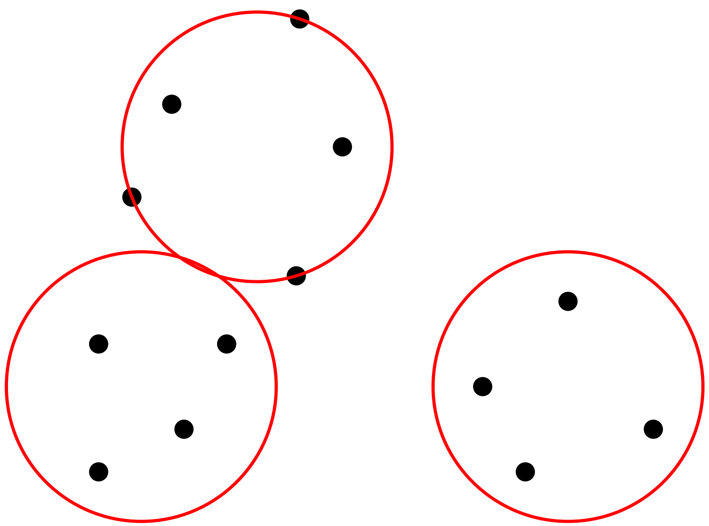
Universal convex covering problems under affine dihedral group actions

M. K. Jung, S. D. Yoon, H. -K. Ahn, T. Tokuyama

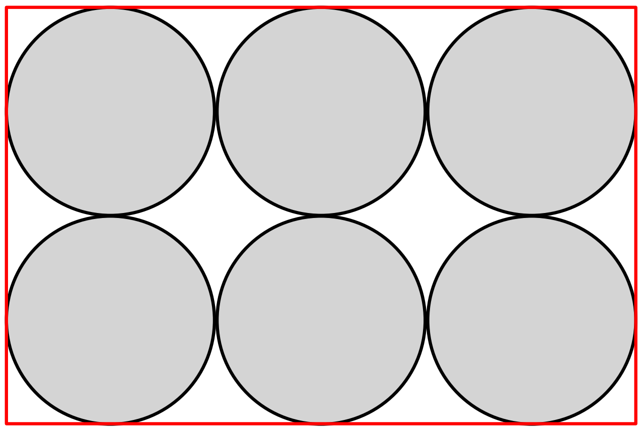
Presenter: Mook Kwon Jung

CCCG2023
2023.08.04

< Covering Problem >



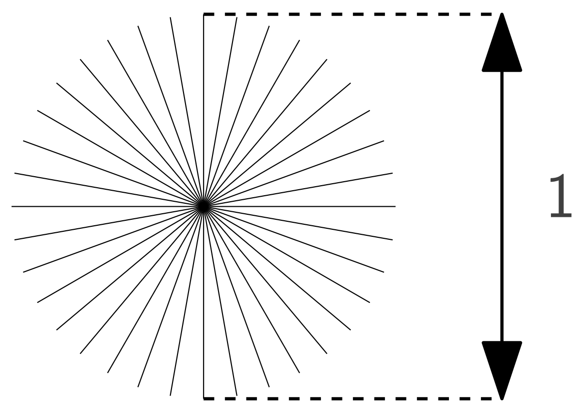
k-center problem



Rectangle covering

- Input: All unit line segments
- Geometric transformation: Translation

What is the smallest-area convex hull of them?



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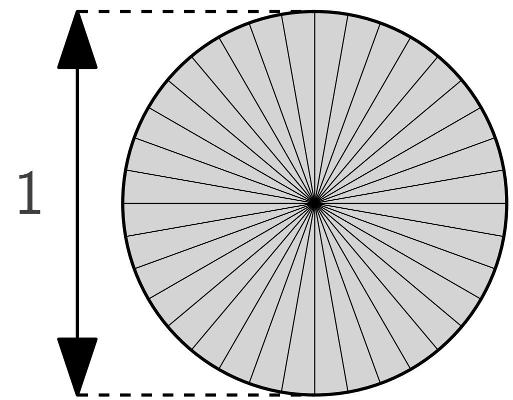
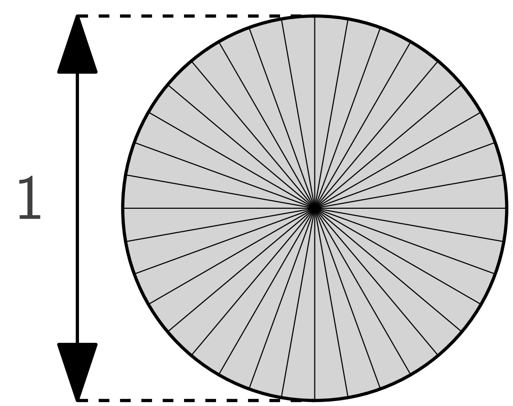


Area: $\pi/4 \approx 0.785$

- Input: All unit line segments
- Geometric transformation: Translation

What is the smallest-area convex hull of them?



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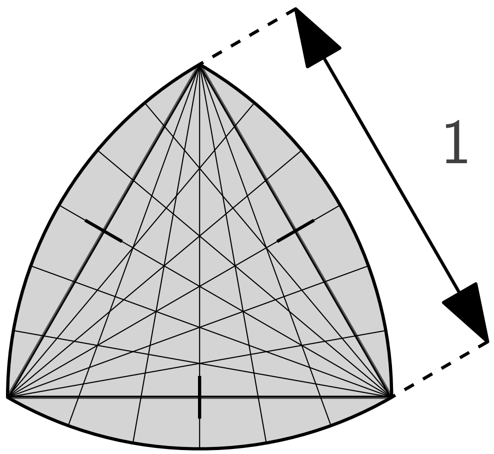
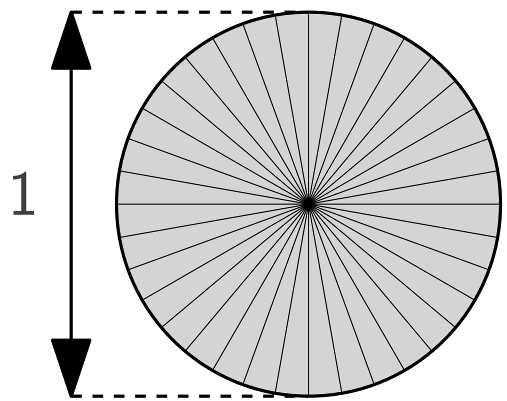


Area: $(\pi - \sqrt{3})/2 \approx 0.705$

Introduction

- Input: All unit line segments
- Geometric transformation: Translation
- Solution: The equilateral triangle with height 1*



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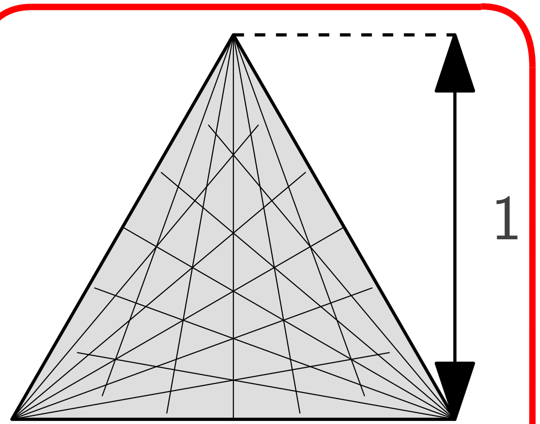
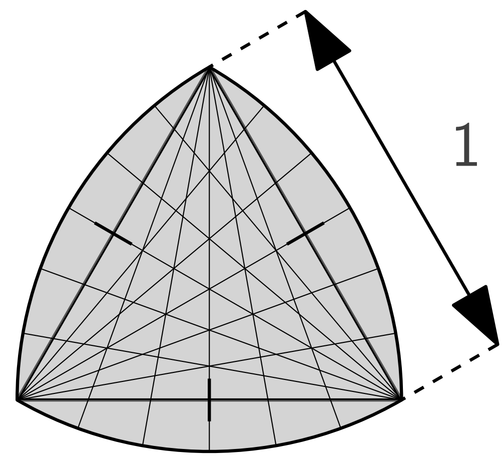


Area: $1/\sqrt{3} \approx 0.577$

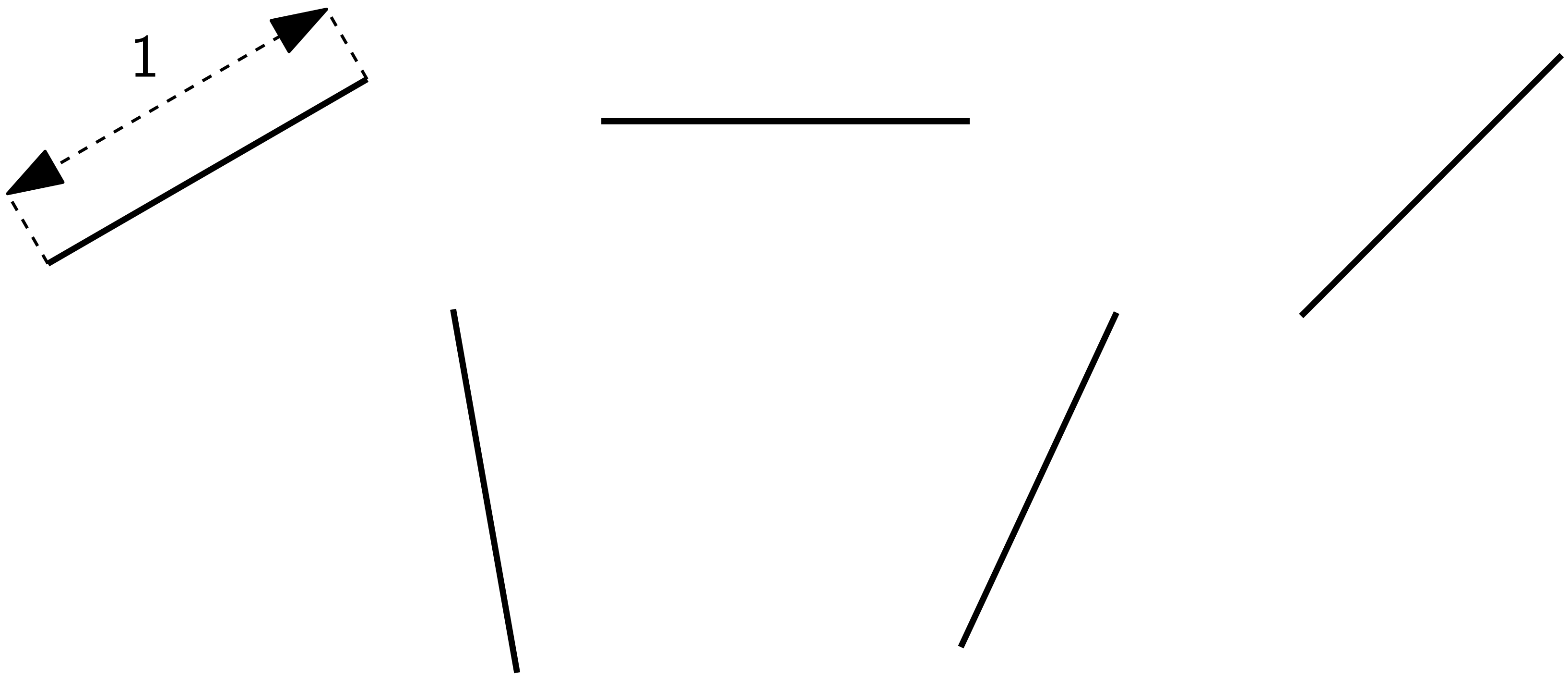


Area: $(\pi - \sqrt{3})/2 \approx 0.705$

*J. Pal, Ein Minimumproblem für Ovale, Mathematische Annalen, 83:311-319, 1921.

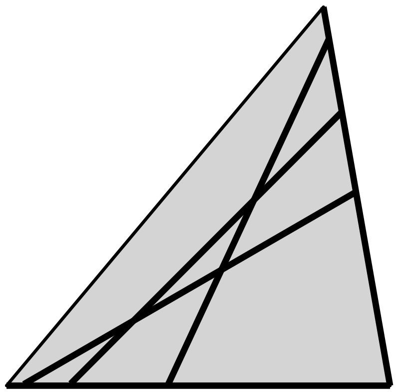
- Input: *n* unit line segments
- Geometric transformation: Translation

What is the smallest-area convex hull of them?



