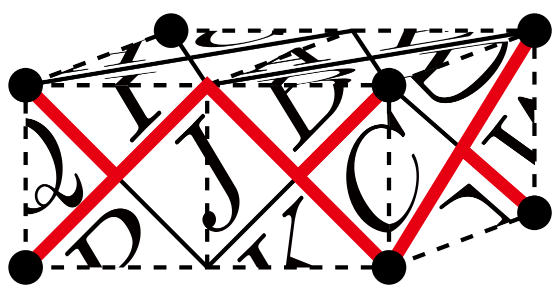
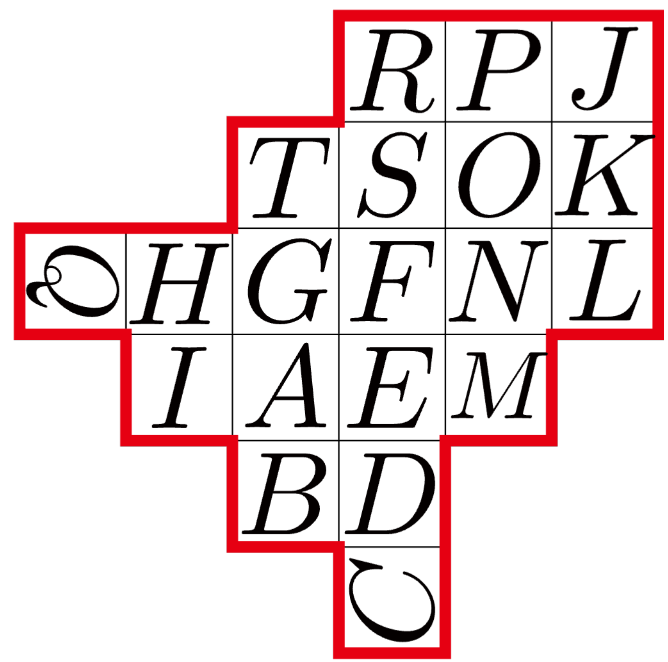
Overlapping of lattice unfolding

Let's consider unfolding a cuboid into a polyomino.

[Note] A *polyomino* is a polygon made by connecting multiple squares along their edges.



Unfold



- Let's call this type of polyomino "Lattice unfolding".

Overlapping of lattice unfolding

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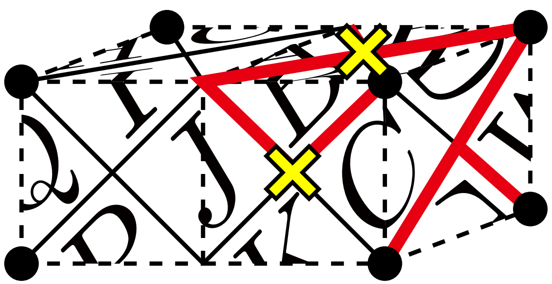
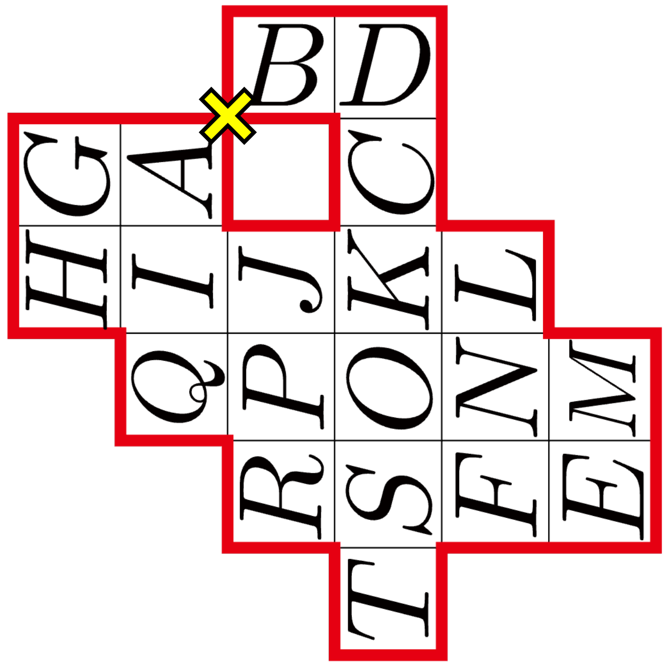


Unfold



- ➢ We call this type of unfolding "Vertices-in-touch unfolding".

Overlapping of lattice unfolding

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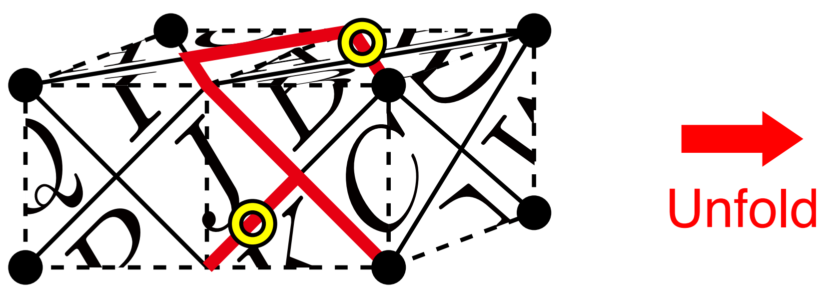

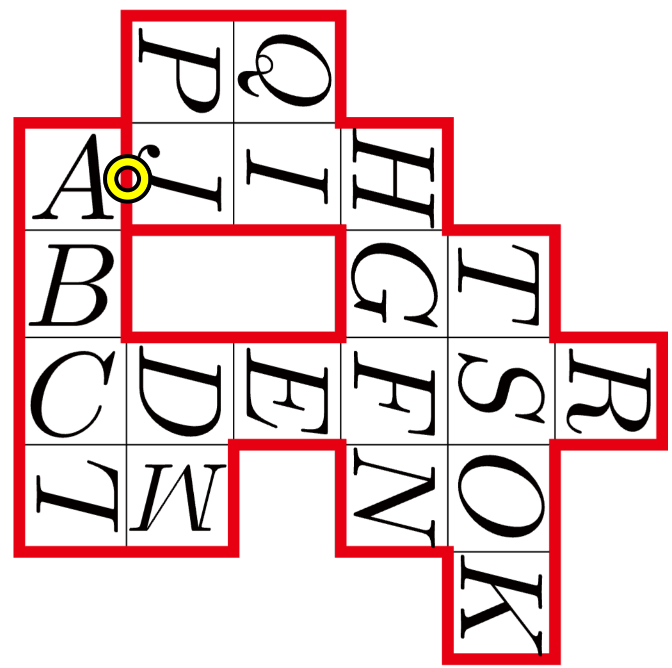

- We call this type of unfolding "Edges-in-touch unfolding".

Overlapping of lattice unfolding

Let's consider unfolding a cuboid into a polyomino.

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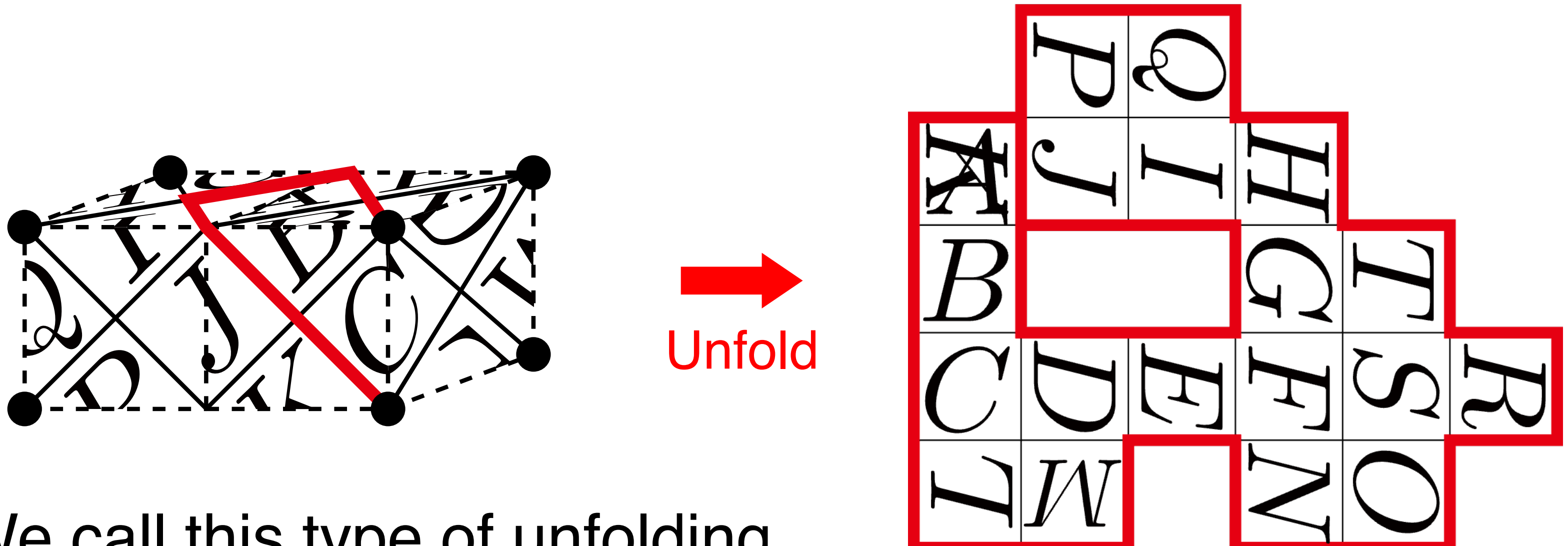


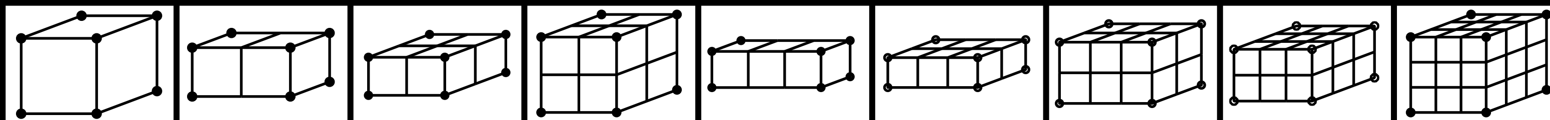
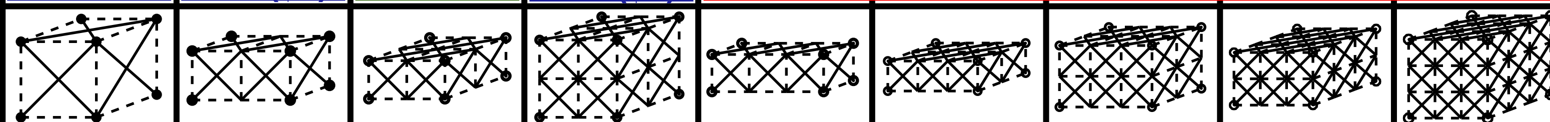
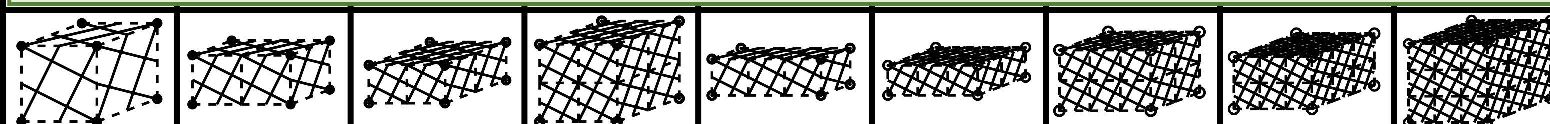
- We call this type of unfolding "Faces-in-touch unfolding".
- Please look at the distributed 3D models.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

(a,b) ※ $\gcd(a,b)=1$

(a,b)		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Open	Open	Open	**Yes** $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		Open		Open						
	F		No (†1)		No (†2)						
$(1,1)$											$\cdots$
	V	Open									
	E										
	F										
$(2,1)$											$\cdots$
	V	Open									
	E										
	F										
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Open									
	E										
	F										

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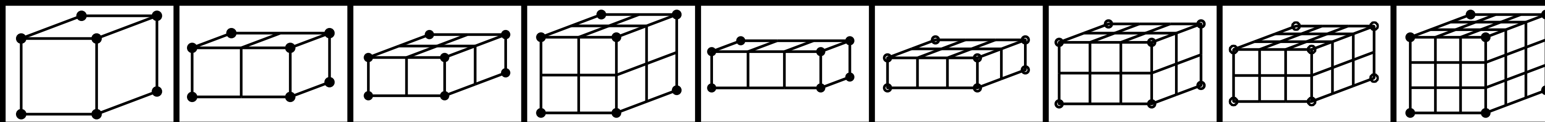
Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a, b) ※ $\gcd(a, b) = 1$

(a,b)		(x,y,z): $(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes		Yes: $(1\times1\times z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
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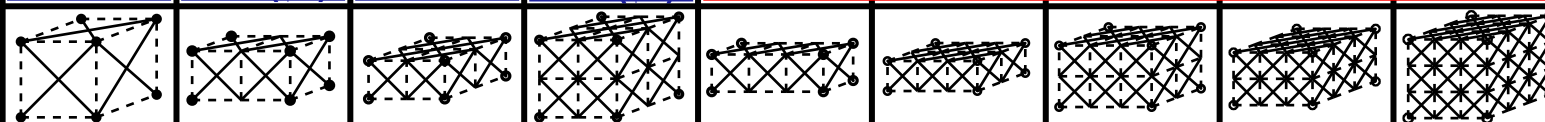
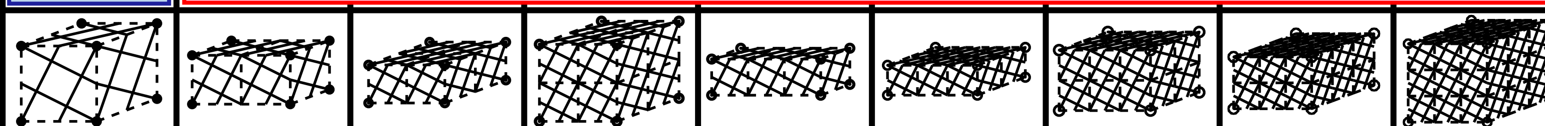
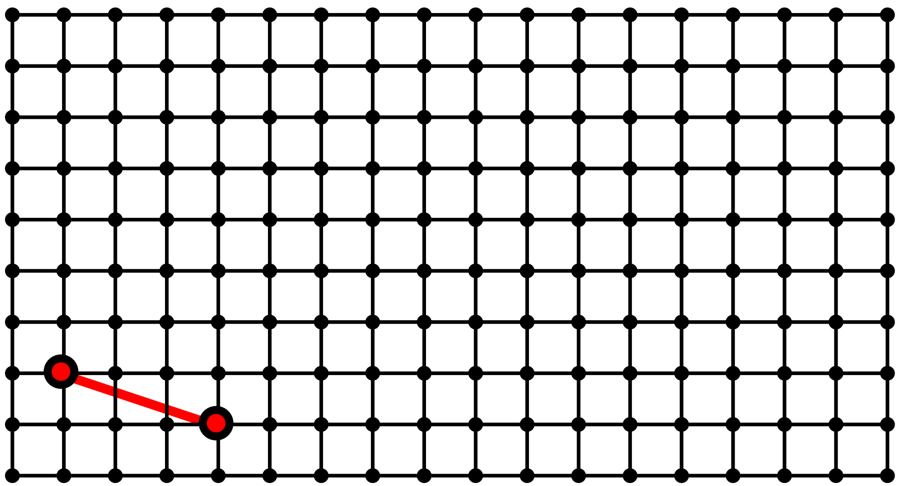
$(1,1)$											⋯
	V	No	Yes								
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

Lattice cubes

Definition 1

Choose two points on a square lattice and construct a square with these two points as one side. The cuboid assembled with this square as one face is called a lattice cube.



The square lattice

Lattice cubes

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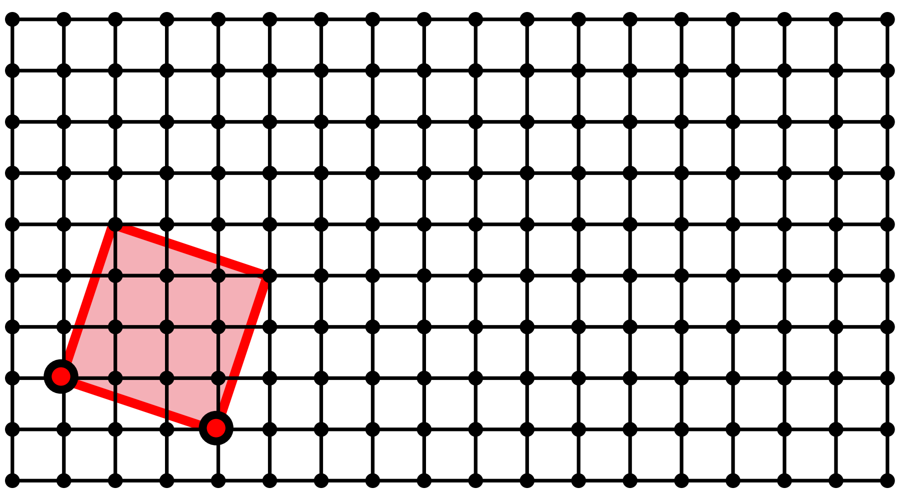

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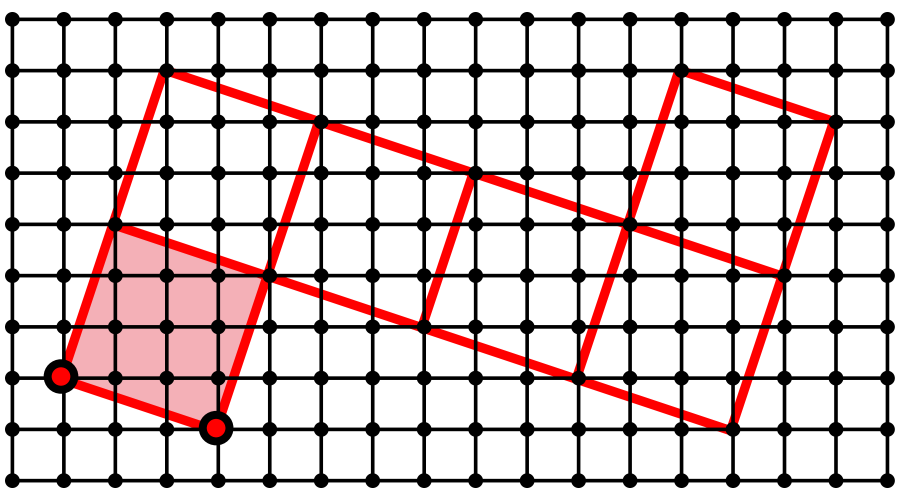

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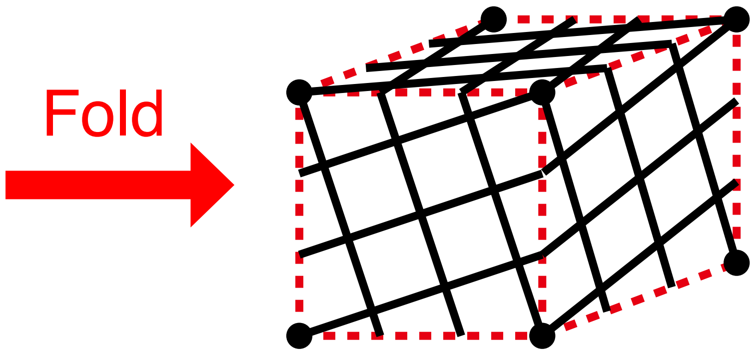


The lattice cube

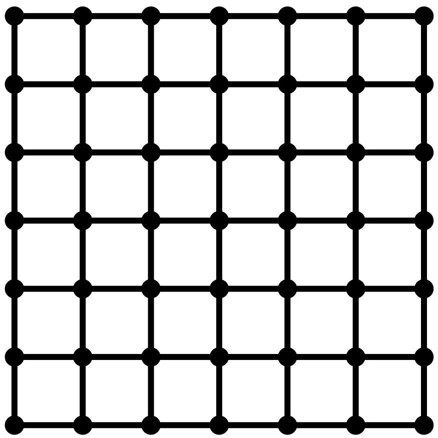
The length of one edge of a cube

We assume a square lattice of unit length (=1).

I. Choose a point $O(0,0)$ on the square lattice.

II. Let the coordinates of point A be $(a, 0)$ and B be $(0, b)$ $(\mathrm{a} \in \mathbb{N},\ \mathrm{b} \in \mathbb{N}^{+}, \mathrm{a} \geq \mathrm{b})$.

III. Let $L = |AB| = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$ be the length of one edge of a lattice cube.



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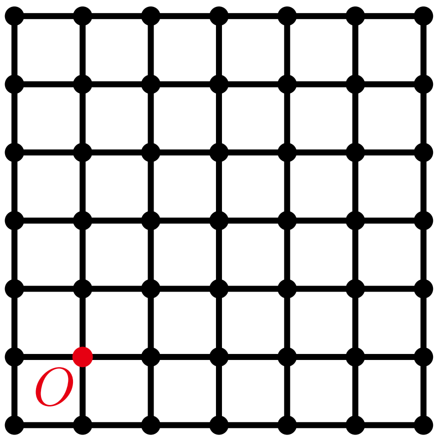

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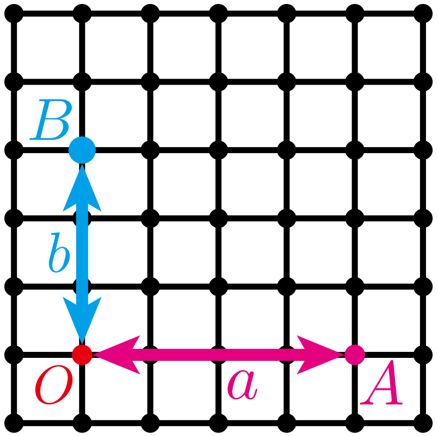

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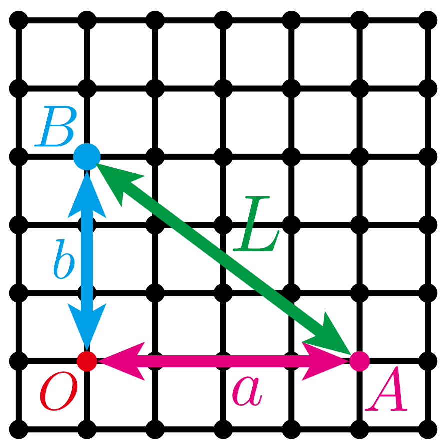
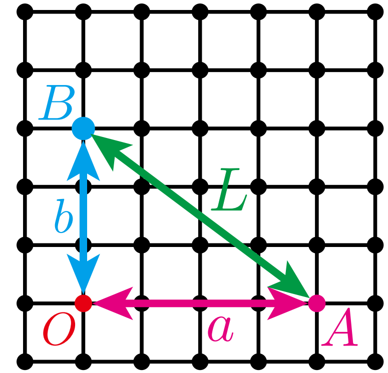
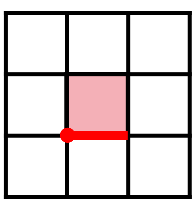
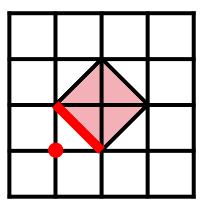
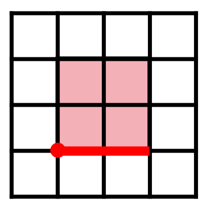
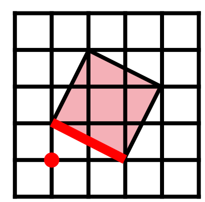
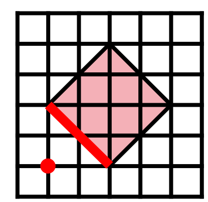
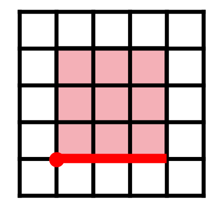
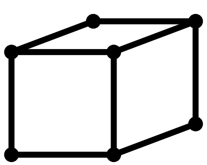
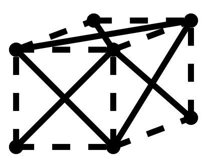
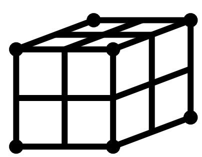
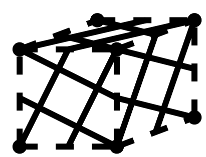
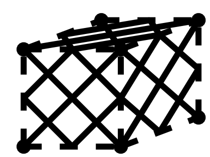
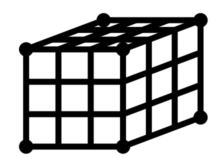


The side length of a cube



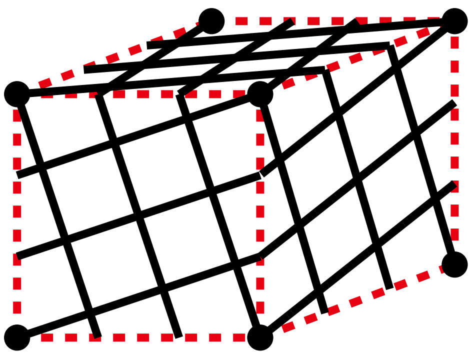
List of lattice cubes

$\boldsymbol{a}$	1	1	2	2	2	3	$\cdots$
$\boldsymbol{b}$	0	1	0	1	2	0	$\cdots$
$\boldsymbol{L}$	1	$\sqrt{2}$	2	$\sqrt{5}$	$2\sqrt{2}$	3	$\cdots$
$\boldsymbol{L\times L}$ square							$\cdots$
$\boldsymbol{L\times L\times L}$ cube							$\cdots$

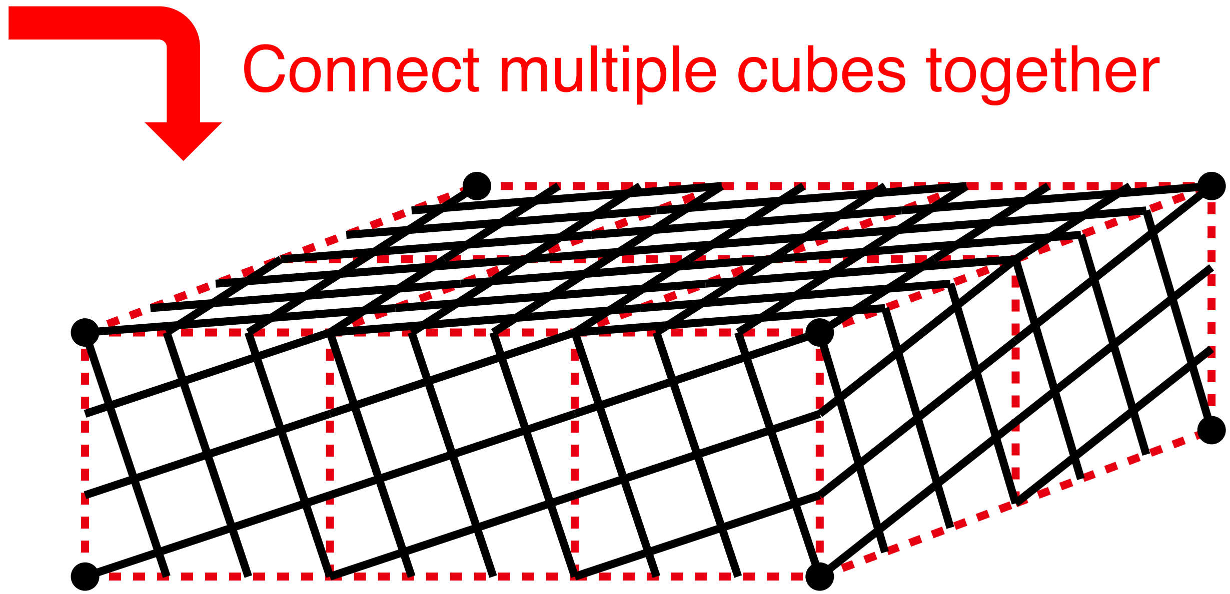
Lattice cuboids

Definition 2

A cuboid made by connecting multiple lattice cubes is called a lattice cuboid. (Note: Lattice cubes ⊂ Lattice cuboids)



The lattice cube



The lattice cuboid

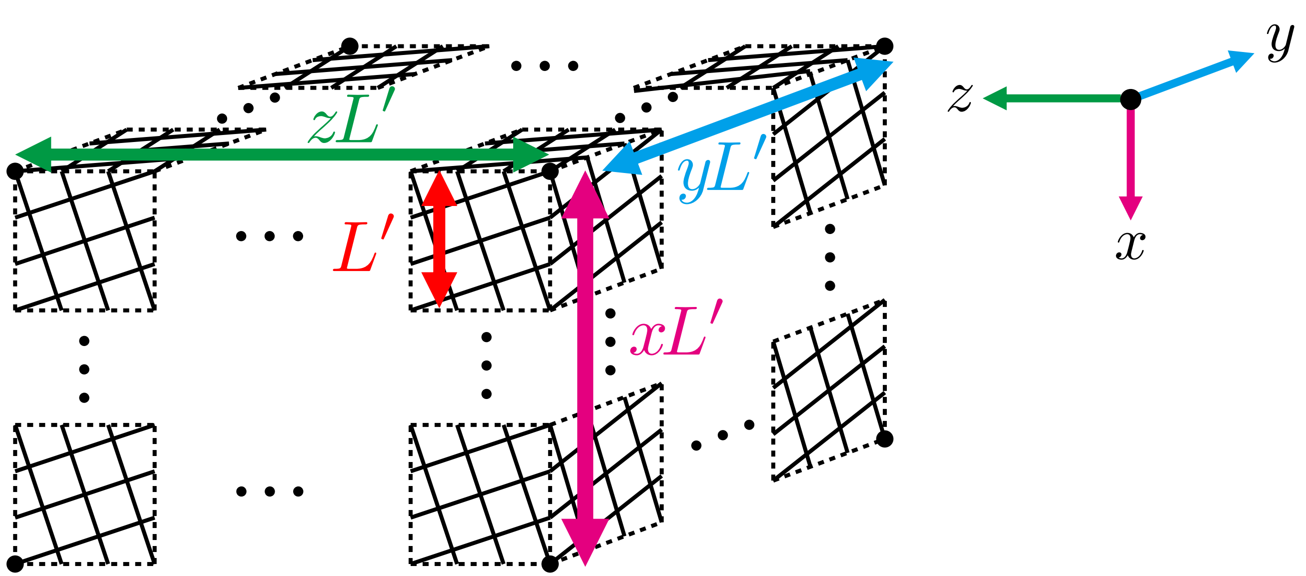
The three side lengths of a cuboid

Let L' be the length of one edge of a lattice cube.

$$L' = \sqrt{a^2 + b^2} \ (a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b, \gcd(a, b) = 1)$$

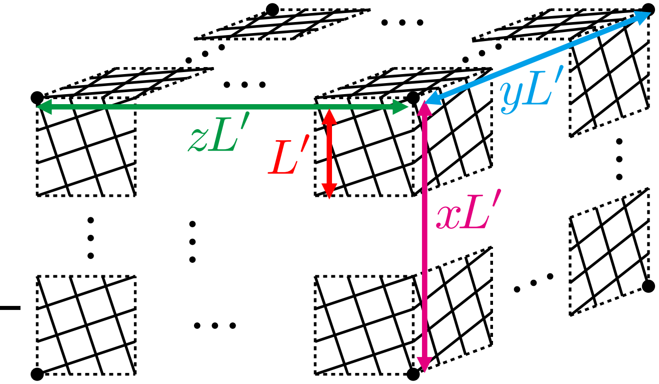
Denote the lattice cuboid as "(xL', yL', zL')-cuboid".