- Input: n unit line segments
- Geometric transformation: Translation

What is the smallest-area convex hull of them?

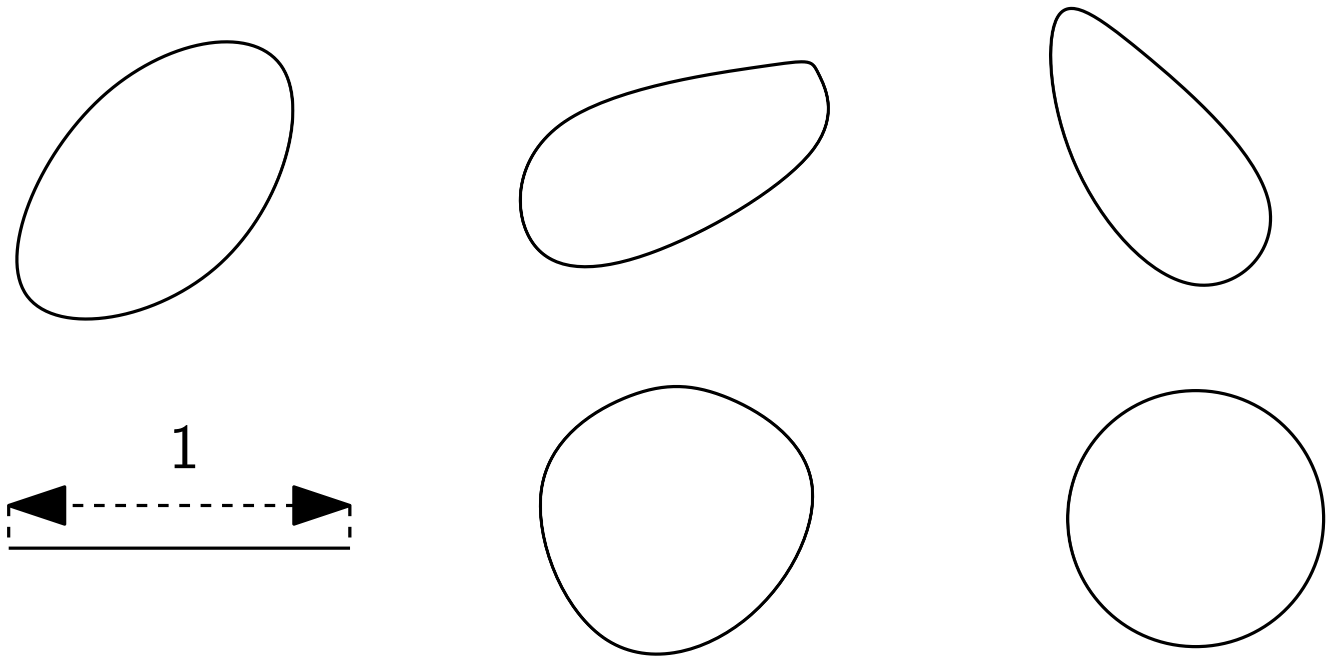


Triangle

$\Rightarrow$ Construct in $O(n \log n)$ time*

H.-K. Ahn, S.-W. Bae, O. Cheong, J. Gudmundsson, T. Tokuyama, A. Vigneron, A Generalization of the Convex Kakeya Problem, Algorithmica, 70:152-170, 2014.

- Input: All closed curves with length 2
- Transformation: Translation



- Input: All closed curves with length 2
- Transformation: Translation

Open Problem!!

$$0.620^* \leq (\text{Area}) \leq 0.657^*$$

*P. Brass, W. Moser, J. Pach, Research Problems in Discrete Geometry, Springer Verlag, 2005.

- Input: All closed curves with length 2
- Transformation: Translation and rotation

Open Problem!!

$$0.4^{1} \leq \text{(Area)} \leq 0.441^{2}$$

[1] B. Grechuk, S. Som-am, A convex cover for closed unit curves has area at least 0.1, Discrete Optimization, 38, article 100608: 1-15, 2020.

[2] W. Wichiramala, A smaller cover for closed unit curves, Miskolc Mathematical Notes, 19(1):691-698, 2018.

Introduction

- Input: All closed curves with length 2
- Transformation: Translation and discrete rotation

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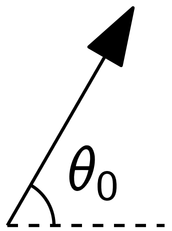
ex) translation and $\frac{\pi}{4}$ rotations



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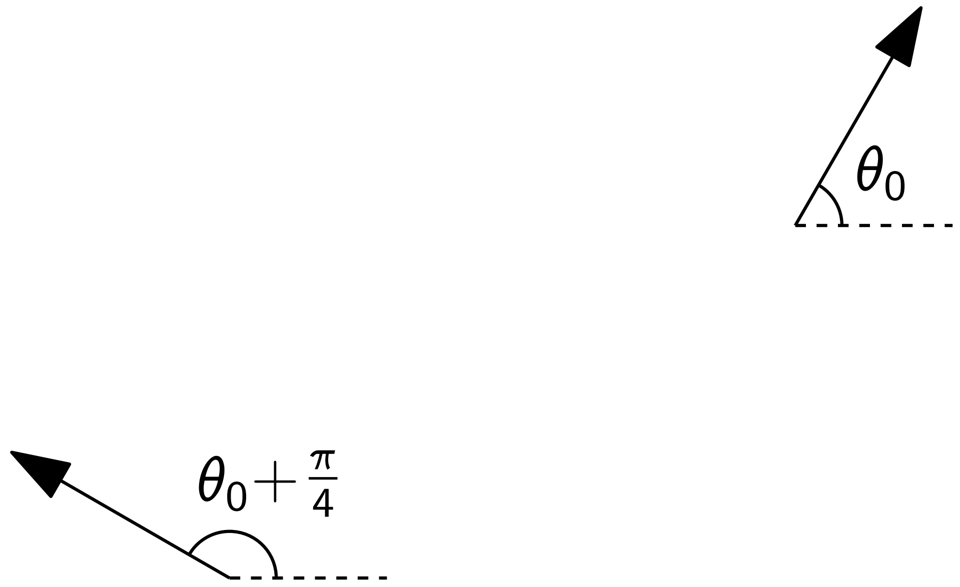

<$\frac{\pi}{4}$ rotation>

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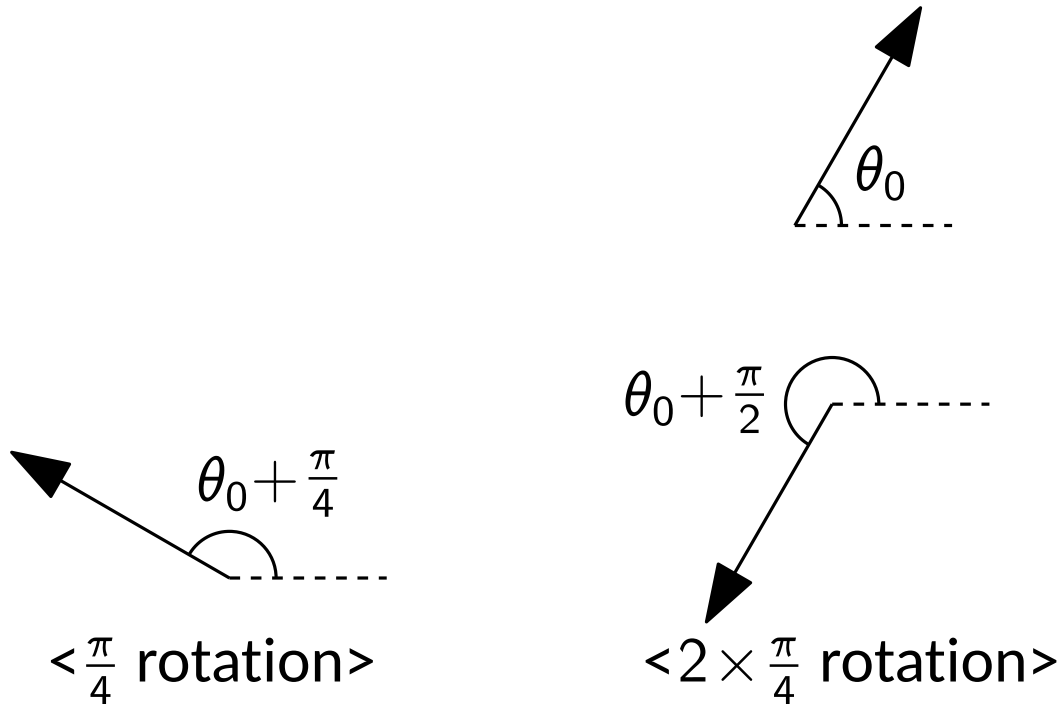
- Transformation: Translation and discrete rotation

ex) translation and $\frac{\pi}{4}$ rotations



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- Transformation: Translation and discrete rotation

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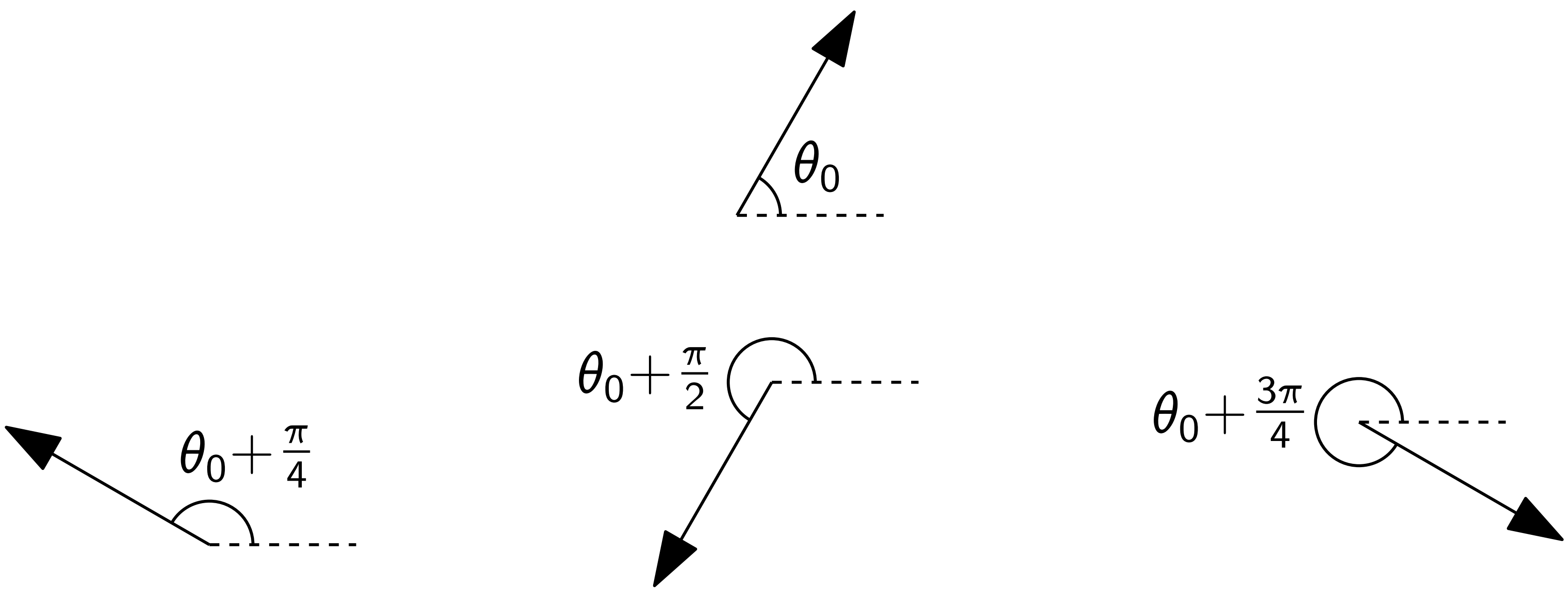
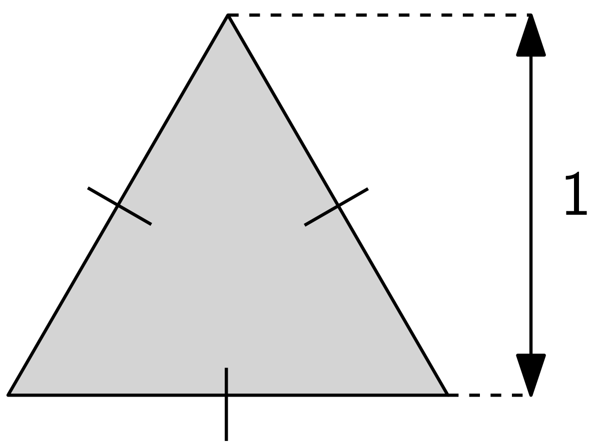


<$\frac{\pi}{4}$ rotation> <$2 \times \frac{\pi}{4}$ rotation> <$3 \times \frac{\pi}{4}$ rotation>

- Input: All closed curves with length 2
- Transformation: Translation and discrete rotation

- $\frac{\pi}{2}$ rotations: The equilateral triangle with height 1*



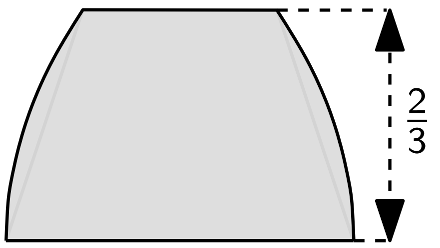
⇐ The smallest-area convex hull of all unit line segments

*M. K. Jung, S. D. Yoon, H.-K. Ahn, T. Tokuyama, Universal convex covering problems under translation and discrete rotations, arXiv:2211.14807 [cs.CG] , 2022.