$$(x, y, z \in \mathbb{N},\ x \leq y \leq z)$$



The three side lengths of a cuboid

List of lattice cuboids



		(x,y,z)									
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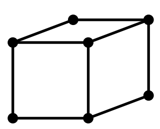
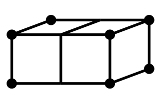
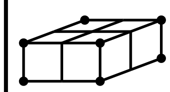
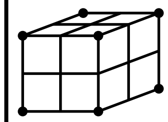
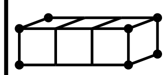
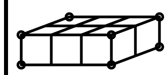
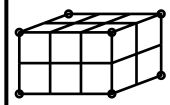
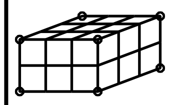
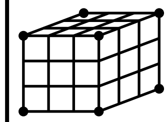
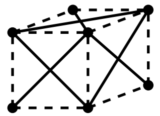
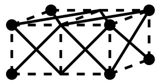
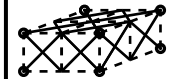
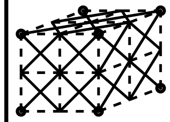
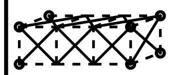
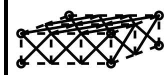
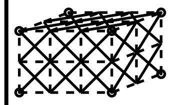
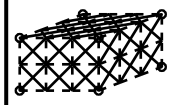
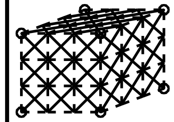
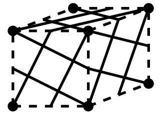
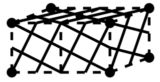
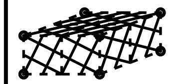
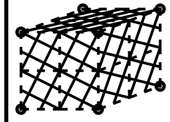
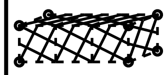
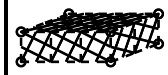
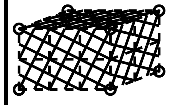
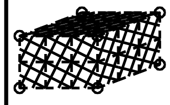
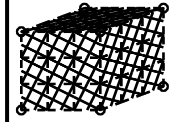
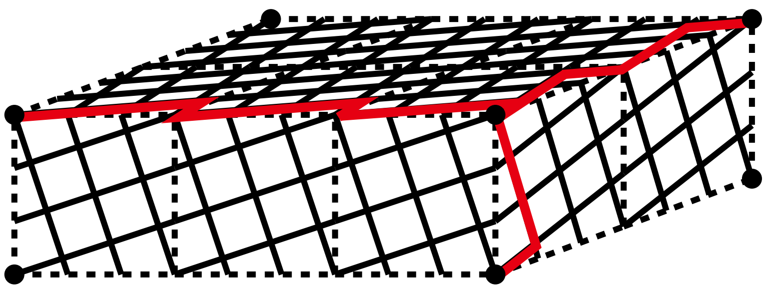
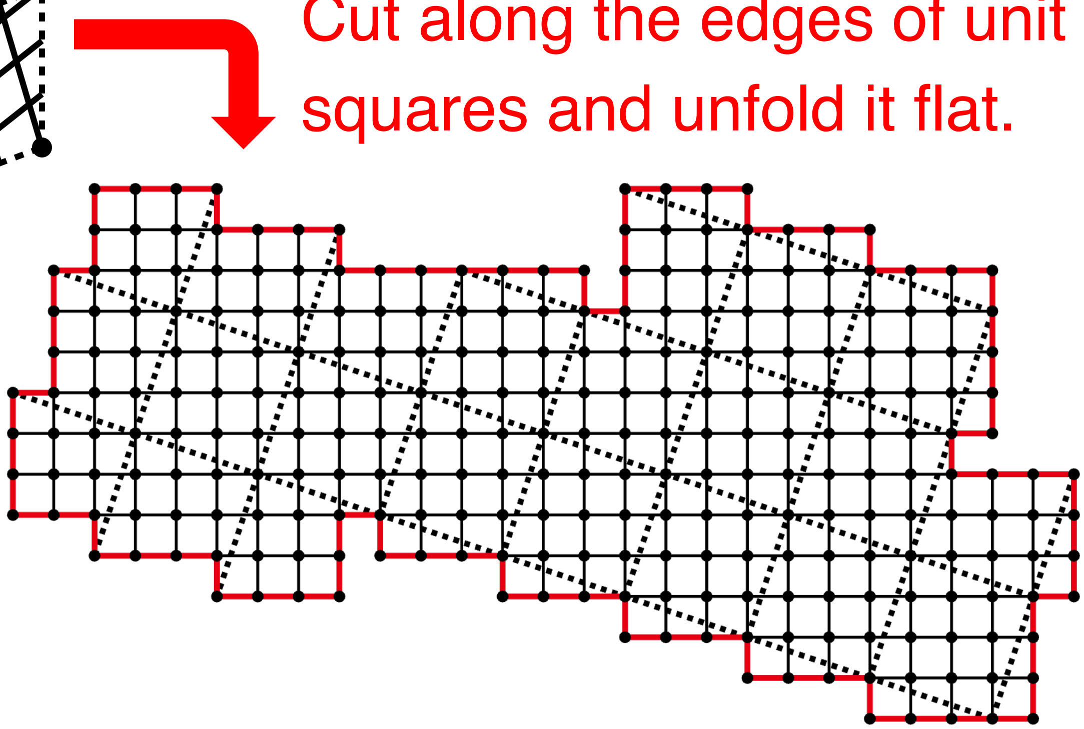
(a,b) ※ $\gcd(a,b)=1$	$(1,0)$										$\cdots$
	$(1,1)$										$\cdots$
	$(2,1)$										$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lattice unfolding for cuboids

Definition 3

A lattice unfolding is a polygon obtained by cutting the face of the cuboid along the edges of unit squares.

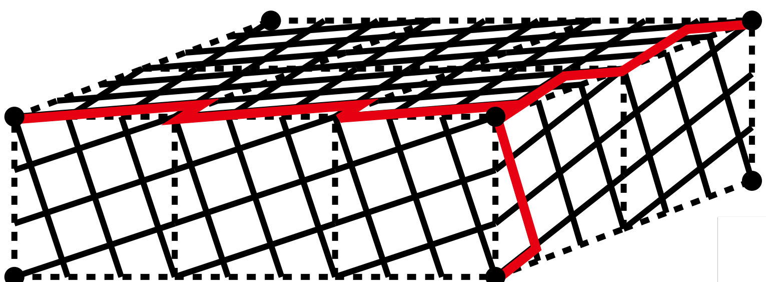

The lattice cuboid



Lattice unfolding for cuboids

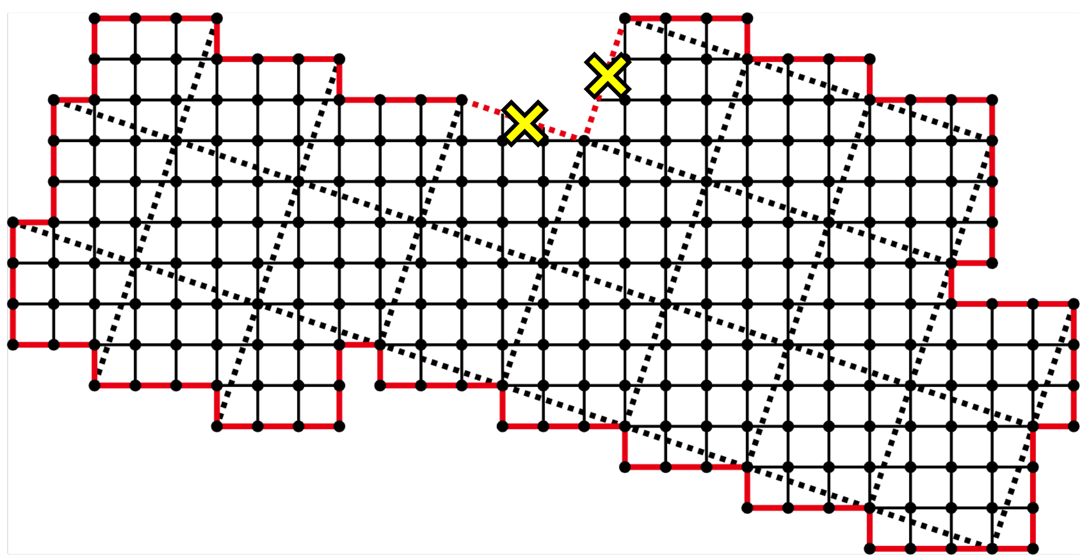
Definition 3

A lattice unfolding is a polygon obtained by cutting the face of the cuboid along the edges of unit squares.



The lattice cuboid

Cut along the edges of unit squares and unfold it flat.

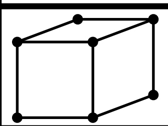
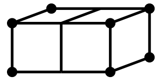
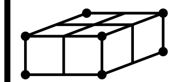
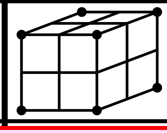
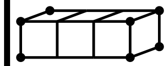
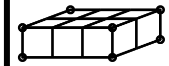
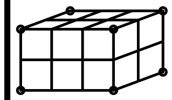
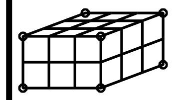
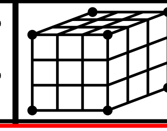
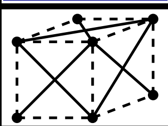
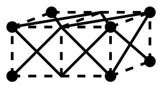
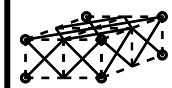
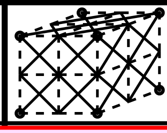
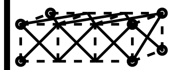
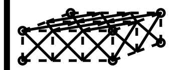
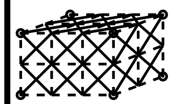
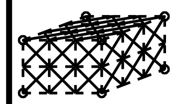
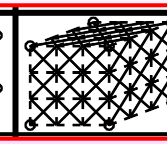
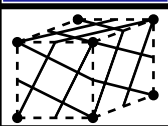
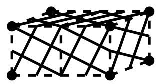
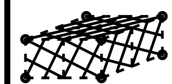
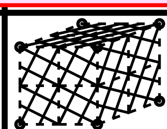
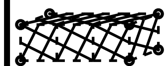
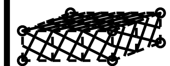
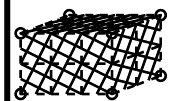
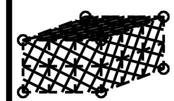
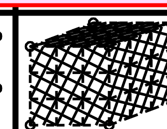


(Note)
Dotted lines ------ are folding lines (No cut)

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

(a,b) ※ $\gcd(a,b)=1$		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes		Yes — $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No	Yes								
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

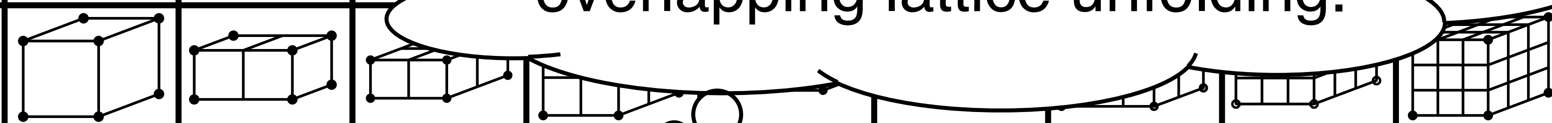
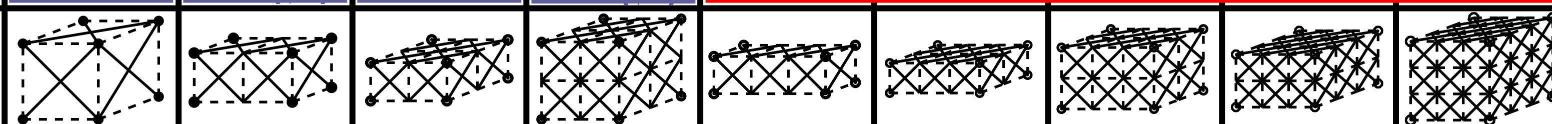
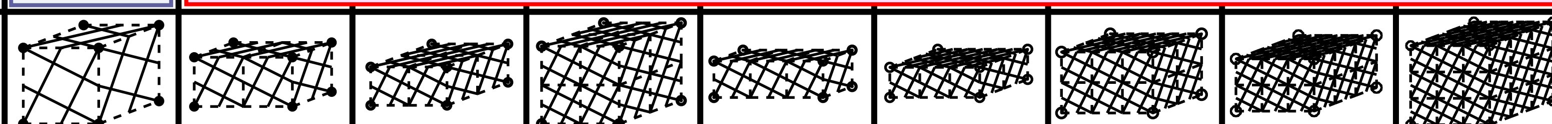
†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

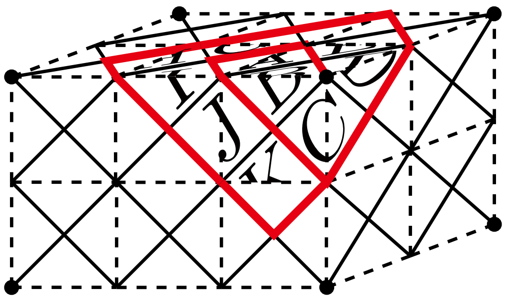
Background and our results

Demonstrate an example of overlapping lattice unfolding.

(a, b) ※ $\gcd(a, b) = 1$

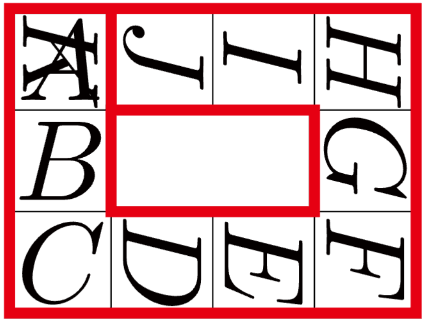
(a,b)		$(1,1,1)$	$(1,1,2)$			…
$(1,0)$						…
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]
	E		No			
	F		No (†1)	No	No (†2)	
$(1,1)$						…
	V	No	Yes			
	E					
	F					
$(2,1)$						…
	V	Yes				
	E					
	F					
⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes				
	E					
	F					

Technique to show the existence

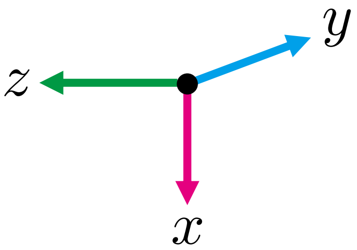


$\left(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}\right)$-cuboid

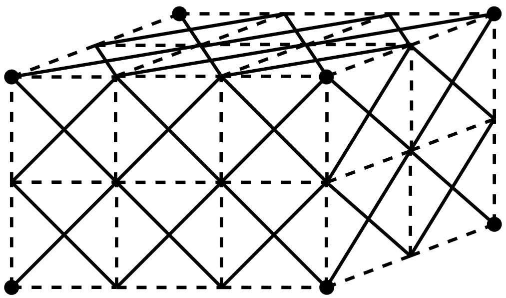
Unfold



Lattice unfolding Q_1

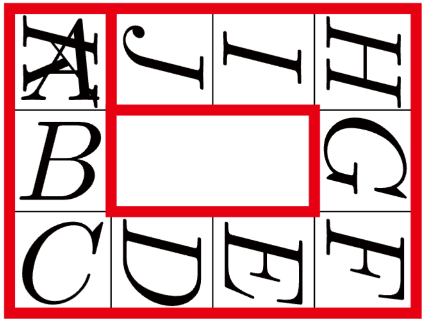


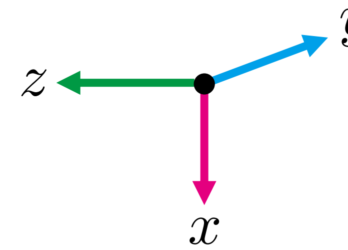
Technique to show the existence



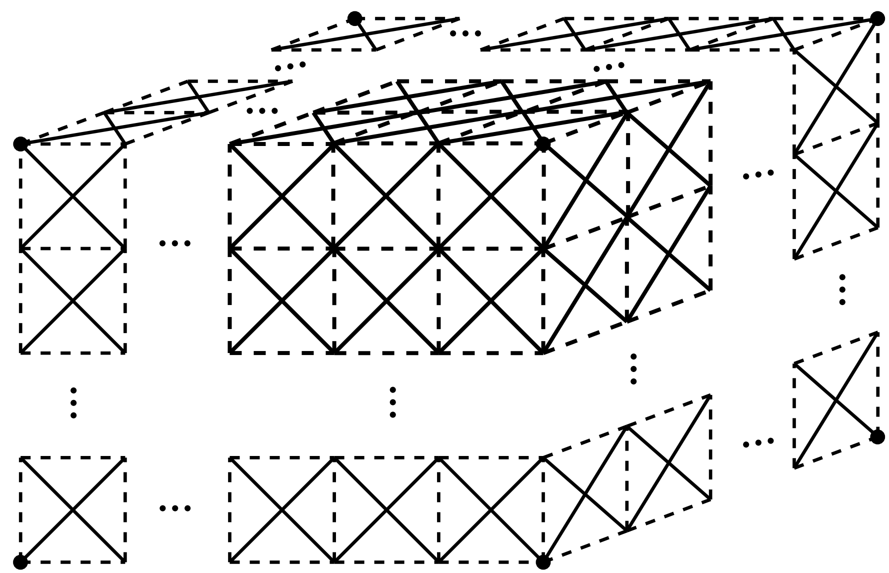
$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

Unfold



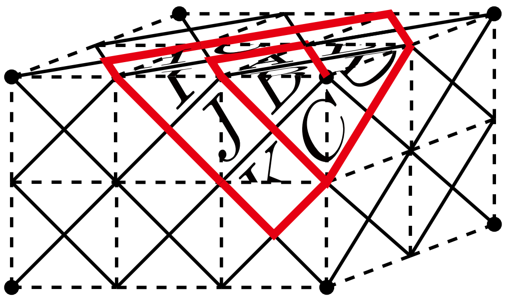


Lattice unfolding Q_1

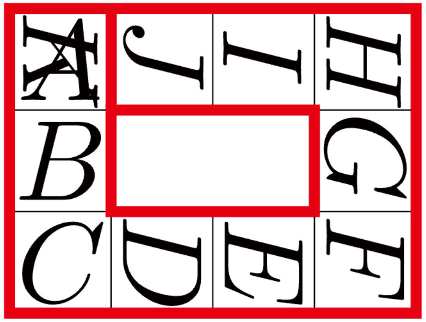


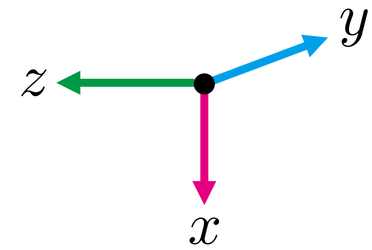
$(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid $(x \geq 2, y \geq 2, z \geq 3)$

Technique to show the existence



Unfold

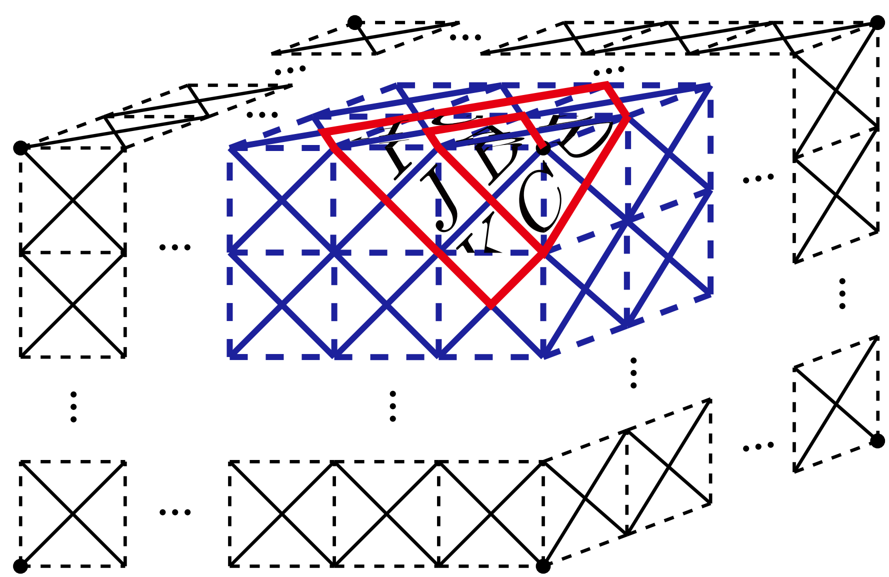




$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

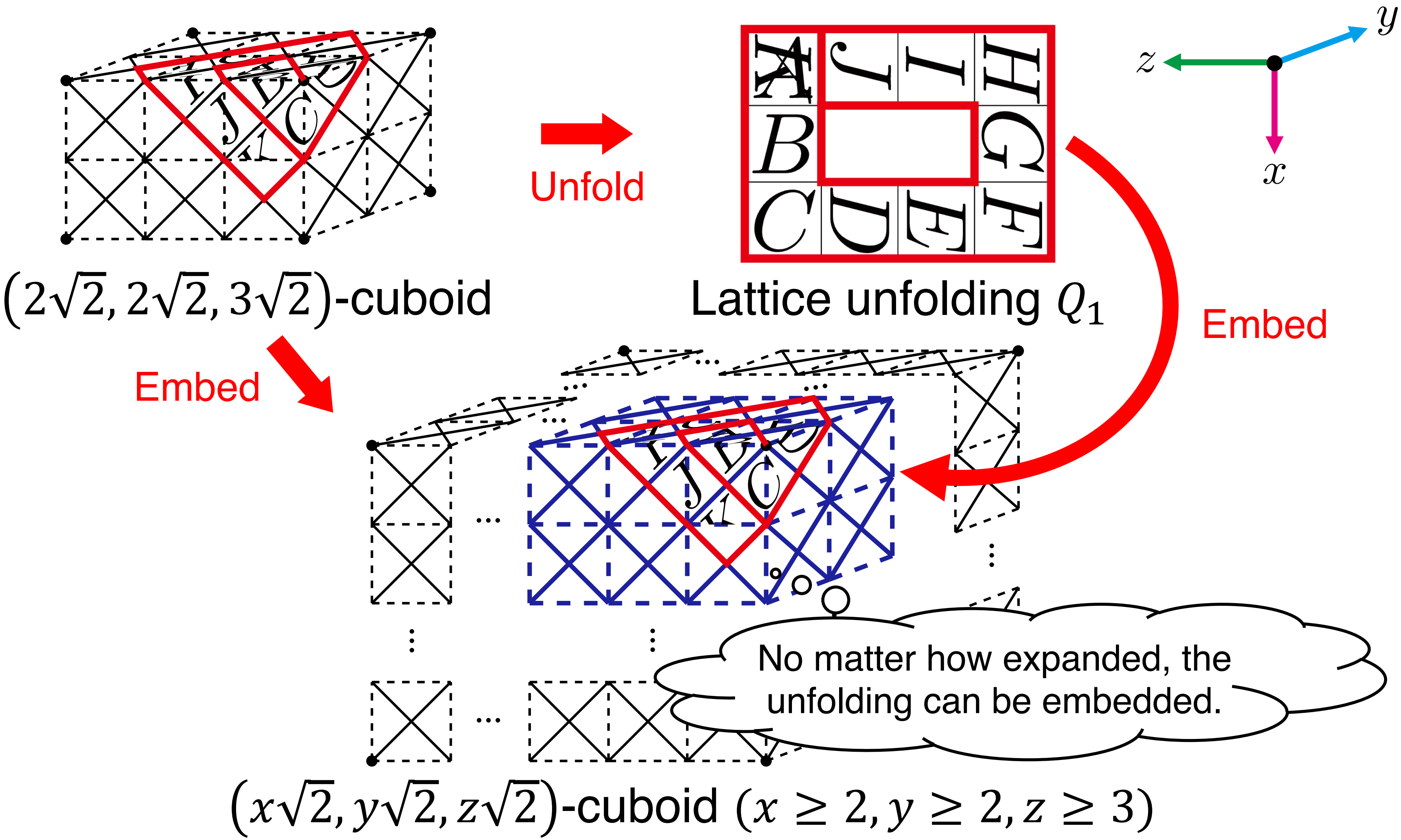
Lattice unfolding Q_1

Embed

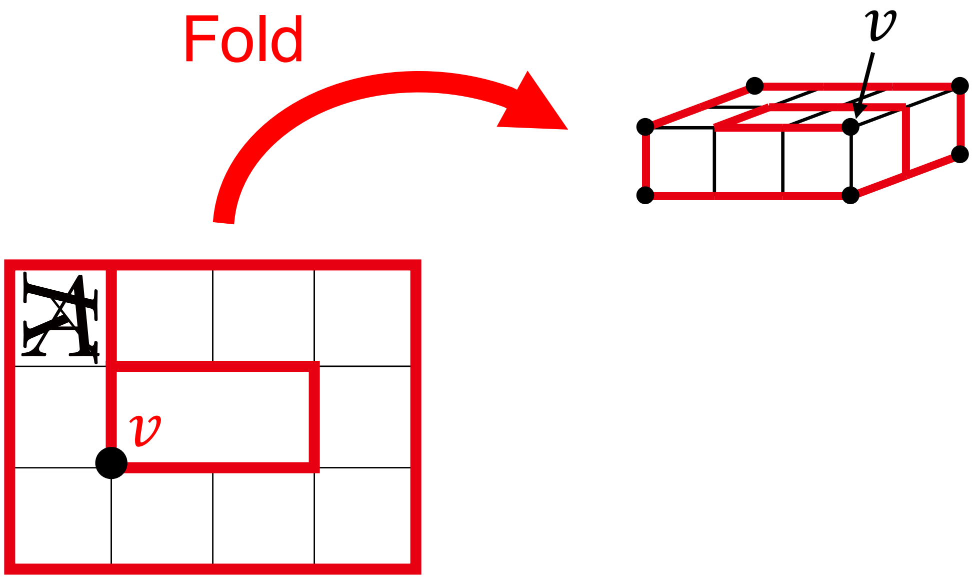


$(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid $(x \geq 2, y \geq 2, z \geq 3)$

Technique to show the existence

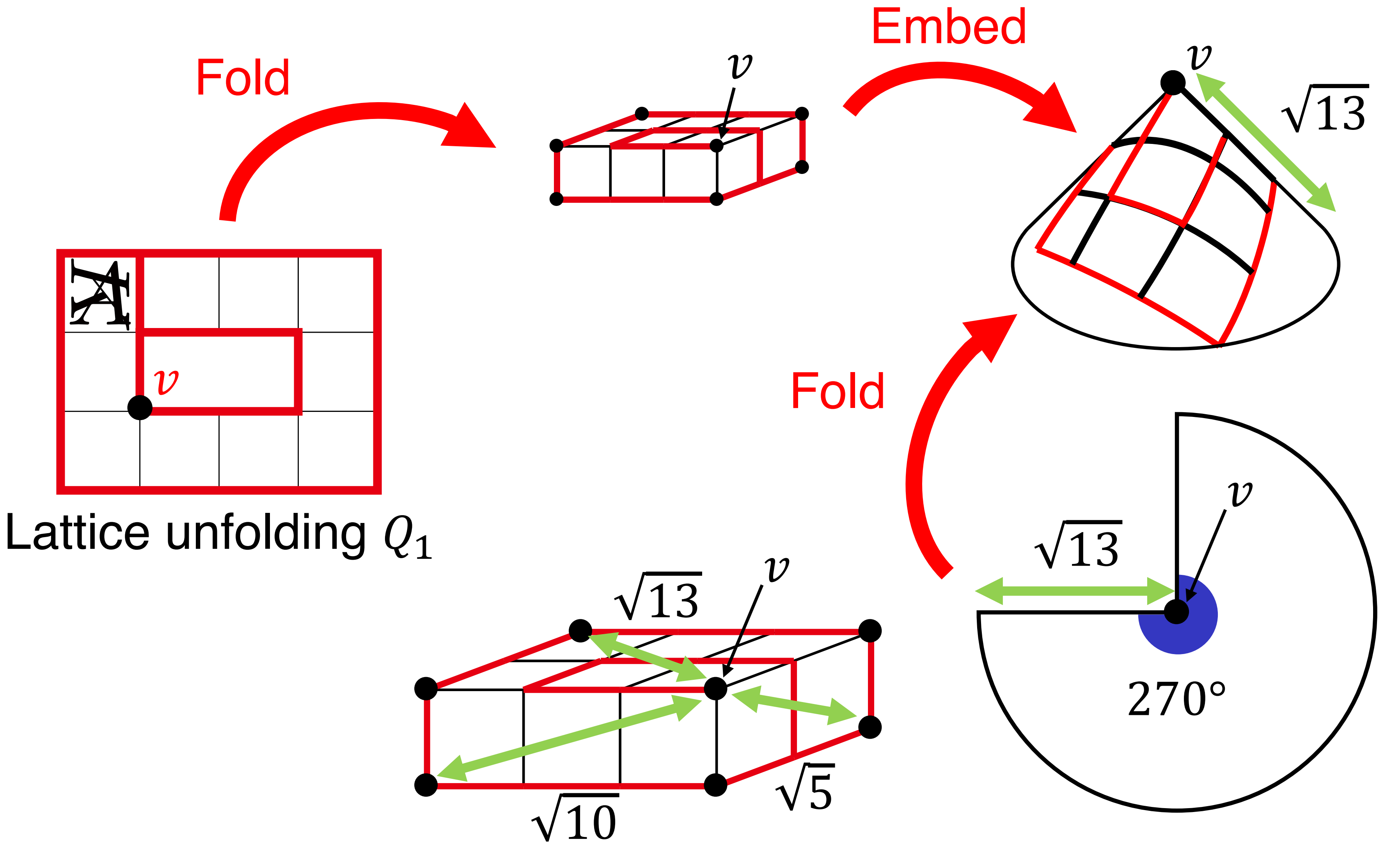


Technique to show the existence

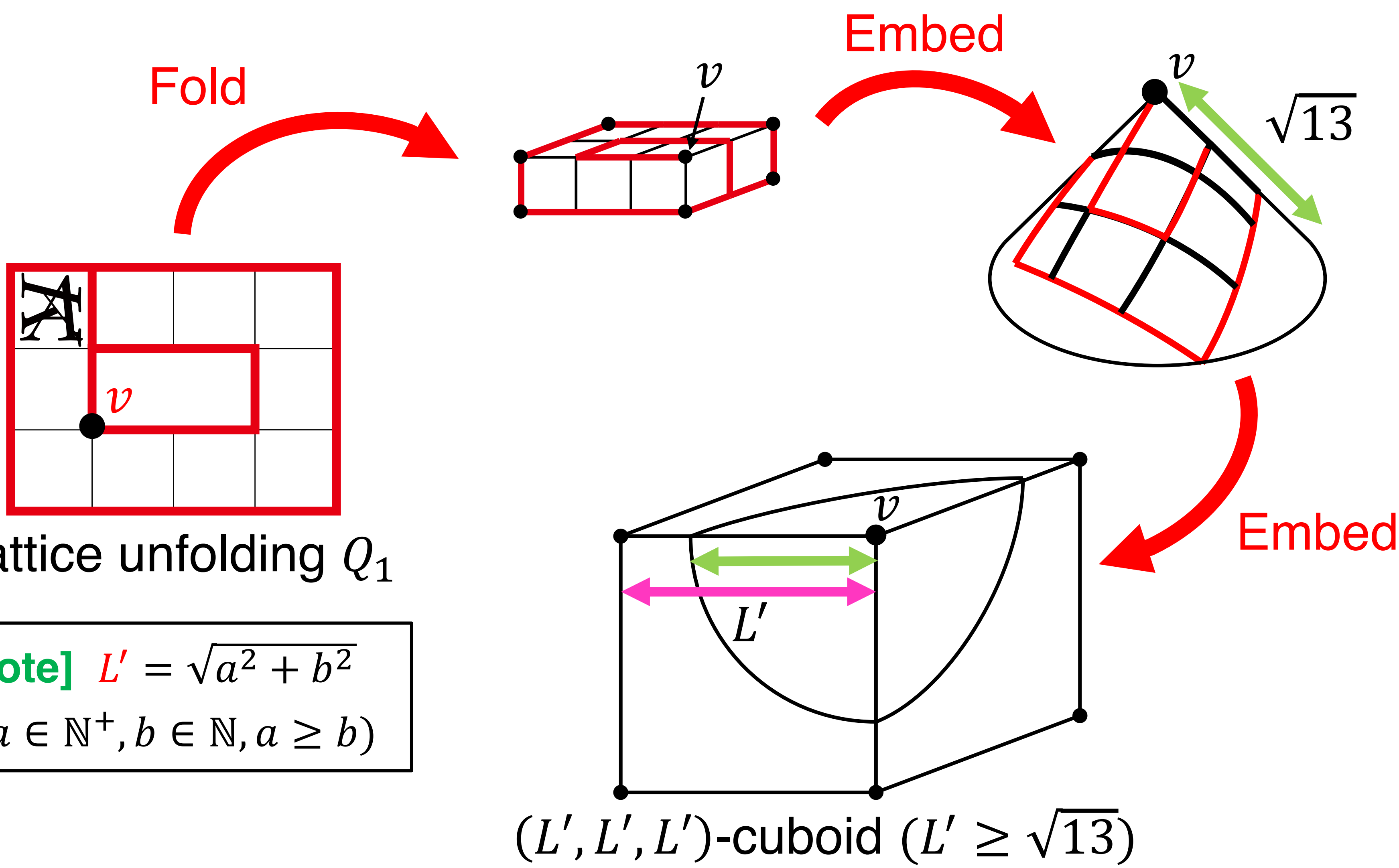


Lattice unfolding Q_1

Technique to show the existence



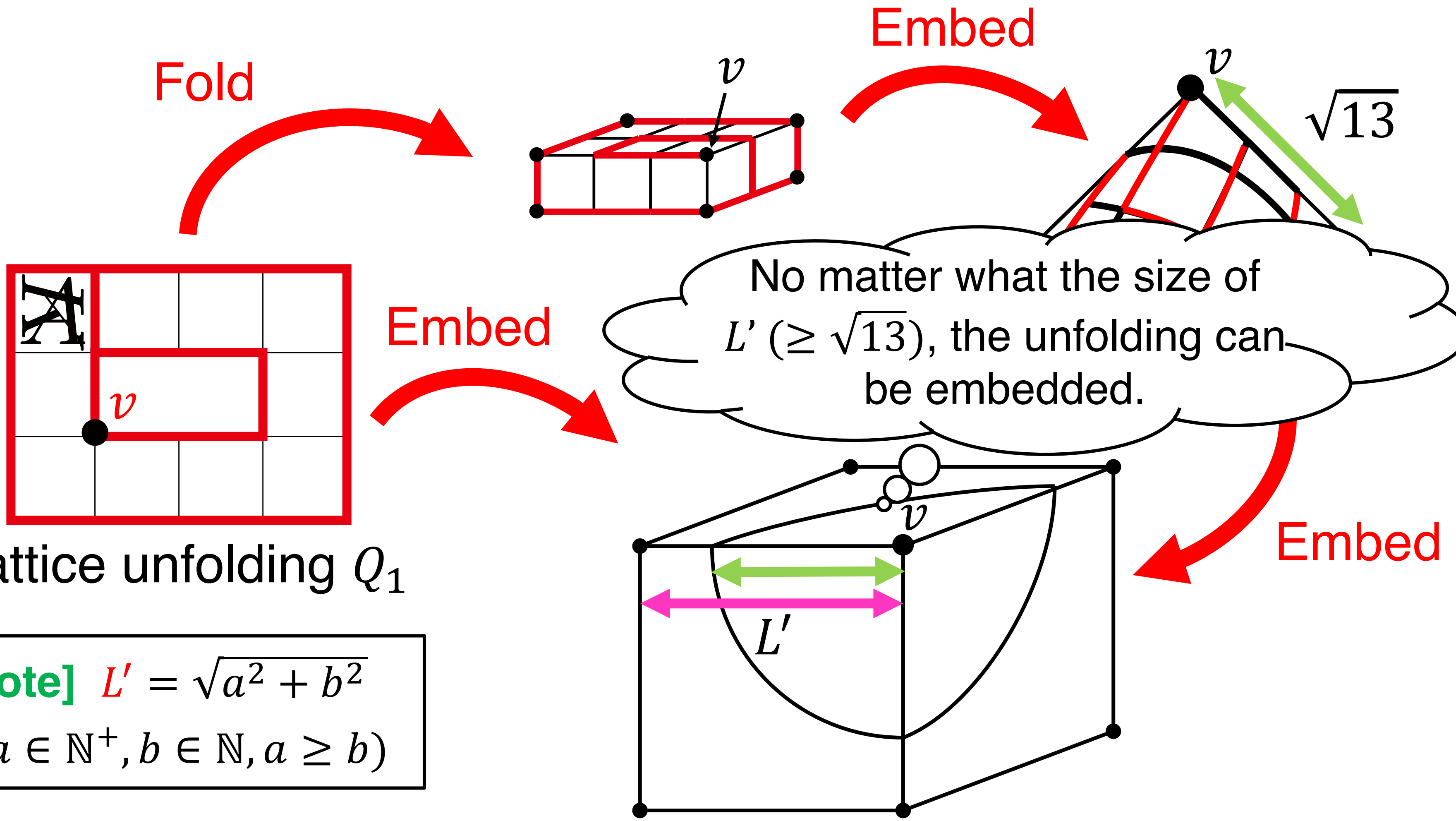
Technique to show the existence



[Note] $L' = \sqrt{a^2 + b^2}$

$(a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b)$

Technique to show the existence



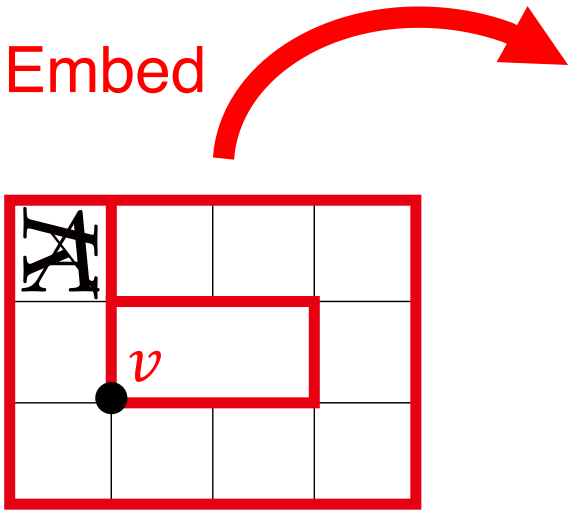
Lattice unfolding Q_1

(L', L', L')-cuboid $(L' \geq \sqrt{13})$

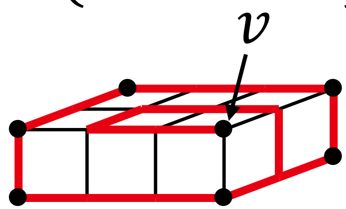
[Note] $L' = \sqrt{a^2 + b^2}$
$(a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b)$