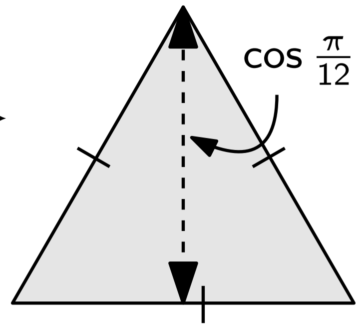
Introduction

- Input: All closed curves with length 2
- Transformation: Translation and discrete rotation

- $\frac{2}{3}\pi$ rotations: (Area) $\leq 0.568^* \Rightarrow$



- $\frac{\pi}{4}$ rotations: (Area) $\leq 0.539^* \Rightarrow$



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If we add the reflection?

If we add the reflection?

- Input: All closed curves with length 2

- Transformation: Affine dihedral group

What is the smallest-area convex hull of them?

- Geometric Transformation

- G_k: Translation and $2\pi/k$ rotations

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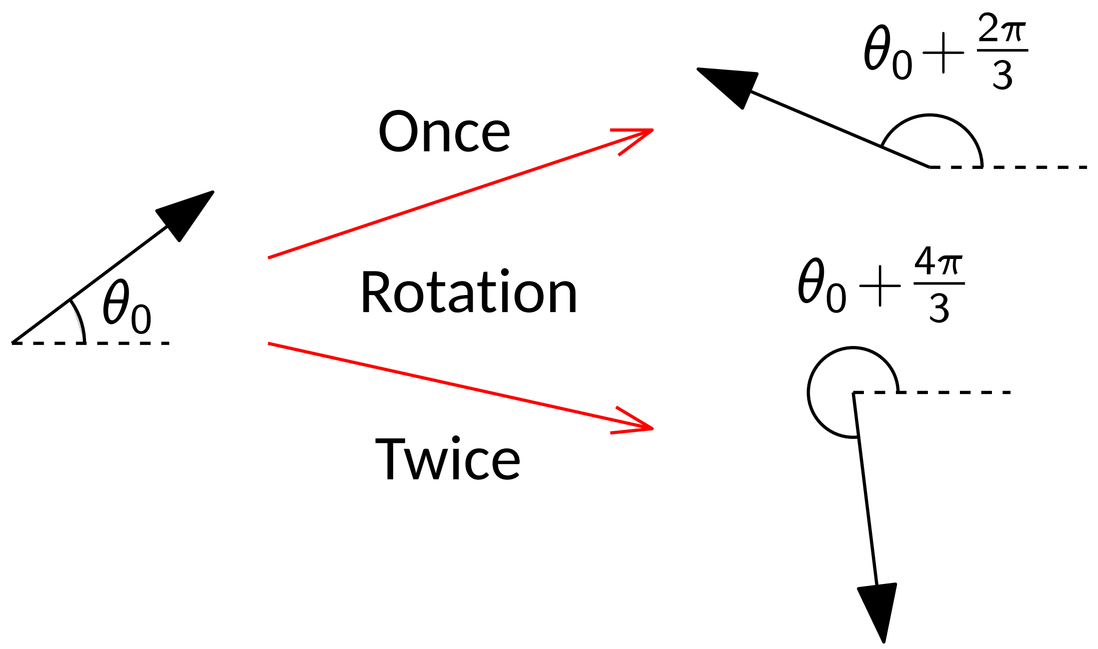
- G_k: Translation and $2\pi/k$ rotations

ex) G_3:Translation and $\frac{2}{3}\pi$ rotations



Preliminary

- Geometric Transformation

- G_k: Translation and $2\pi/k$ rotations

- H_k: Translation, the x-axis reflection and $2\pi/k$ rotations

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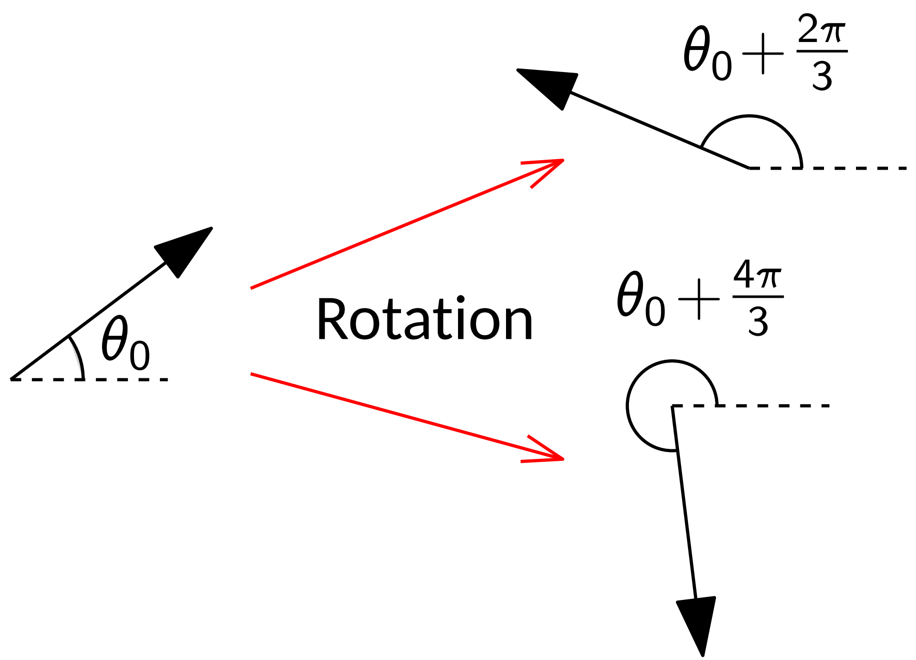
ex) H_3:Translation, x-axis reflection, and $\frac{2}{3}\pi$ rotations



- Geometric Transformation

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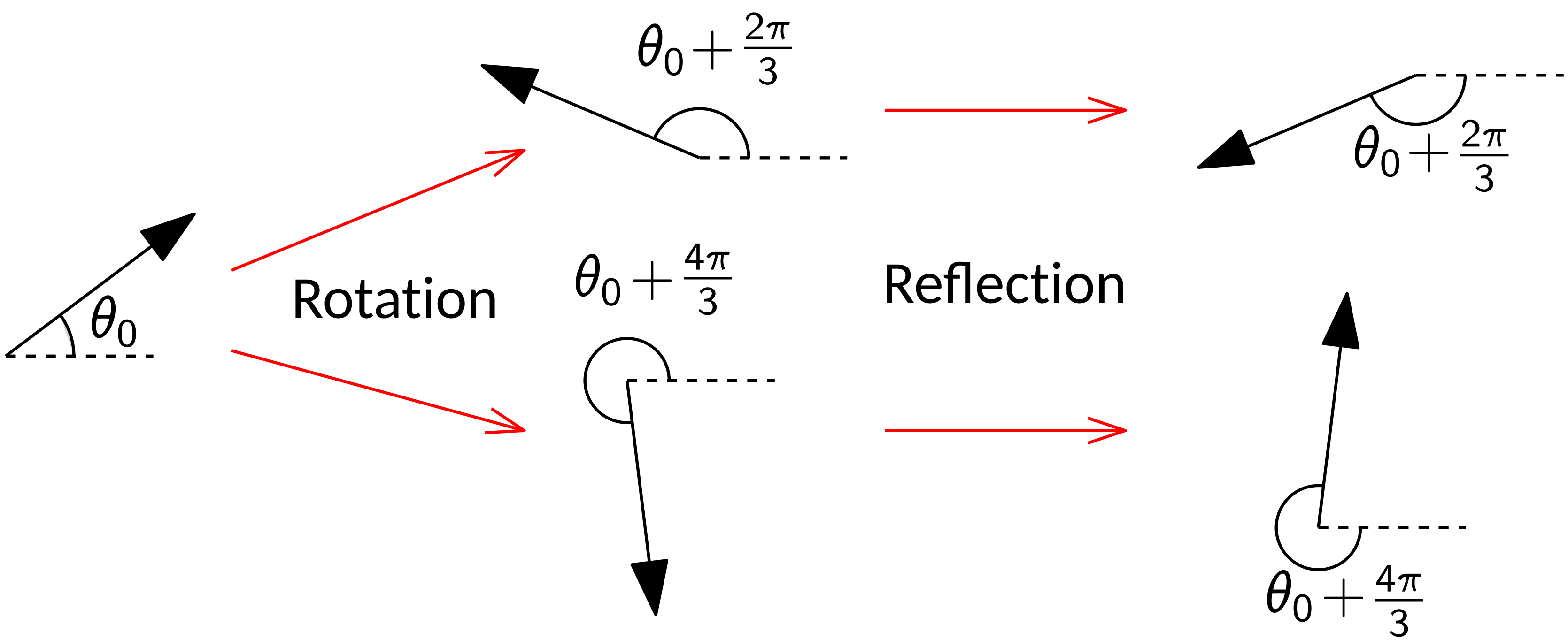

- Geometric Transformation

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$$H_k : G_k + \text{The x-axis reflection}$$

Preliminary

- Geometric Transformation

- G_k: Translation and $2\pi/k$ rotations

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- T: Translation

Preliminary

- Geometric Transformation

- G_k: Translation and $2\pi/k$ rotations

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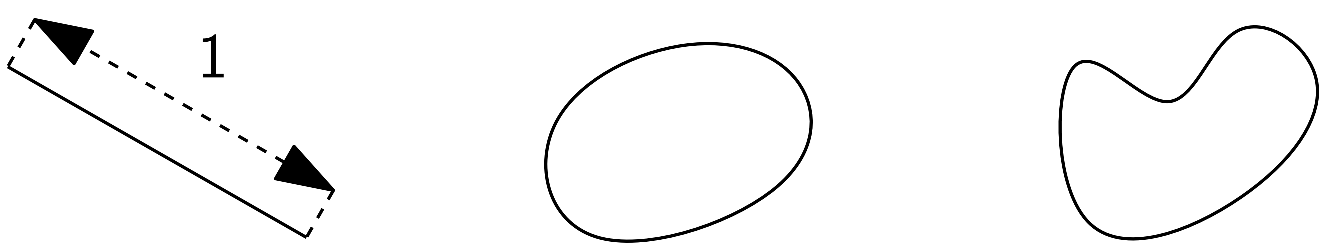
- T: Translation

- Object Set

- S_c: The set of all closed curves with length 2



Preliminary

G: Geometric Transformation

S: Object set

Def. A covering K is a *G-covering of S* if $\forall \gamma \in S, \exists g \in G$ such that $g\gamma \subseteq K$