Technique to show the existence

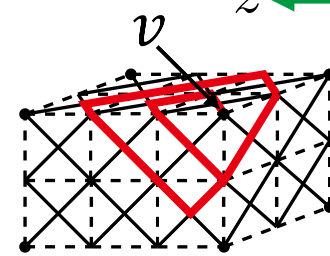
For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$

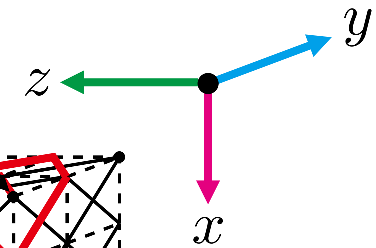


Lattice unfolding Q_1

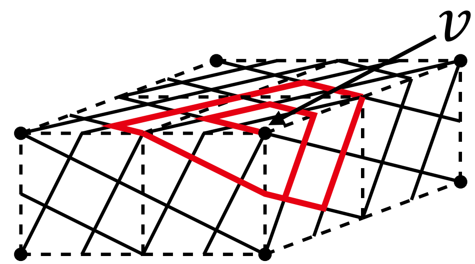


$(1,2,3)$-cuboid
[J. Mitani et al., 2008]

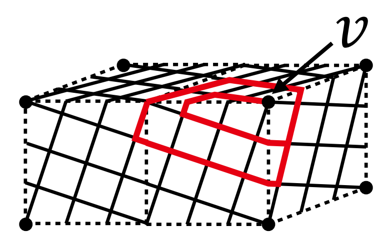




$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid



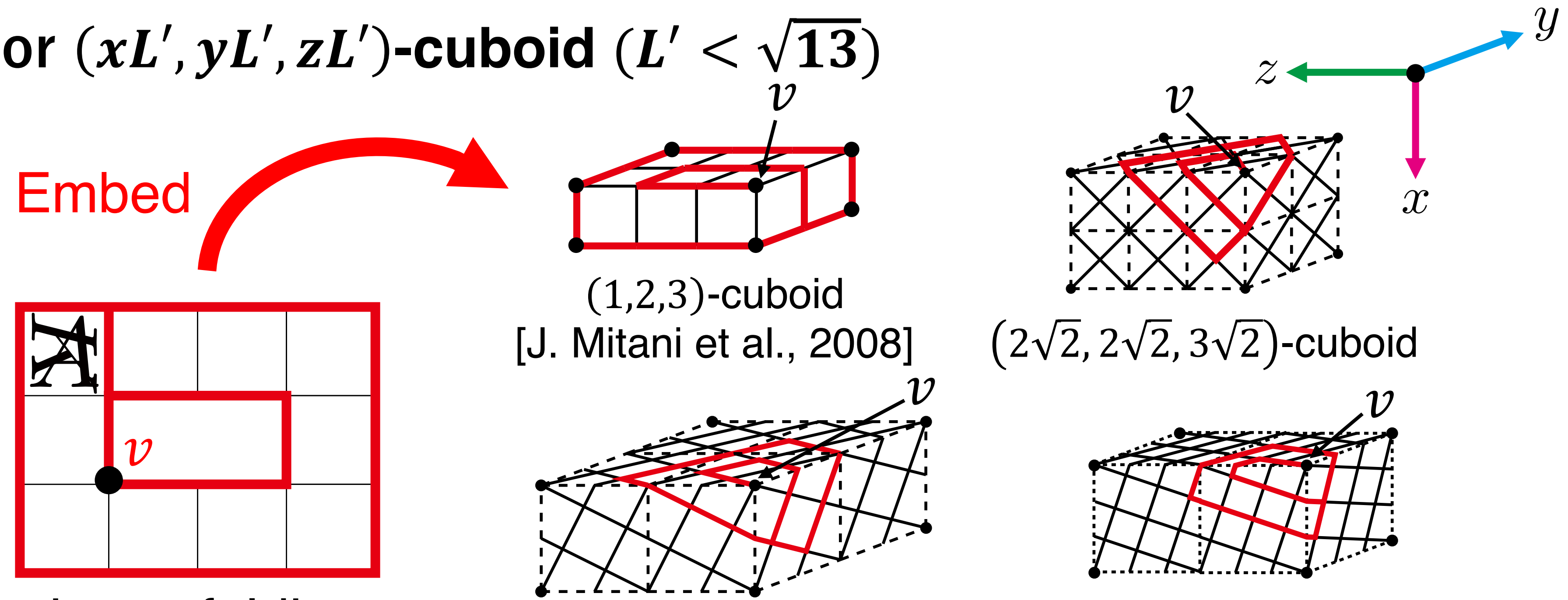
$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid



$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid

Technique to show the existence

For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$



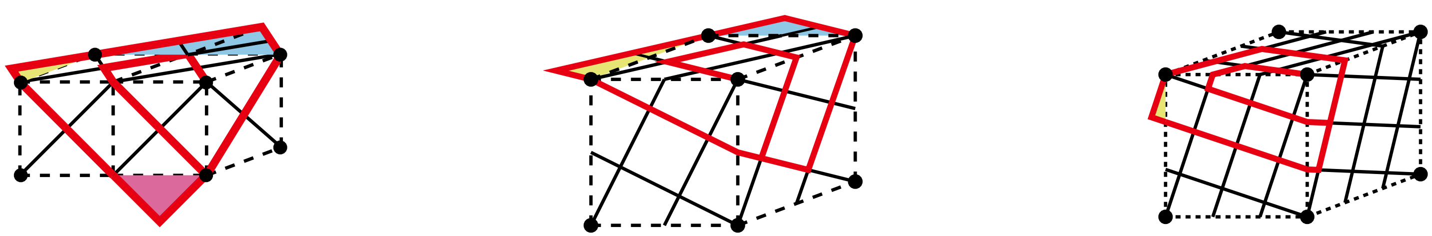
Lattice unfolding Q_1

(1,2,3)-cuboid
[J. Mitani et al., 2008]

$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid

$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid



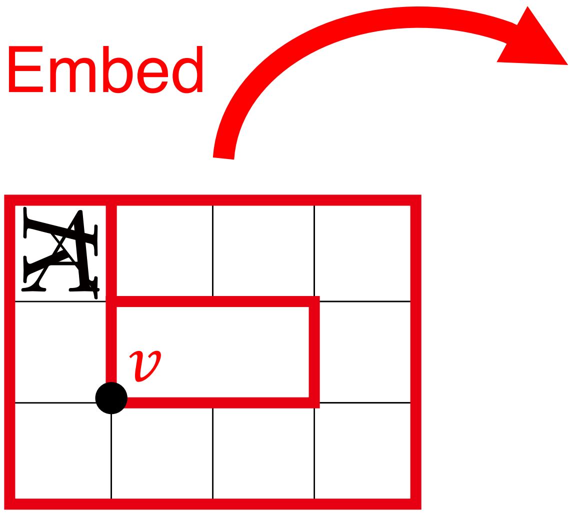
$(\sqrt{2}, \sqrt{2}, 2\sqrt{2})$-cuboid

$(\sqrt{5}, \sqrt{5}, \sqrt{5})$-cuboid

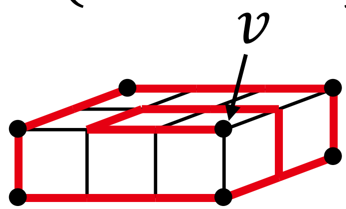
$(\sqrt{10}, \sqrt{10}, \sqrt{10})$-cuboid

Technique to show the existence

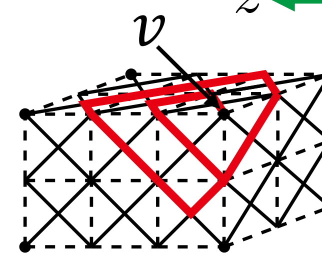
For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$



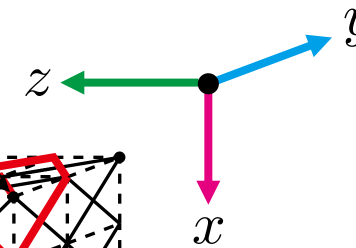
Lattice unfolding Q_1

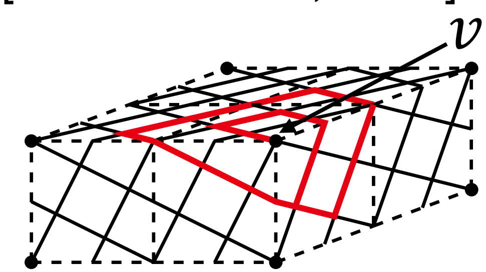


(1,2,3)-cuboid
[J. Mitani et al., 2008]

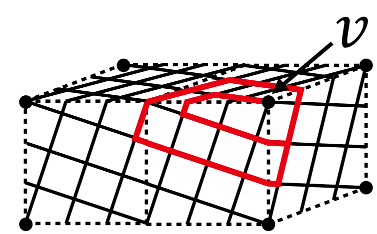


$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

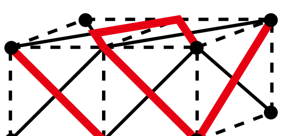




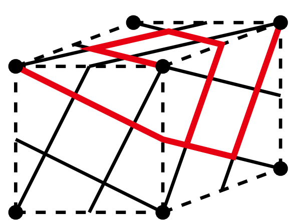
$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid



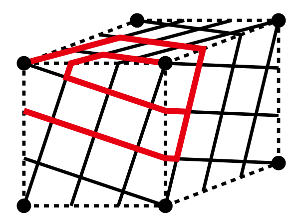
$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid



$(\sqrt{2}, \sqrt{2}, 2\sqrt{2})$-cuboid



$(\sqrt{5}, \sqrt{5}, \sqrt{5})$-cuboid



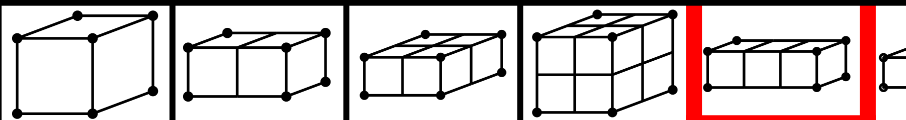
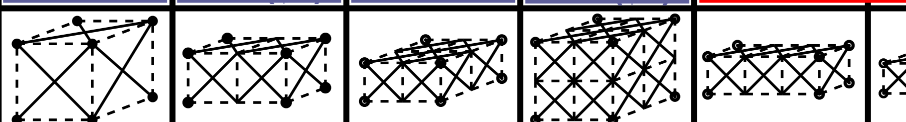
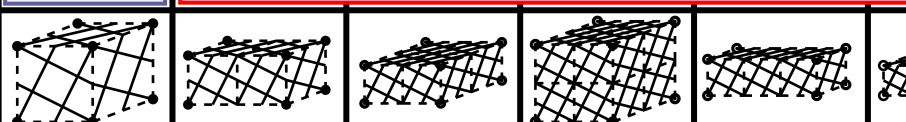
$(\sqrt{10}, \sqrt{10}, \sqrt{10})$-cuboid

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

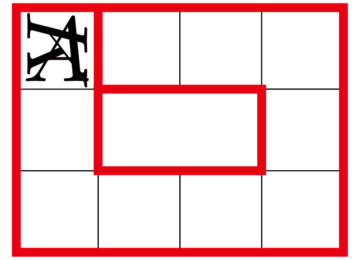
Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a, b) ※ $\gcd(a, b) = 1$

(a,b)		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	(1
		(x,y,z)					
$(1,0)$							
	V	No (Obvi.)	Yes	Yes		Yes (1× Othe	
	E		No				
	F		No (†1)	No	No (†2)		
$(1,1)$							
	V	No	Ye				
	E						
	F						
$(2,1)$							
	V	Yes					
	E						
	F						
⋮		⋮	⋮	⋮	⋮	⋮	
	V	Yes					
	E						
	F						

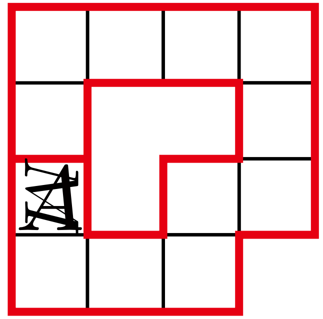
Gadgets for Faces-in-touch unfolding



Lattice unfolding Q_1

[Except]

$(1,1,z)$-cuboid $(z \geq 3)$

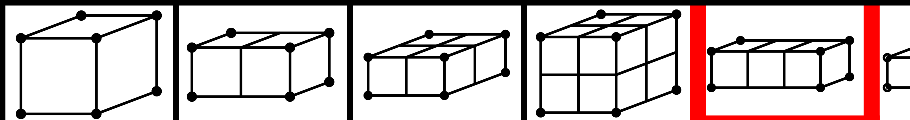
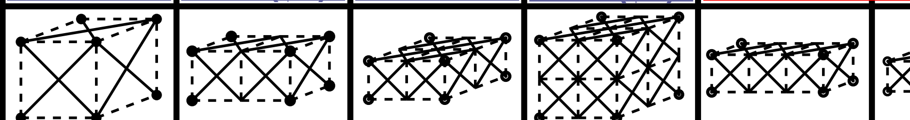
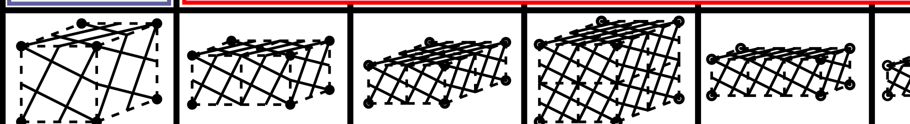


[T.Uno, 2008]

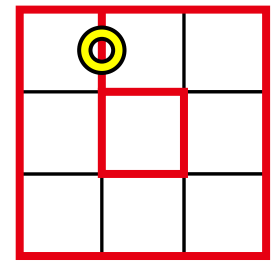
†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a,b) ※ $\gcd(a,b)=1$		(x,y,z) $(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1$
$(1,0)$							
	V	No (Obvi.)	Yes	Yes		Yes $(1\times$ Othe	
	E		No				
	F		No (†1)	No	No (†2)		
$(1,1)$							
	V	No	Ye				
	E						
	F						
$(2,1)$							
	V	Yes					
	E						
	F						
⋮		⋮	⋮	⋮	⋮	⋮	
	V	Yes					
	E						
	F						

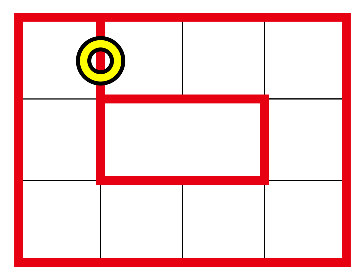
Gadgets for Edges-in-touch unfolding



Lattice unfolding Q_2

[Except]

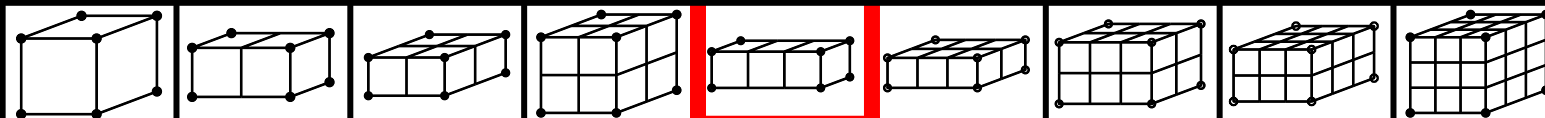
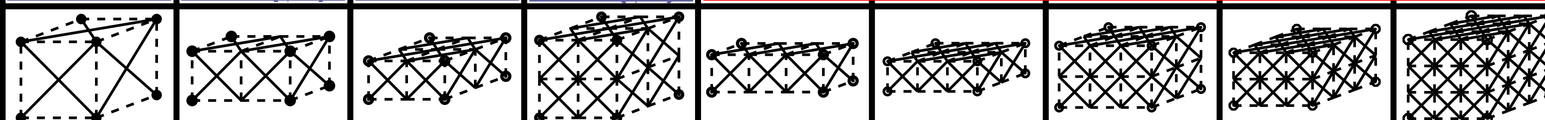
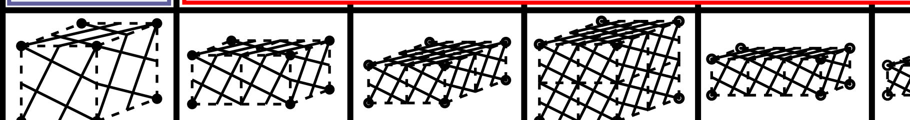
$(1,1,z)$-cuboid $(z \geq 3)$