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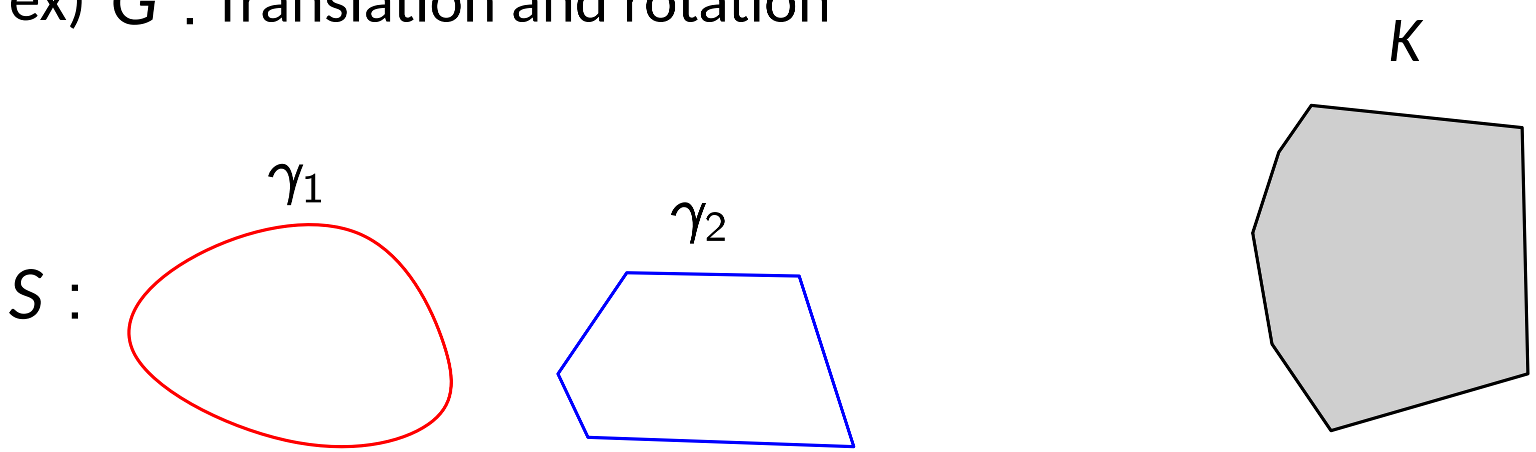
ex) G : Translation and rotation



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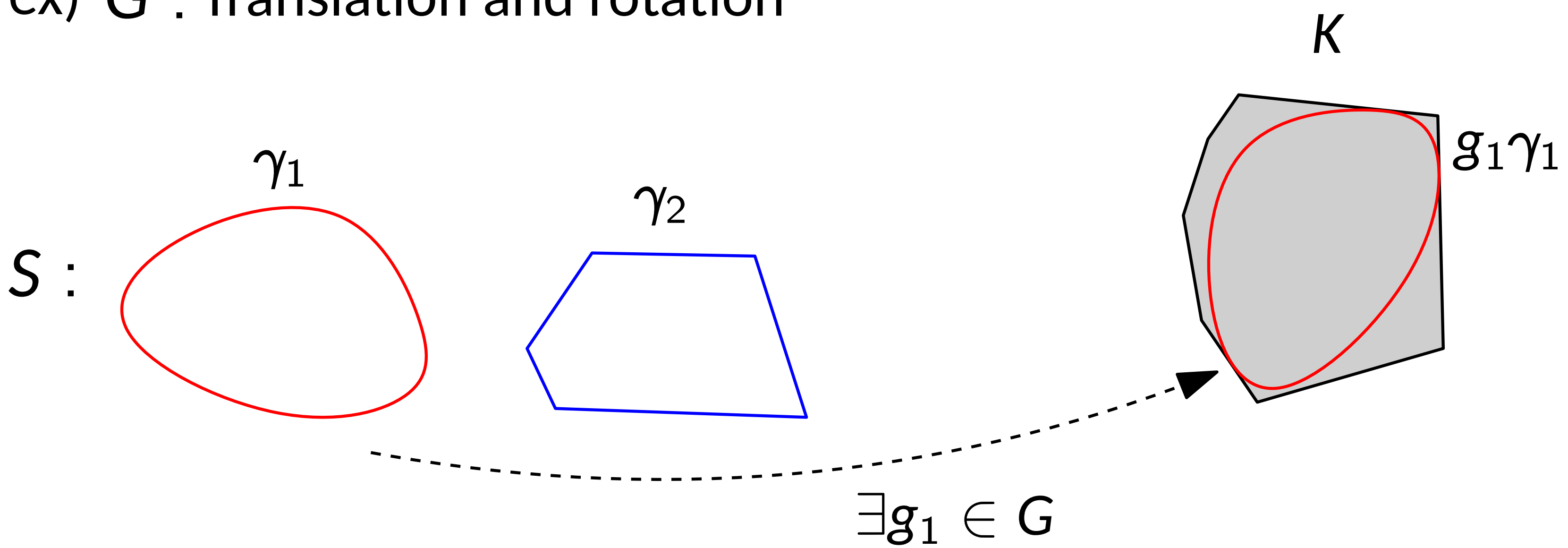

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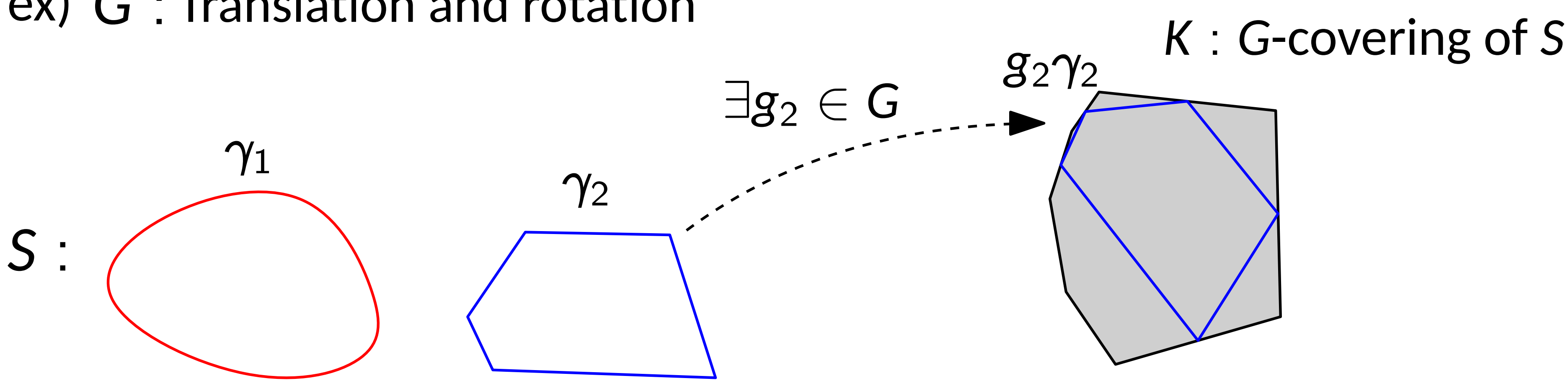

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Smallest area $\Longrightarrow$ / $\not\Longleftarrow$ Minimality

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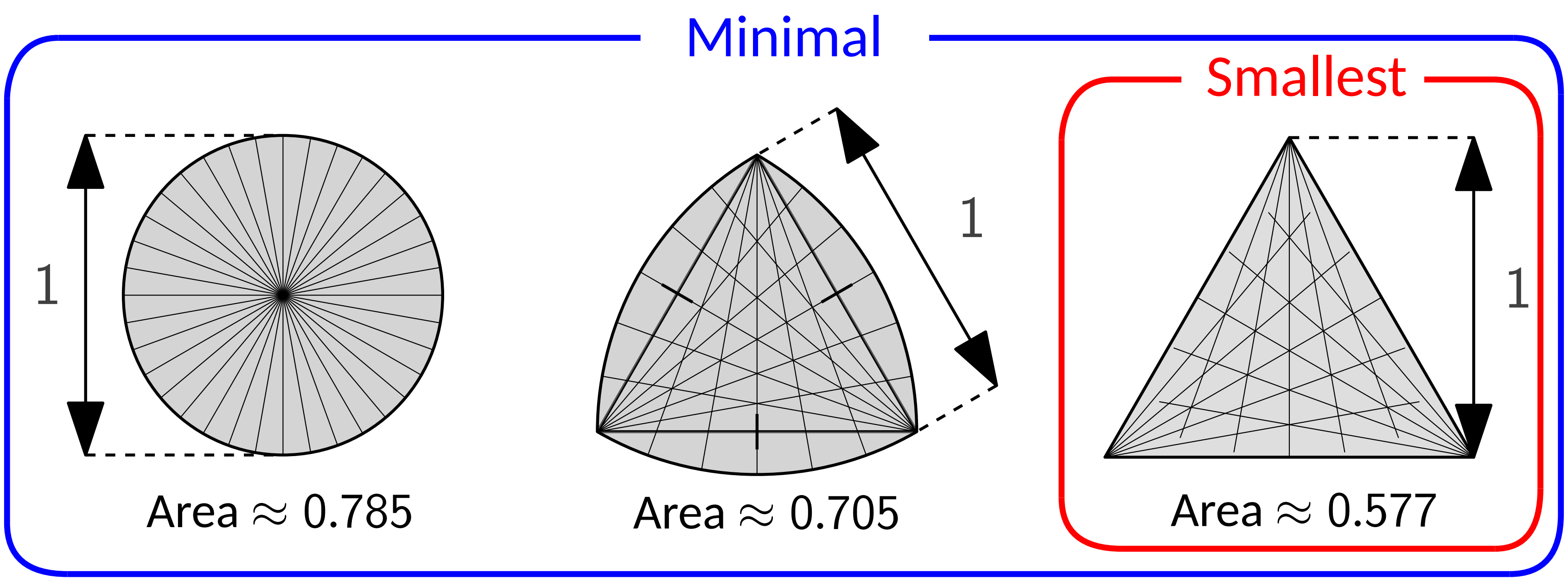
Def. *The smallest-area* G-covering K of S has the smallest area among all G-coverings of S.

Ex. T-covering of all unit line segments



H_k-covering of S_c

G_k: translation, $2\pi/k$ rotations

H_k: translation, the x-axis reflection $2\pi/k$ rotations

Observation. G_k-covering is also H_k-covering. ($\because G_k \subseteq H_k$)

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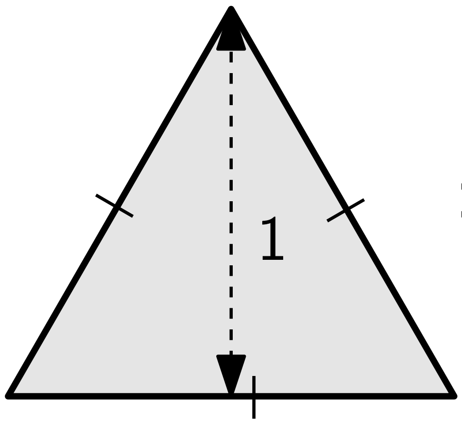

The smallest-area G_2-covering*

$\Rightarrow H_2$-covering

$\Rightarrow H_1$-covering??

*M. K. Jung, S. D. Yoon, H.-K. Ahn, T. Tokuyama, Universal convex covering problems under translation and discrete rotations, arXiv:2211.14807 [cs.CG] , 2022.

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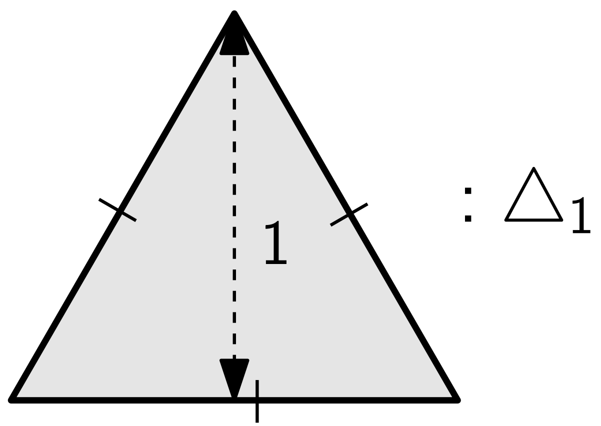

$\Rightarrow H_2$-covering

$\Rightarrow H_1$-covering!!!

The smallest-area G_2-covering*

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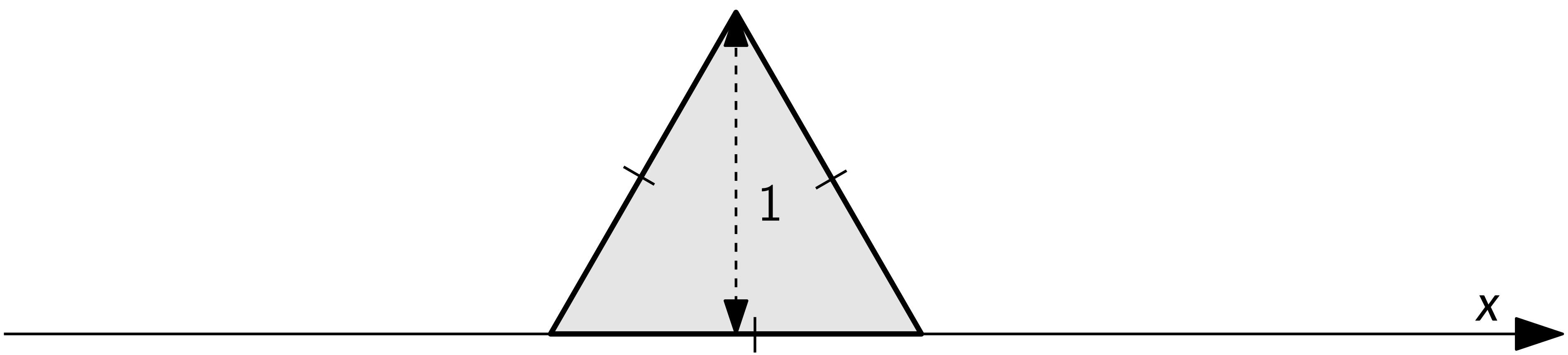
G_2-covering vs. H_1-covering

G_k: translation, $2\pi/k$ rotations

H_k: translation, the x-axis reflection $2\pi/k$ rotations

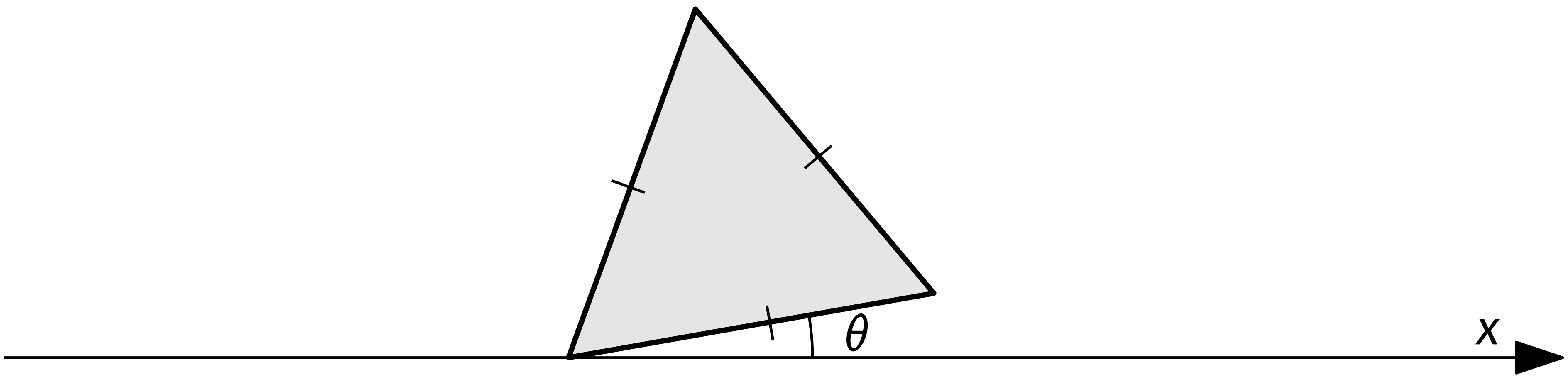


G_2-covering of S_c? YES

H_1-covering of S_c? YES

G_k: translation, $2\pi/k$ rotations

H_k: translation, the x-axis reflection $2\pi/k$ rotations



G_2-covering of S_c? YES

H_1-covering of S_c? NO

G_k: translation, $2\pi/k$ rotations

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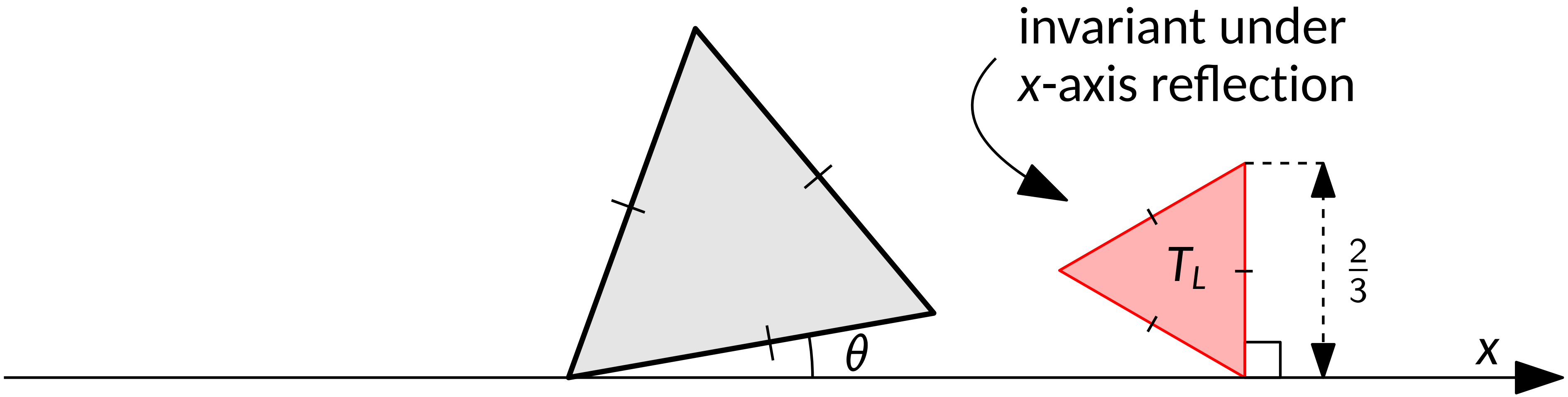

G_2-covering of S_c? YES

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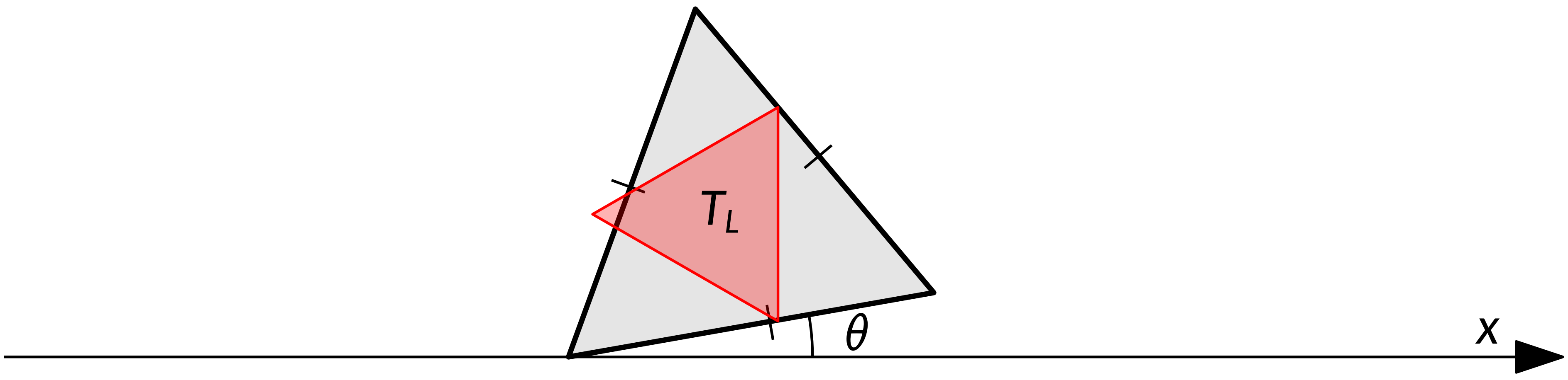


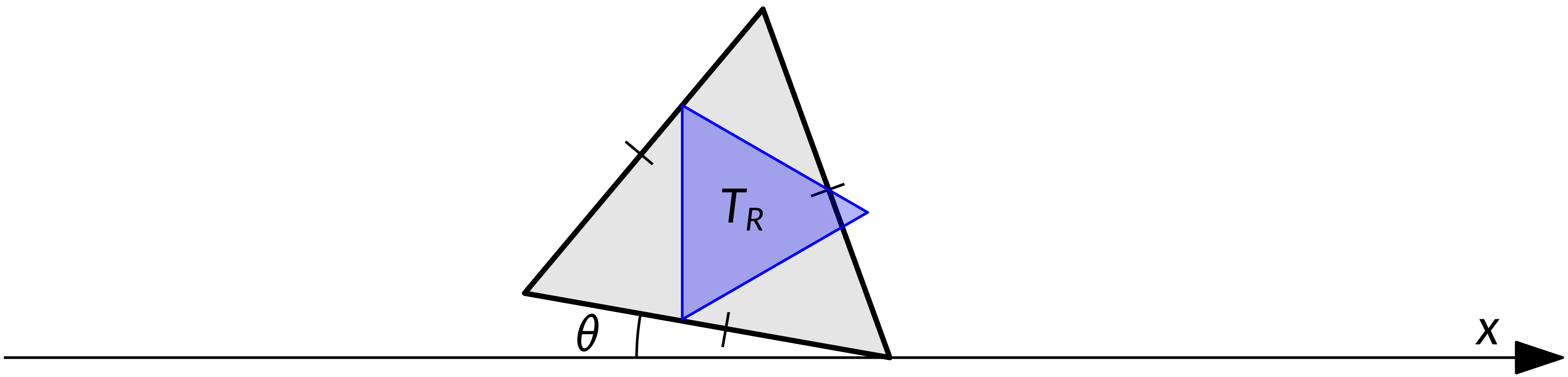
G_2-covering of S_c? YES

H_1-covering of S_c? NO

G_2-covering vs. H_1-covering

G_k: translation, $2\pi/k$ rotations

H_k: translation, the x-axis reflection $2\pi/k$ rotations



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G_2-covering vs. H_1-covering

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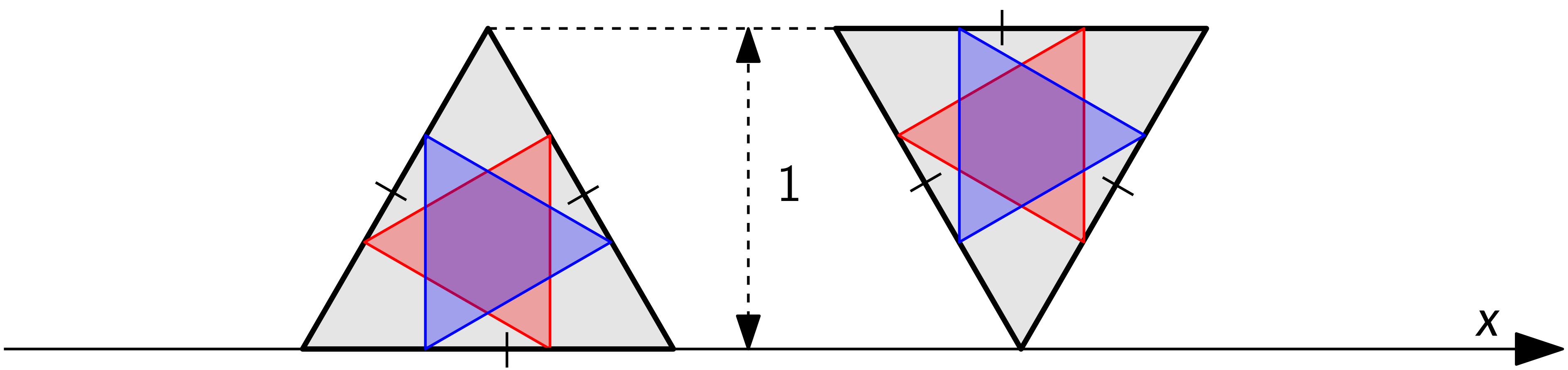

G_2-covering of S_c? YES

H_1-covering of S_c? YES

G_k: translation, $2\pi/k$ rotations

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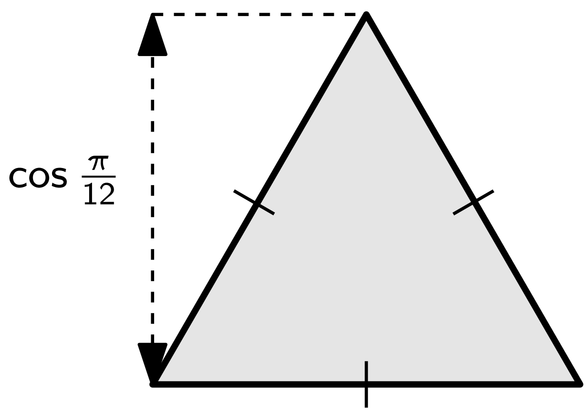

Minimal G_4-covering*

$\Rightarrow H_4$-covering

$\Rightarrow H_2$-covering??