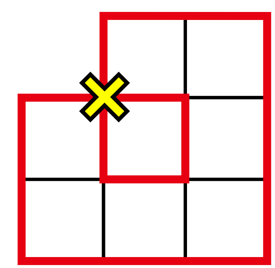
Lattice unfolding Q_1

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

(a,b) ※ $\gcd(a,b)=1$		(1,1,1)	(1,1,2)	(1,2,2)	(2,2,2)	(1,1,3)	(1,2,3)	(2,2,3)	(2,3,3)	(3,3,3)	⋯
(1,0) 											⋯
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
(1,1) 											⋯
	V	No	Ye								
	E										
	F										
(2,1) 											
	V	Yes									
	E										
	F										
	⋮	⋮	⋮	⋮	⋮	⋮	⋮				
	V	Yes									
	E										
	F										

Gadgets for Vertices-in-touch unfolding

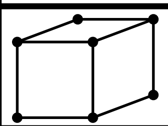
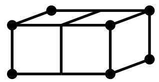
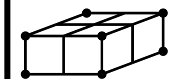
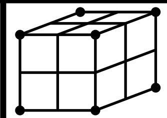
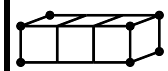
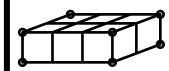
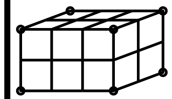
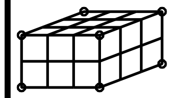
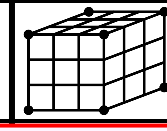
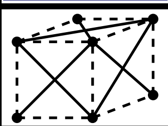
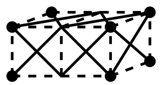
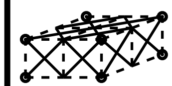
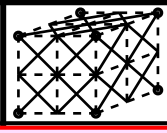
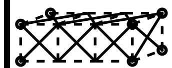
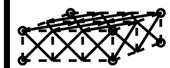
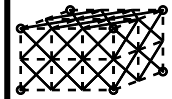
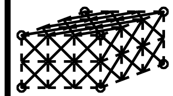
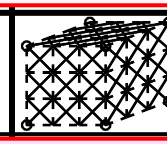
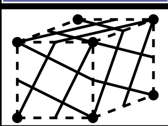
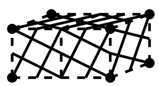
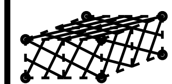
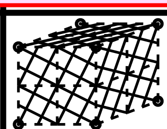
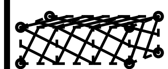
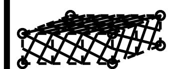
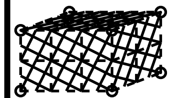
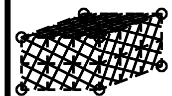
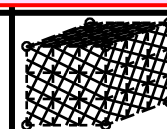


Lattice unfolding Q_3

Background and our results

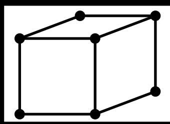
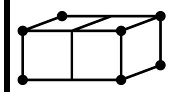
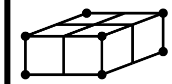
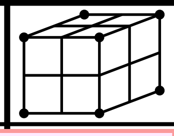
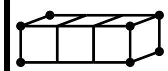
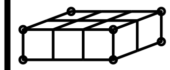
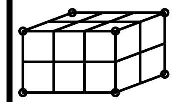
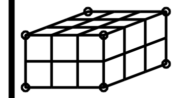
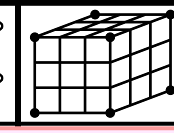
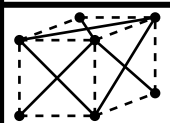
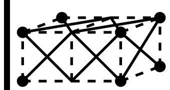
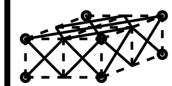
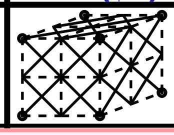
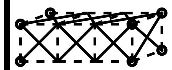
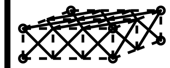
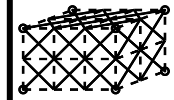
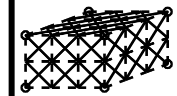
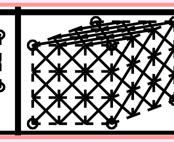
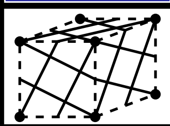
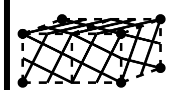
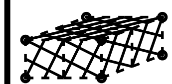
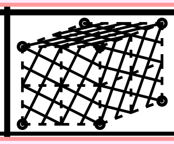
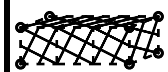
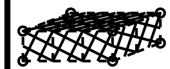
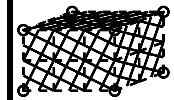
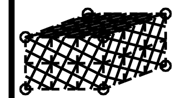
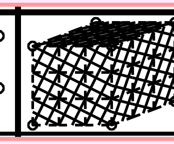
V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a,b) ※ $\gcd(a,b)=1$

(a,b)		(x,y,z): $(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No	Yes								
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

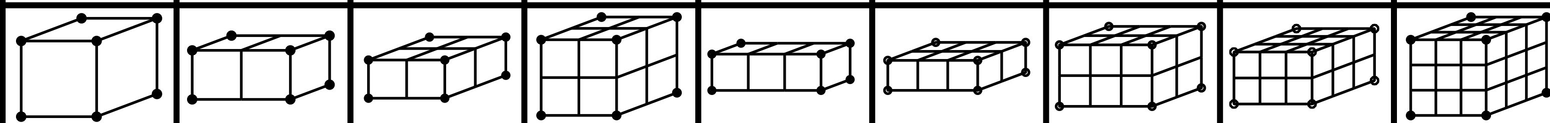
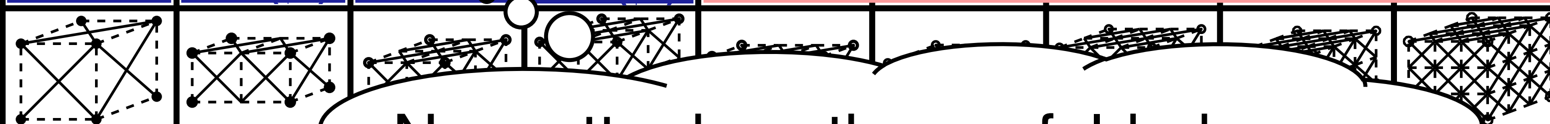
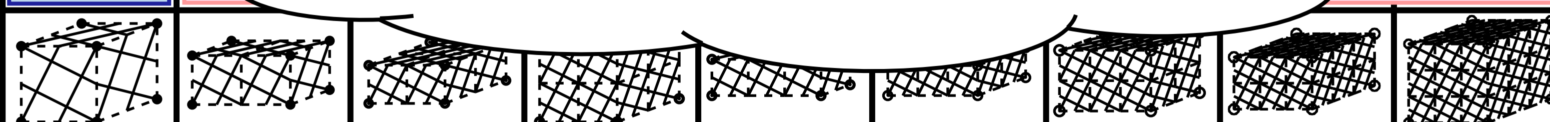
(a,b) ※ $\gcd(a,b)=1$		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No	Yes								
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

Background and our results

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

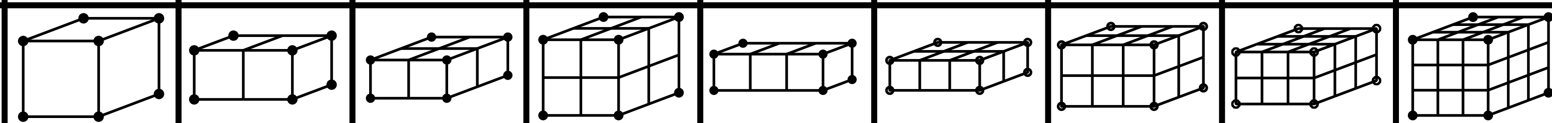
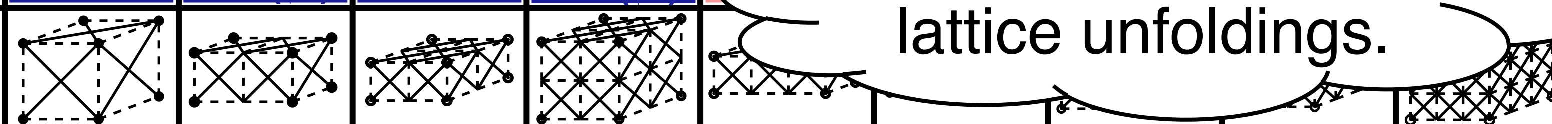
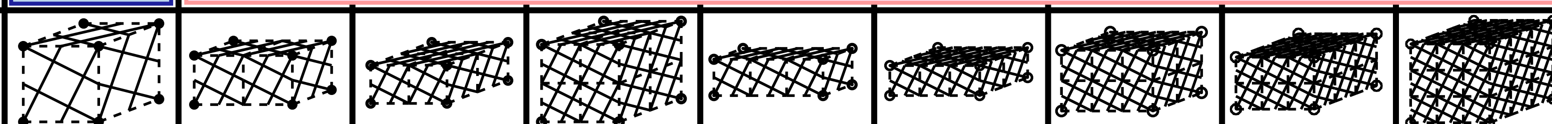
(a,b) ※ $\gcd(a,b)=1$

(a,b)		(x,y,z): (1, 1, 1)	(1, 1, 2)	(1, 2, 2)	(2, 2, 2)	(1, 1, 3)	(1, 2, 3)	(2, 2, 3)	(2, 3, 3)	(3, 3, 3)	⋯
(1, 0)											⋯
	V	No (Obvi.)	Yes	Yes	Yes	Yes — $(1\times1\times z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
(1, 1)											⋯
	V	No									
	E										
	F										
(2, 1)											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

No matter how they unfolded,
they do not overlap.

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

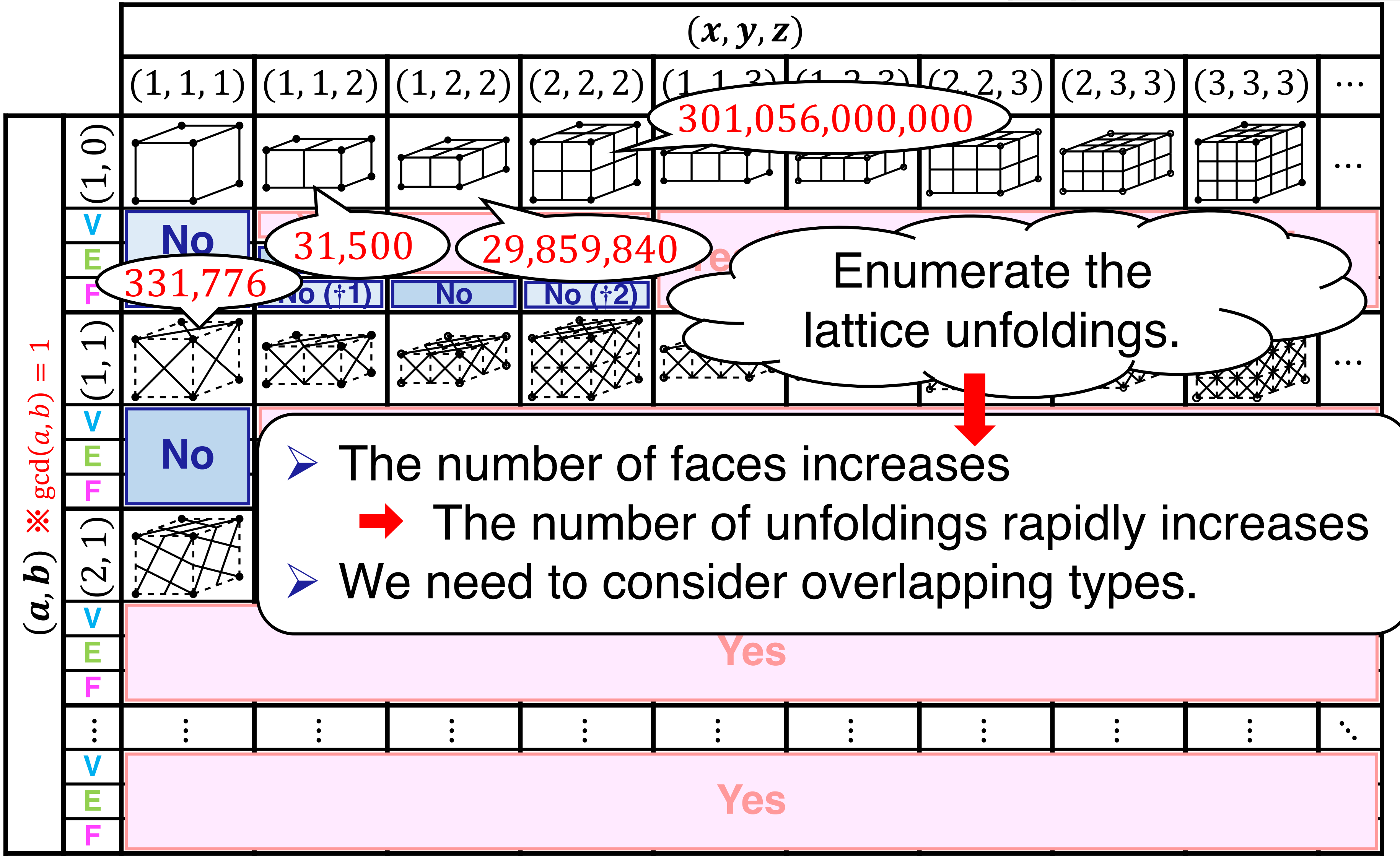
(a,b) ※ $\gcd(a,b)=1$		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes	Yes	Ye…					
	E		No	Yes	Yes						
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	E	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	F	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
$(2,1)$											⋯
	V	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	E	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	F	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	E	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	
	F	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	

Enumerate the lattice unfoldings.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch



†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
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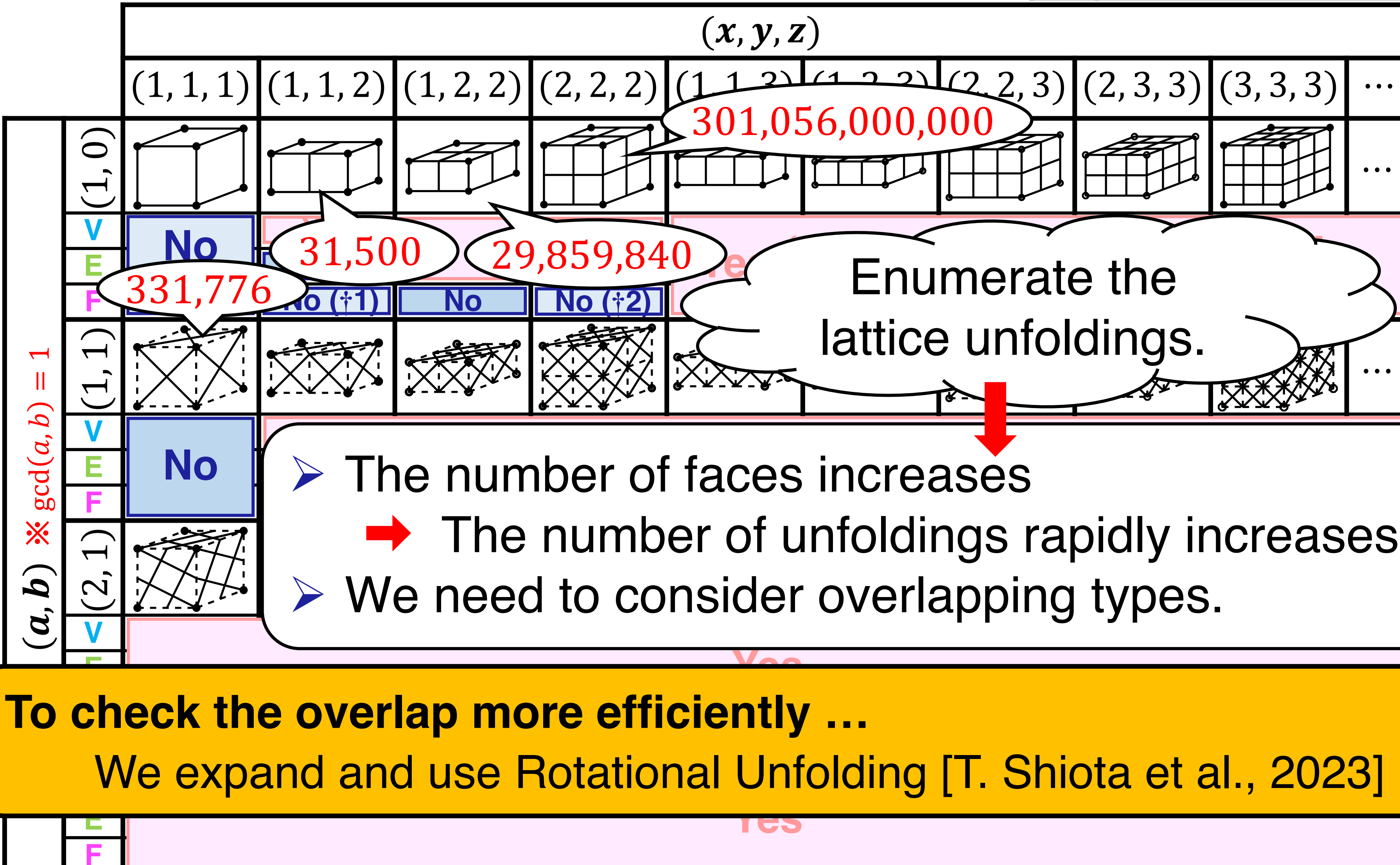
Background and our results



- The number of faces increases
  ➡ The number of unfoldings rapidly increases
- We need to consider overlapping types.