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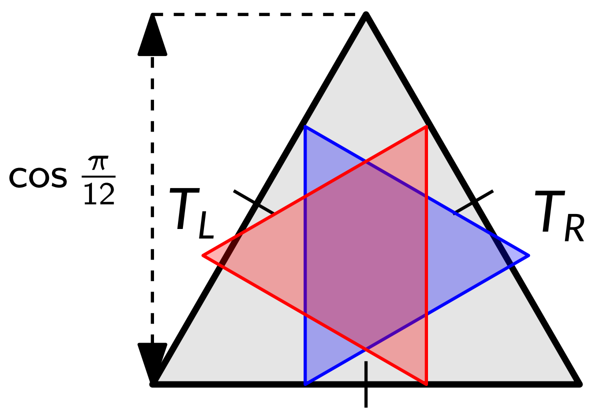

Minimal G_4-covering*

$\Rightarrow$ H_4-covering

$\Rightarrow$ Not H_2-covering!!

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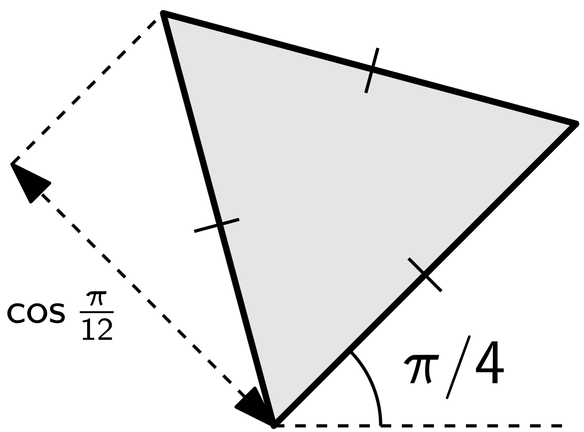

Minimal G_4-covering*

$\Rightarrow$ H_4-covering

$\Rightarrow$ H_2-covering!!

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G_k: translation, $2\pi/k$ rotations

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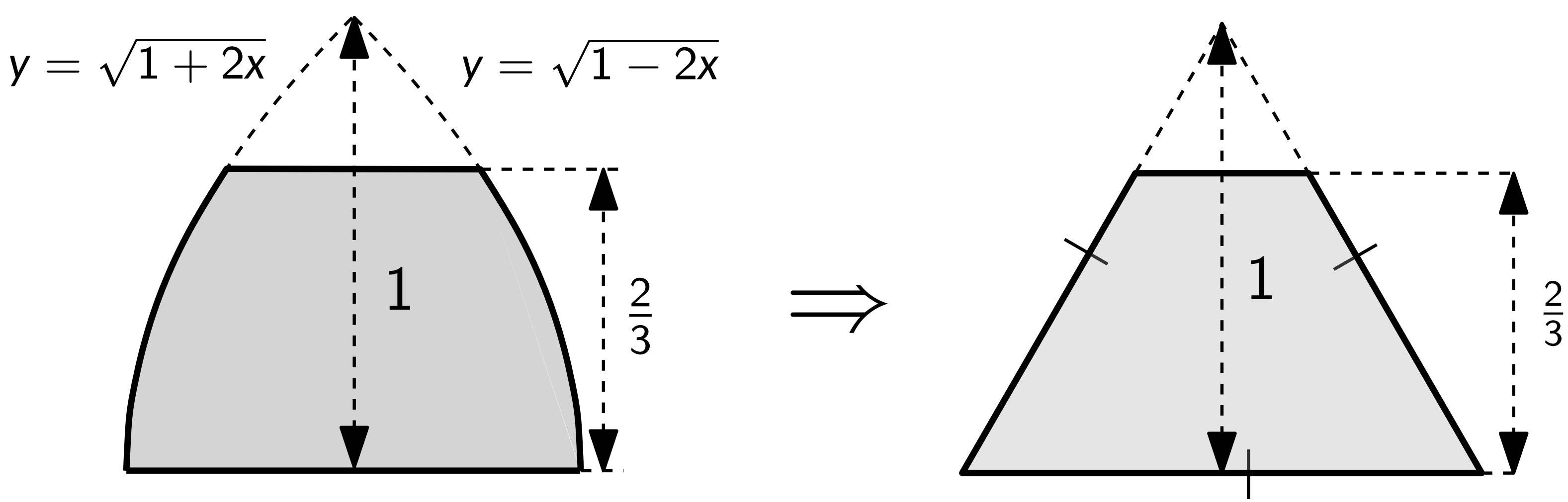
Observation. G_k-covering is also H_k-covering.

Area: 0.5680 | Area: 0.5132



G_3-covering* | H_3-covering

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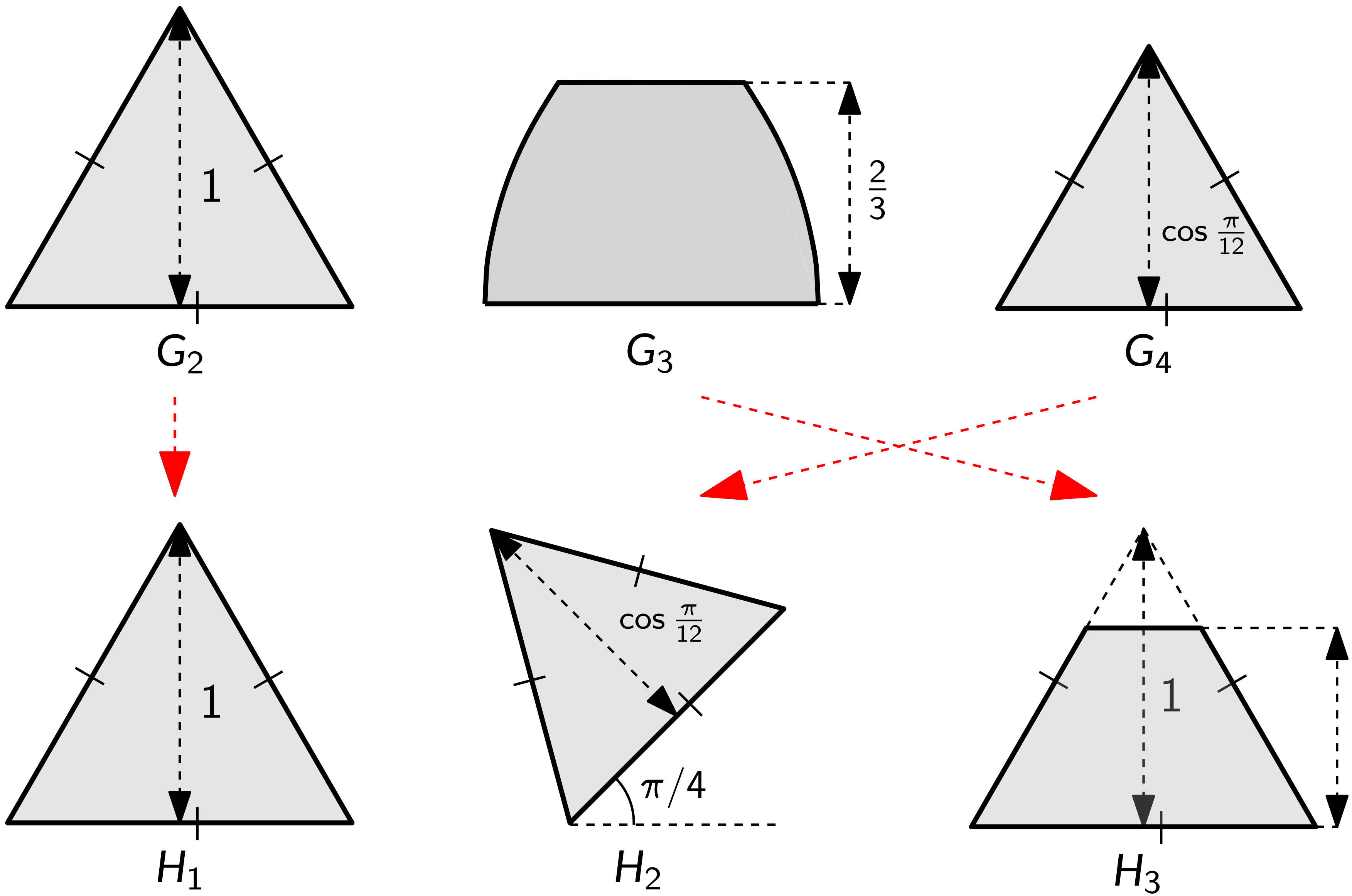
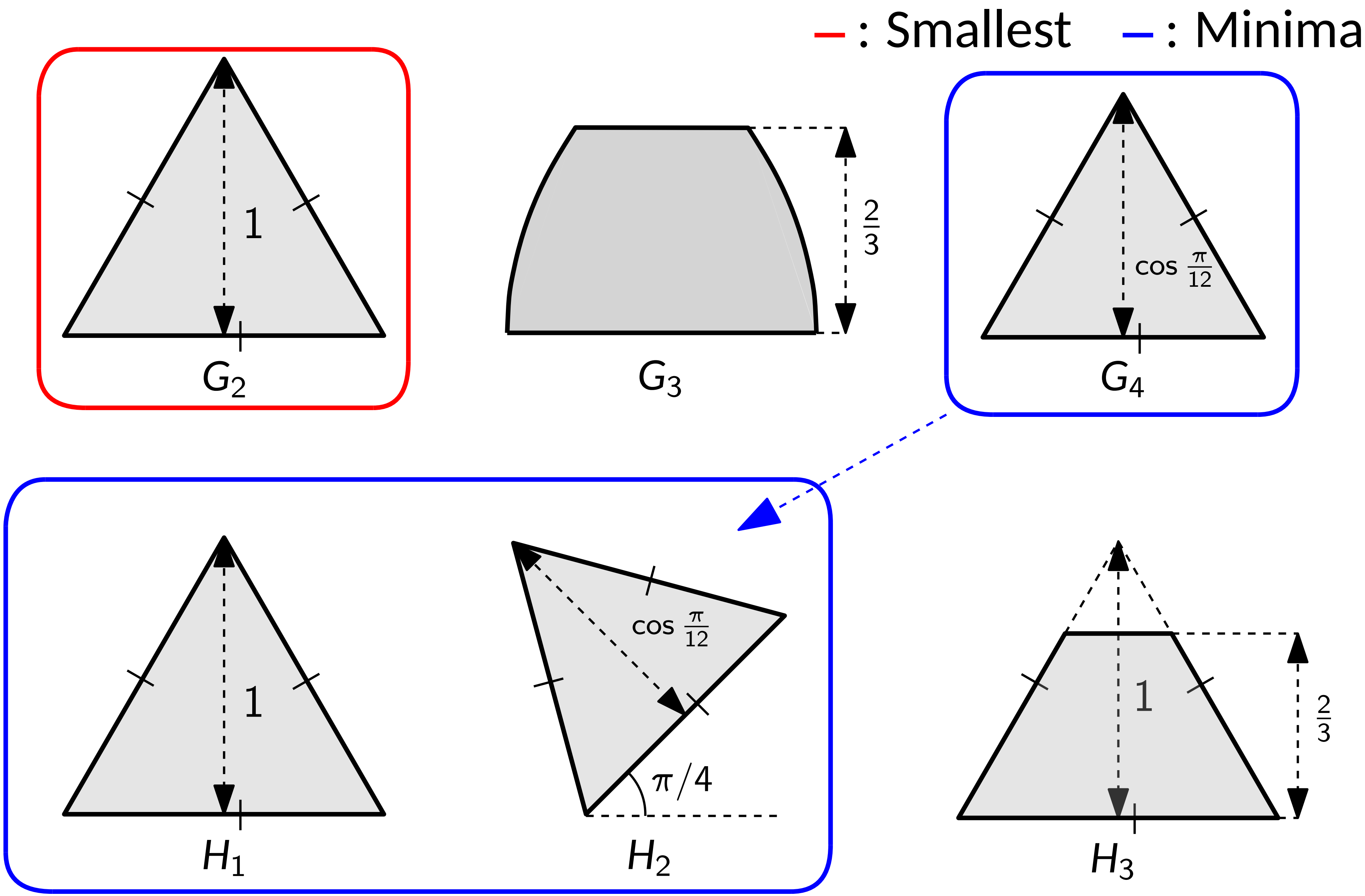
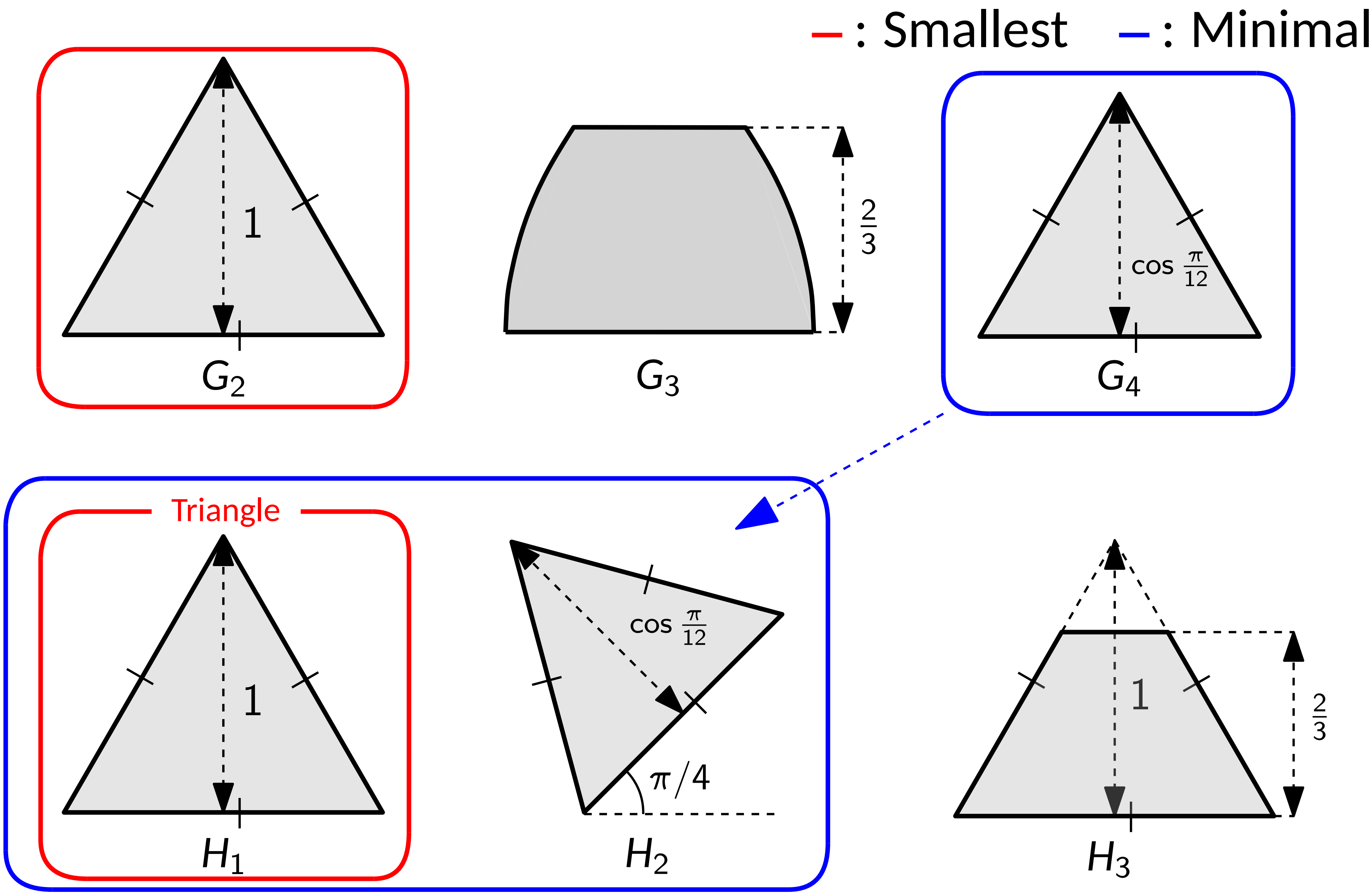
Outline


1
G_2
$\frac{2}{3}$
G_3
$\cos\frac{\pi}{12}$
G_4
1
H_1
$\cos\frac{\pi}{12}$
$\pi/4$
H_2
1
$\frac{2}{3}$
H_3

Outline

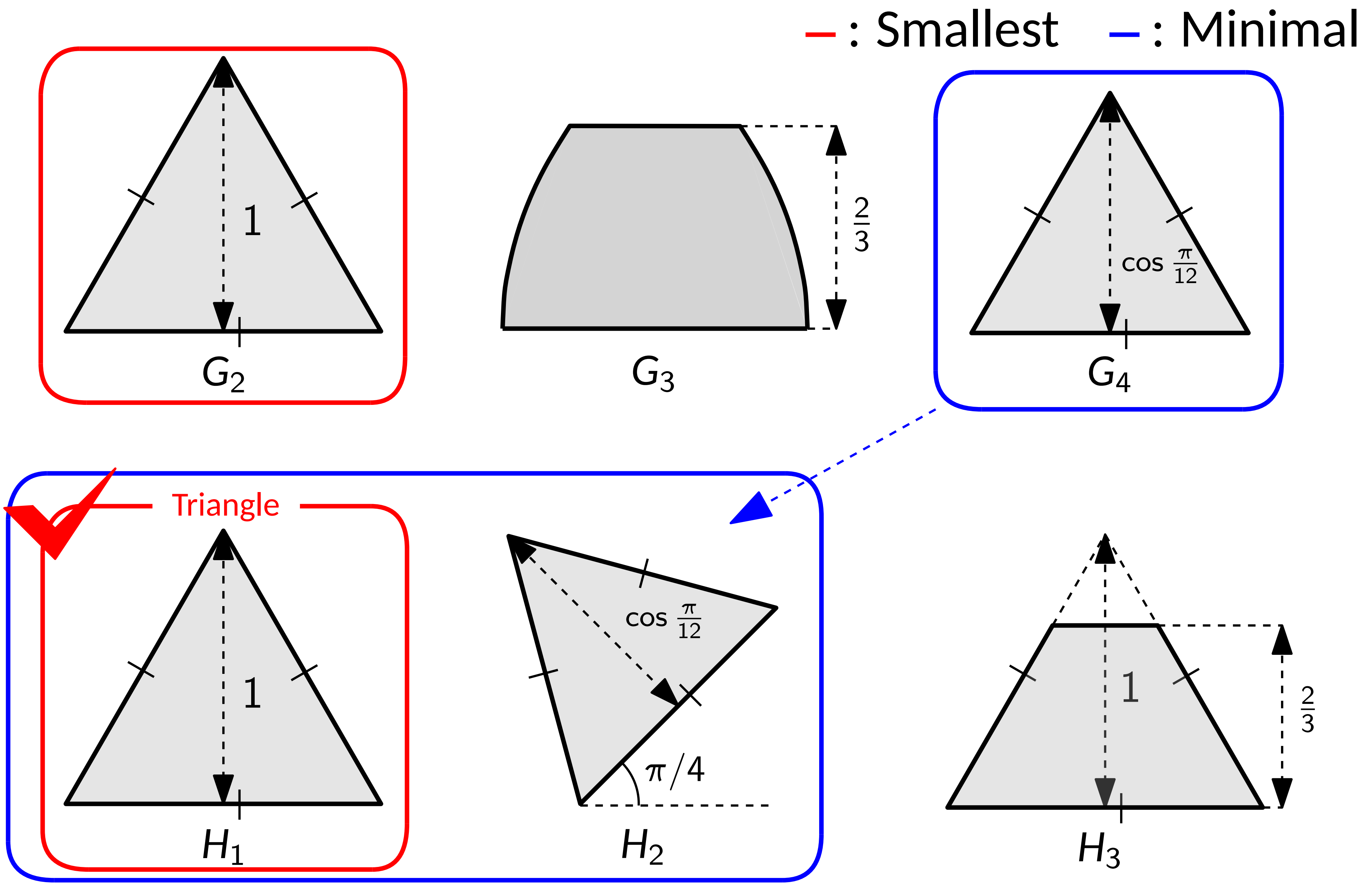

: Smallest
: Minimal
1
G_2
$\frac{2}{3}$
G_3
$\cos\frac{\pi}{12}$
G_4
1
H_1
$\cos\frac{\pi}{12}$
$\pi/4$
H_2
1
$\frac{2}{3}$
H_3

Outline



– : Smallest
– : Minimal
1
G_2
$\frac{2}{3}$
G_3
$\cos\frac{\pi}{12}$
G_4
Triangle
1
H_1
$\cos\frac{\pi}{12}$
$\pi/4$
H_2
1
$\frac{2}{3}$
H_3

Outline



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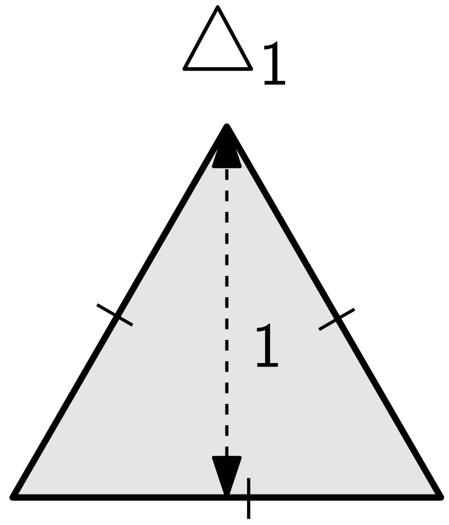
Outline

H_1: translation, the x-axis reflection

1. $\triangle_1$ is an H_1-covering of S_c

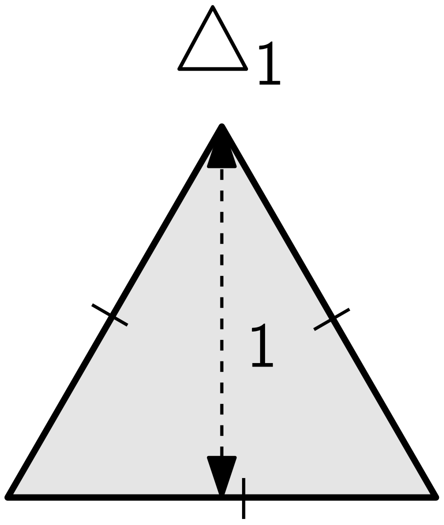


Outline

H_1: translation, the x-axis reflection

1. $\triangle_1$ is an H_1-covering of S_c

2. $\triangle_1$ is a minimal closed H_1-covering of S_c

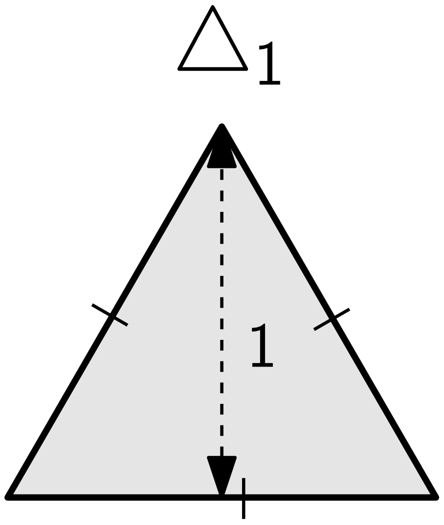


H_1: translation, the x-axis reflection

1. $\triangle_1$ is an H_1-covering of S_c

2. $\triangle_1$ is a minimal closed H_1-covering of S_c

3. $\triangle_1$ is the smallest-area triangle H_1-covering of S_c



$\triangle_1$: H_1-covering

Lemma 1. Suppose that a region P is symmetric with respect to the y-axis. Then P is an H_1-covering of S_c iff it is a G_2-covering of S_c.