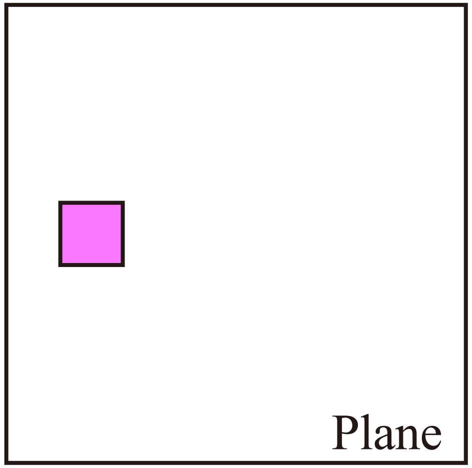
To check the overlap more efficiently …

We expand and use Rotational Unfolding [T. Shiota et al., 2023]

Technique to show the non-existence

Rotational Unfolding [T. Shiota et al., 2023]

- Enumerating the path between any two faces by rolling a polyhedron.
- Checking the overlap of both end-faces of a path.



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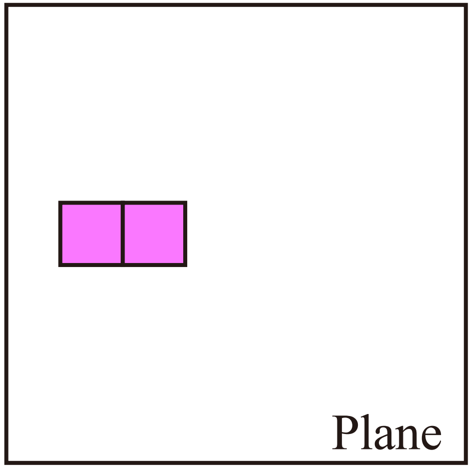

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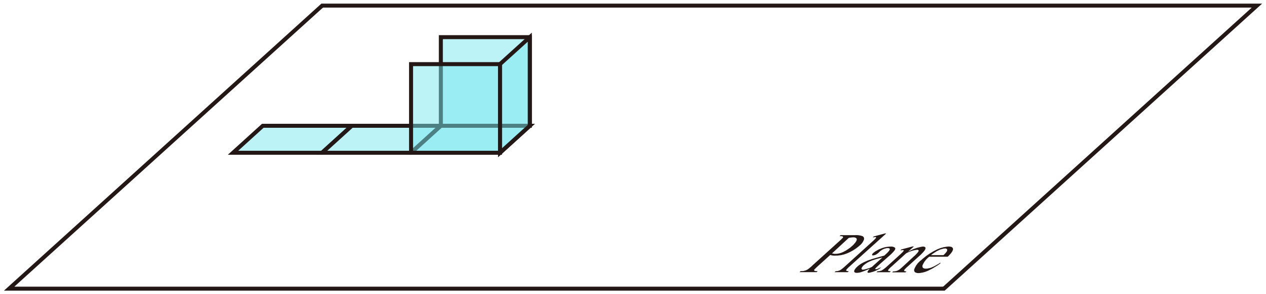

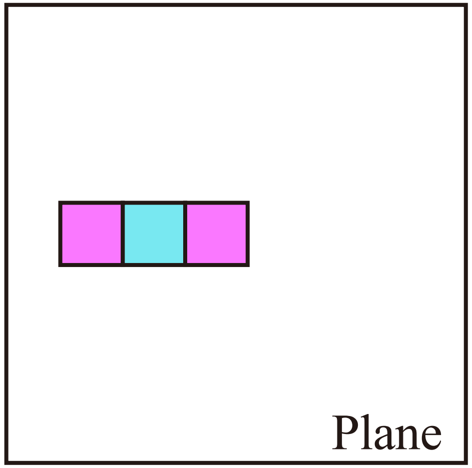

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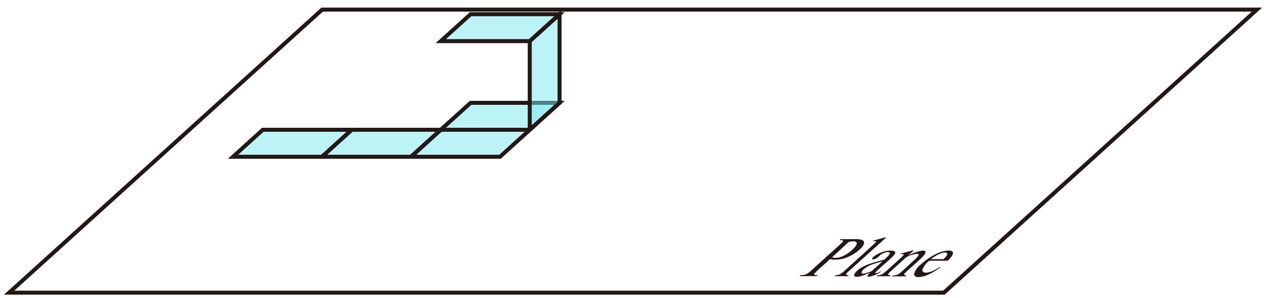

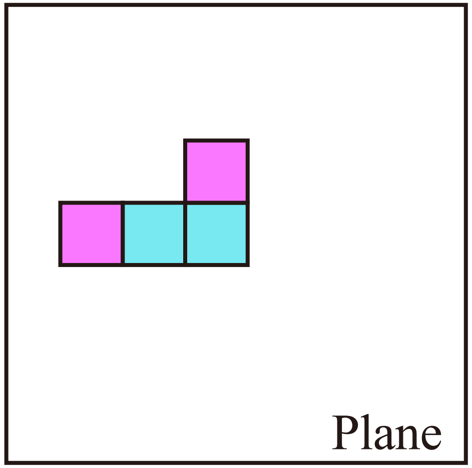

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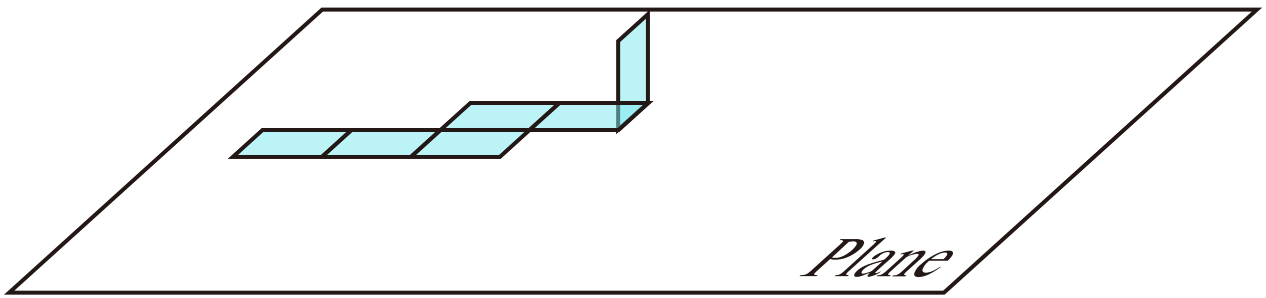

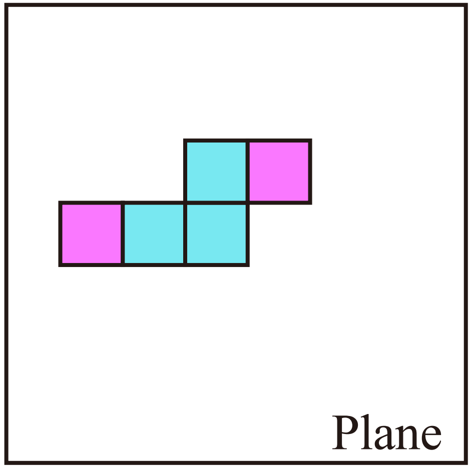

Technique to show the non-existence

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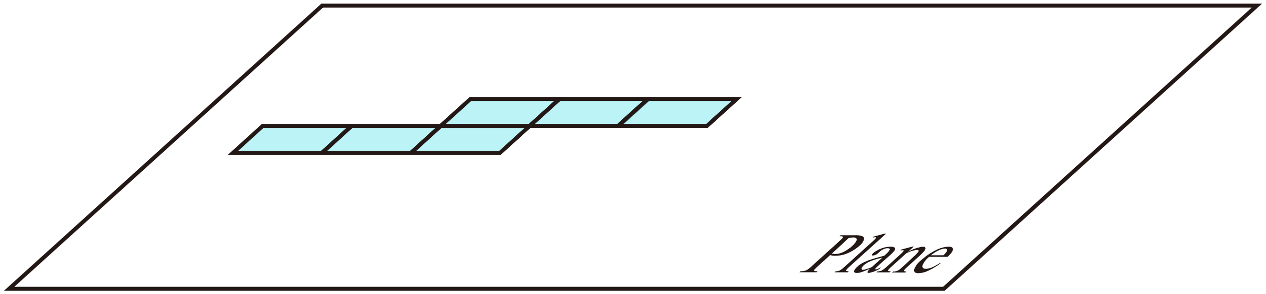

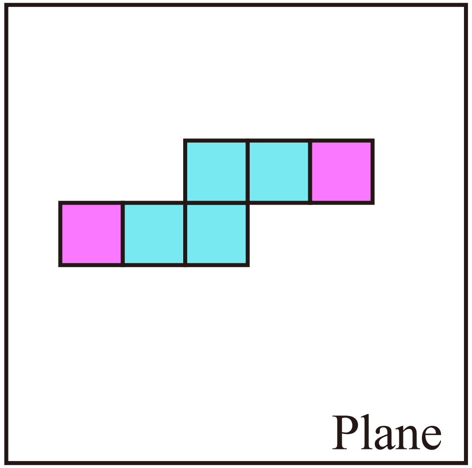

Technique to show the non-existence

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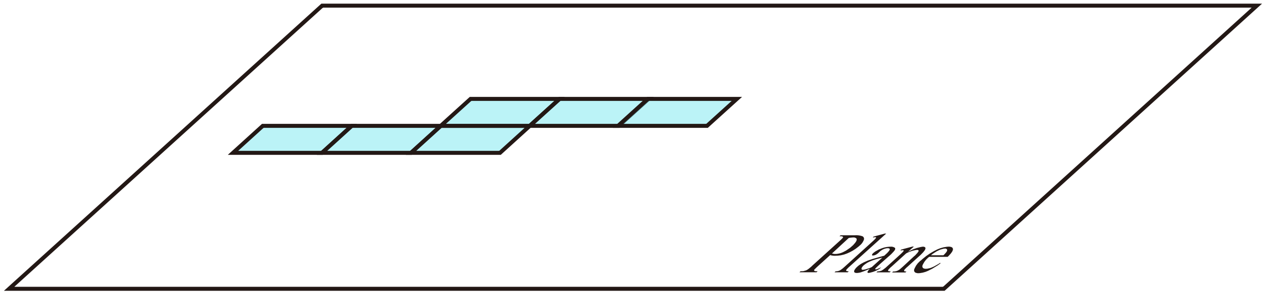
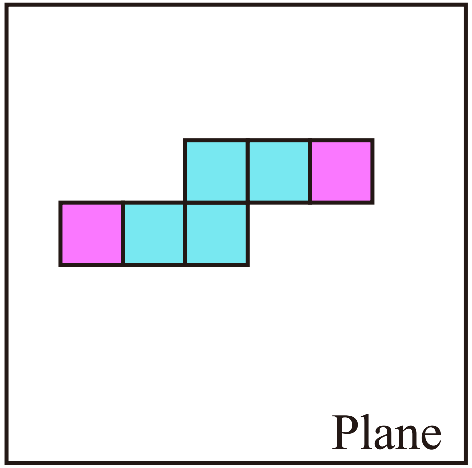


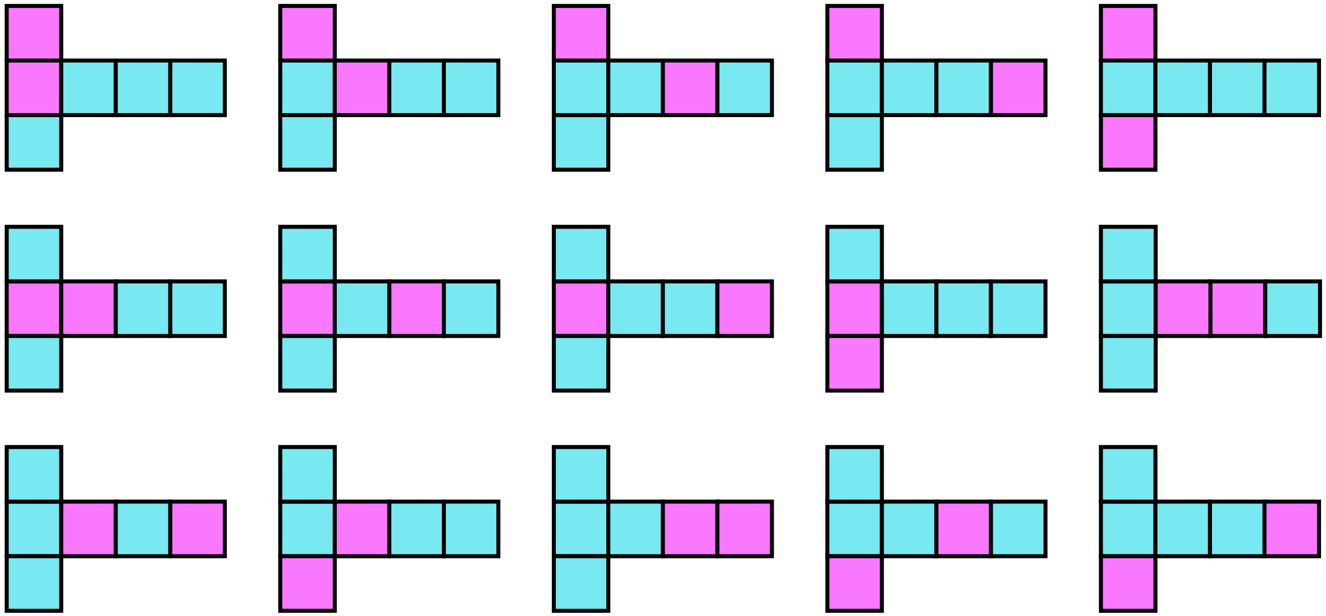


Q. Why only check the overlap of both end-faces in the path?

Technique to show the non-existence

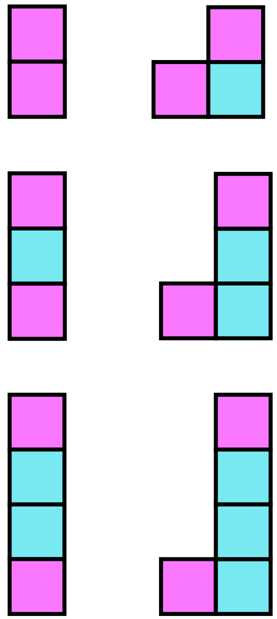
Lemma 1 [T. Shiota et al., 2023]

The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.



$_6C_2 = 15$ ways

Check all combinations of faces [T. Horiyama and W. Shoji, 2011]



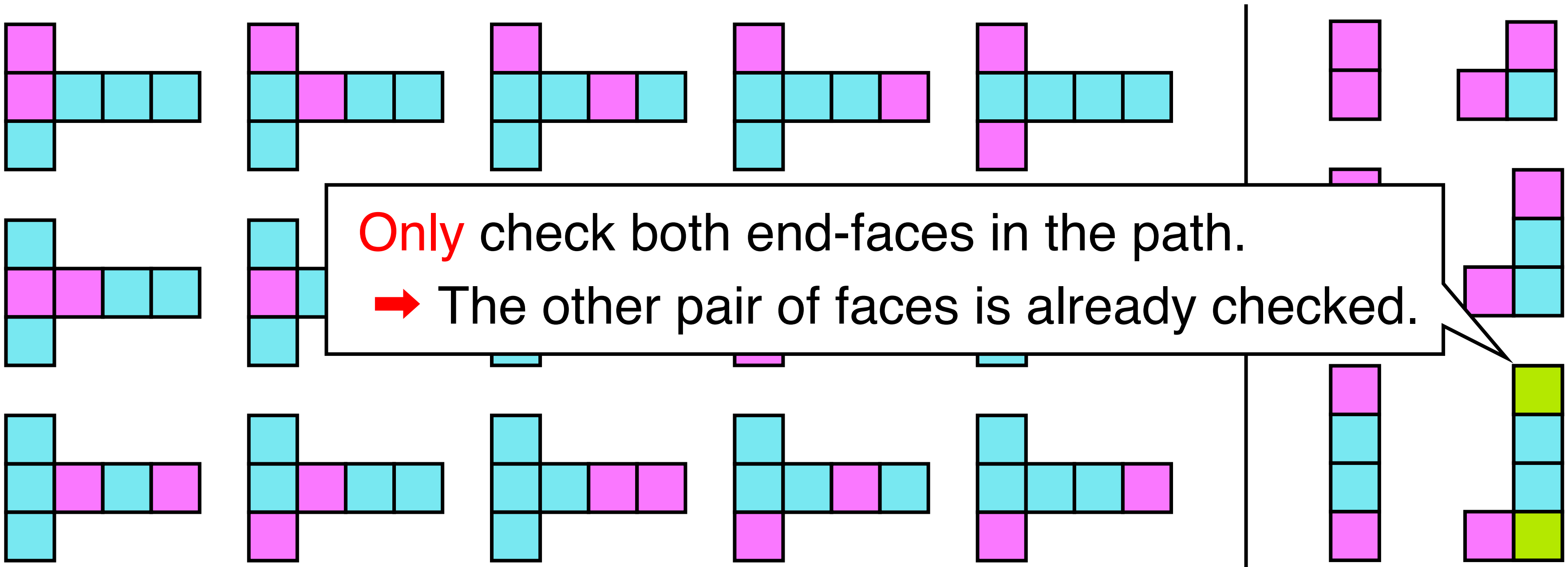
6 ways

Rotational unfolding

Technique to show the non-existence

Lemma 1 [T. Shiota et al., 2023]

The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.



$_6C_2 = 15$ ways

Check all combinations of faces [T. Horiyama and W. Shoji, 2011]

6 ways

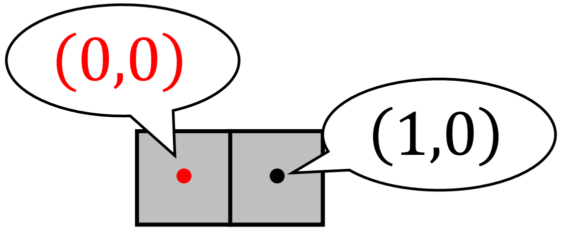
Rotational unfolding

Overlap check in lattice unfoldings

In rotational unfolding, we check for overlaps with each roll.

1. Set the center coordinates of one endpoint of the path to $(x, y) = (0,0)$.
2. While rolling the cuboid, sequentially compute the center coordinates of the other endpoint.

[Note] The length of one side of the cuboid is 1.



The computation process for the other endpoint's coordinates

Overlap check in lattice unfoldings

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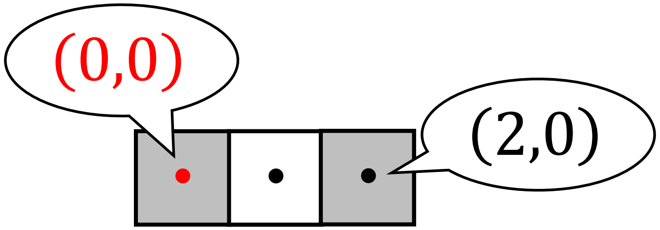

The computation process for the other endpoint's coordinates

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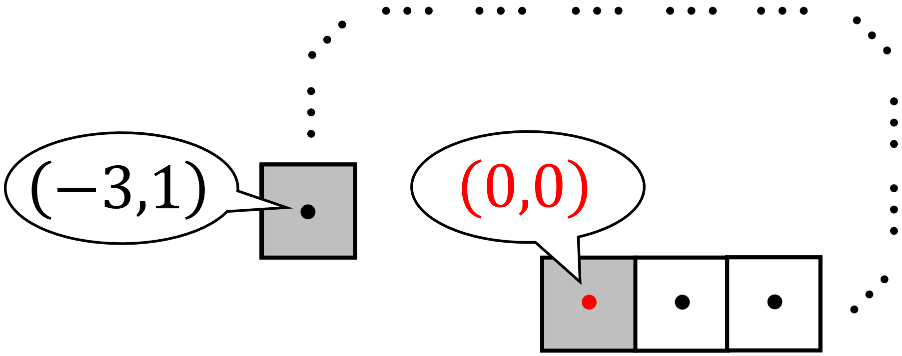

The computation process for the other endpoint's coordinates

Overlap check in lattice unfoldings

In rotational unfolding, we check for overlaps with each roll.

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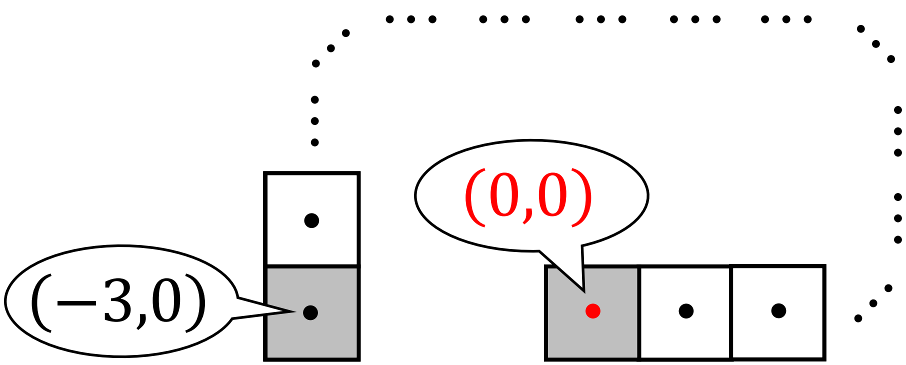


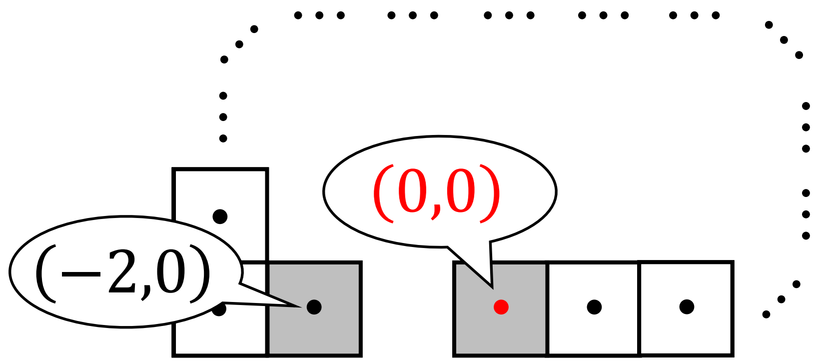
The computation process for the other endpoint's coordinates

Overlap check in lattice unfoldings

In rotational unfolding, we check for overlaps with each roll.

1. Set the center coordinates of one endpoint of the path to $(x, y) = (0,0)$.
2. While rolling the cuboid, sequentially compute the center coordinates of the other endpoint.

[Note] The length of one side of the cuboid is 1.

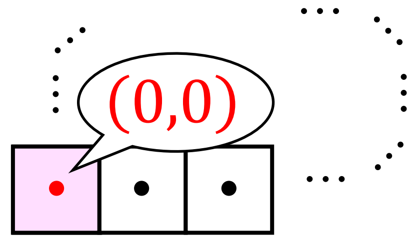


The computation process for the other endpoint's coordinates

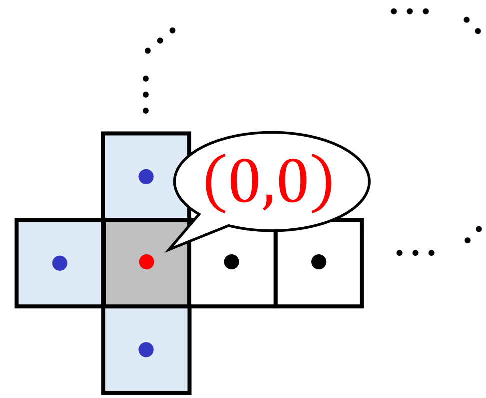
Overlap check in lattice unfoldings

The center coordinates of the other endpoint of the path are...

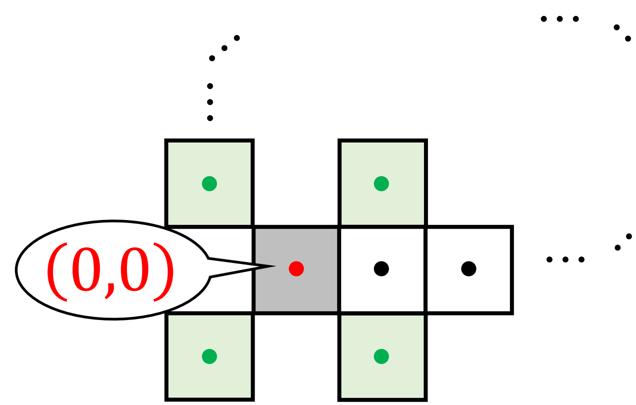
- $(0,0)$
  $\rightarrow$ Faces-in-touch
- $(0,1), (-1,0), (0,-1)$
  $\rightarrow$ Edges-in-touch
- $(1,1), (1,-1), (-1,-1), (-1,1)$
  $\rightarrow$ Vertices-in-touch



Faces-in-touch



Edges-in-touch



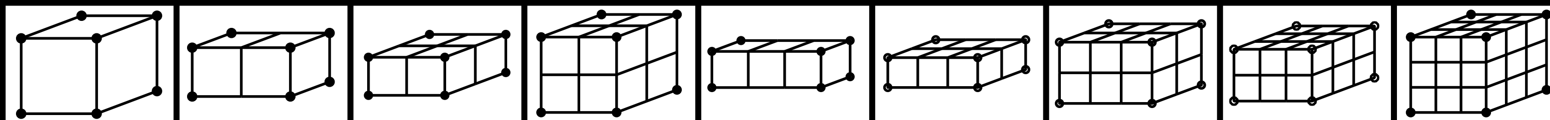
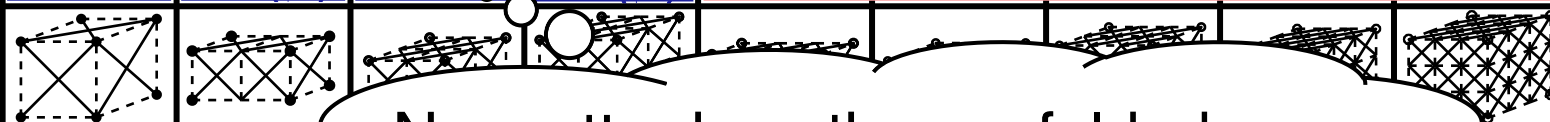
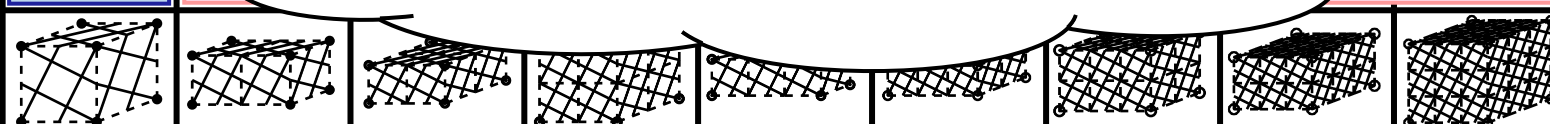
Vertices-in-touch

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a, b) ※ gcd(a, b) = 1

(a, b)		(1,1,1)	(1,1,2)	(1,2,2)	(2,2,2)	(1,1,3)	(1,2,3)	(2,2,3)	(2,3,3)	(3,3,3)	⋯
(1, 0)											⋯
	V	No (Obvi.)	Yes	Yes		(1×1×z)-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
(1, 1)											⋯
	V	No									
	E										
	F										
(2, 1)											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

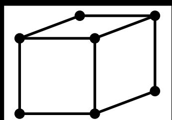
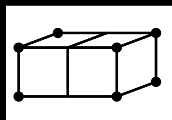
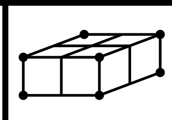
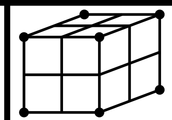
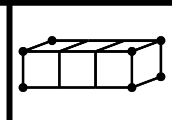
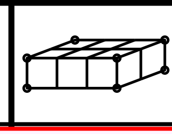
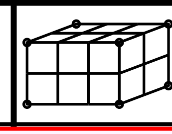
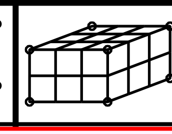
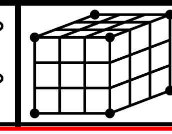
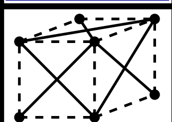
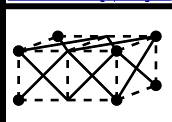
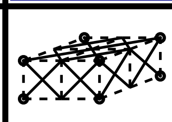
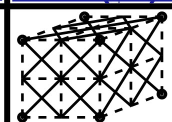
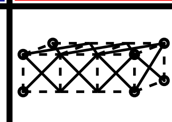
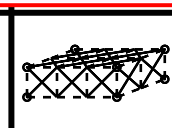
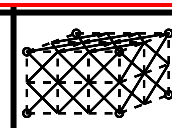
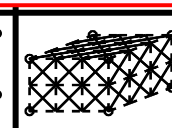
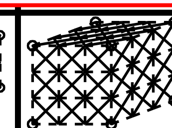
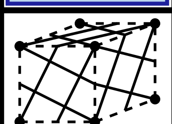
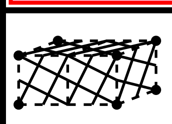
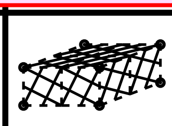
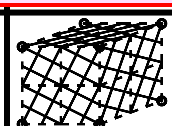
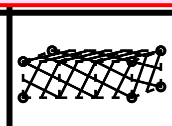
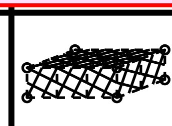
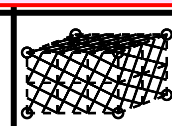
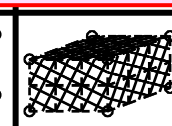
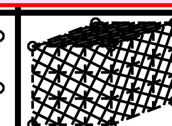
No matter how they unfolded,
they do not overlap.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a,b) ※ $\gcd(a,b)=1$

(a,b)		(x,y,z):	$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$												⋯
	V		No (Obvi.)	Yes	Yes		Yes — $(1{\times}1{\times}z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E			No								
	F			No (†1)	No	No (†2)						
$(1,1)$												⋯
	V		No	Yes								
	E											
	F											
$(2,1)$												⋯
	V		Yes									
	E											
	F											
⋮			⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V		Yes									
	E											
	F											

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
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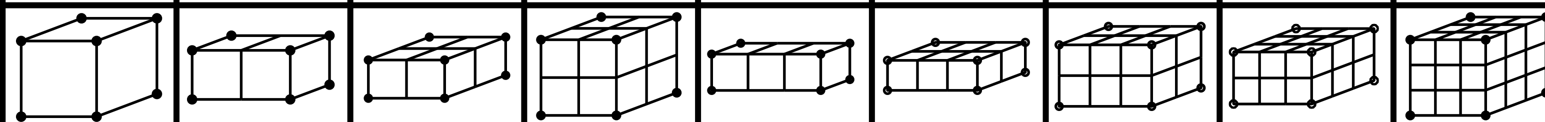
(a,b) ※ $\gcd(a,b)=1$		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
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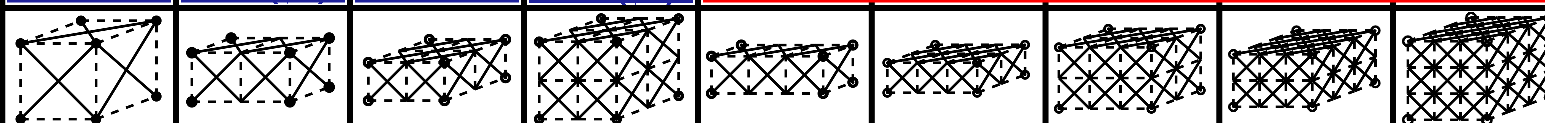
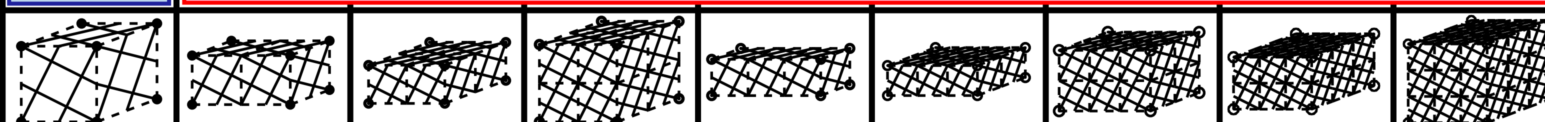
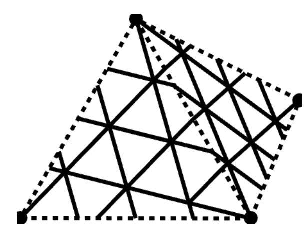
$(1,1)$											⋯
	V	No	Yes								
	E										
	F										
$(2,1)$											⋯

Future work: Clarify the existence of overlapping unfolding for "tetrahedron" or "octahedron" that can be constructed from the triangular lattice.


Tetrahedron