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- G_2: Translation and π rotations

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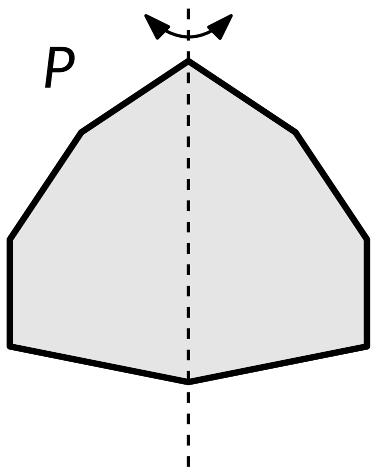

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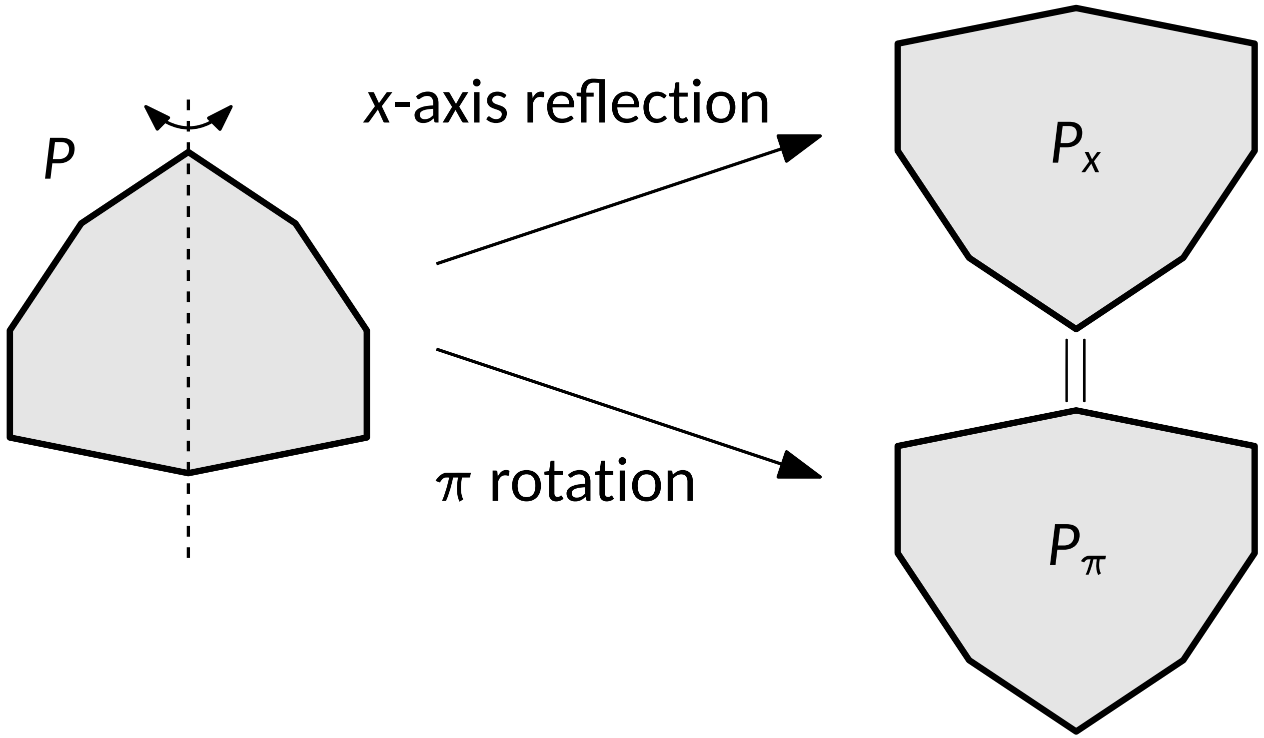

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Lemma 1. Suppose that a region P is symmetric with respect to the y-axis. Then P is an H_1-covering of S_c iff it is a G_2-covering of S_c.

Lemma 2*. $\triangle_1$ is the smallest-area G_2-covering of S_c.

*M. K. Jung, S. D. Yoon, H.-K. Ahn, T. Tokuyama, Universal convex covering problems under translation and discrete rotations, arXiv:2211.14807 [cs.CG] , 2022.

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Lemma 2*. $\triangle_1$ is the smallest-area G_2-covering of S_c.

Since $\triangle_1$ is symmetric with respect to the y-axis and a G_2-covering of S_c, $\triangle_1$ is an H_1-covering of S_c.

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Since $\triangle_1$ is symmetric with respect to the y-axis and a G_2-covering of S_c, $\triangle_1$ is an H_1-covering of S_c.

Corollary. $\triangle_1$ is the smallest-area H_1-covering of S_c among all H_1-covering of S_c that are symmetric to the y-axis.

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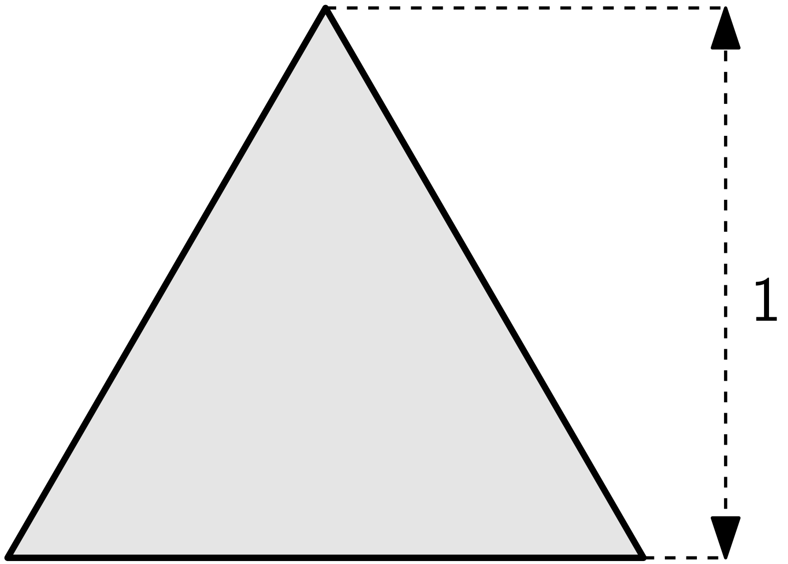
Theorem. $\triangle_1$ is a minimal closed convex H_1-covering of S_c.



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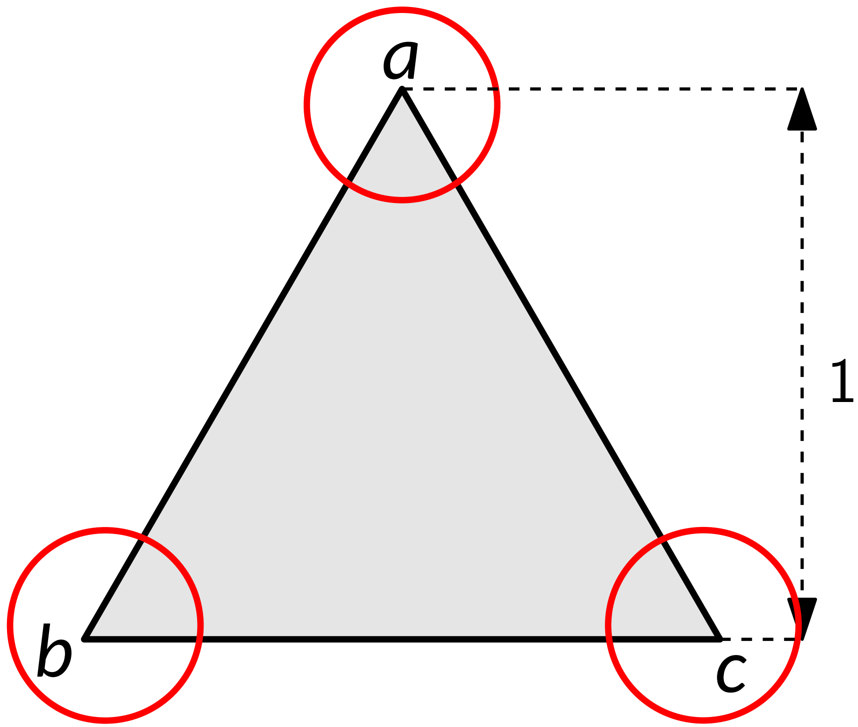

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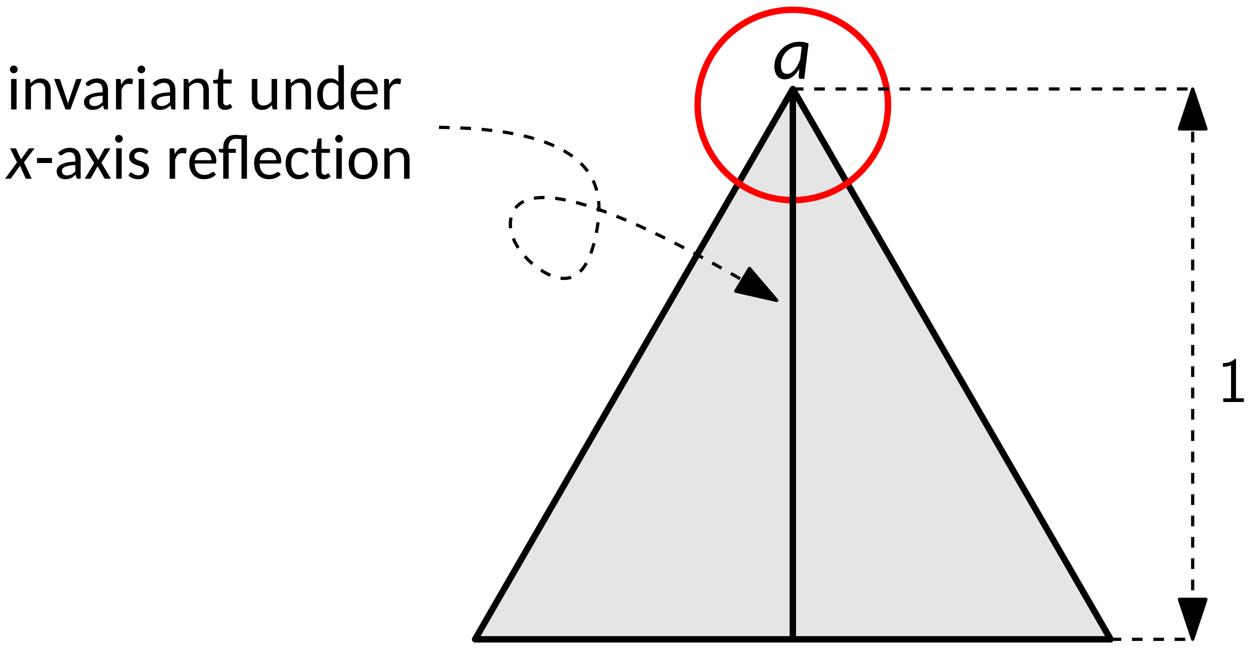

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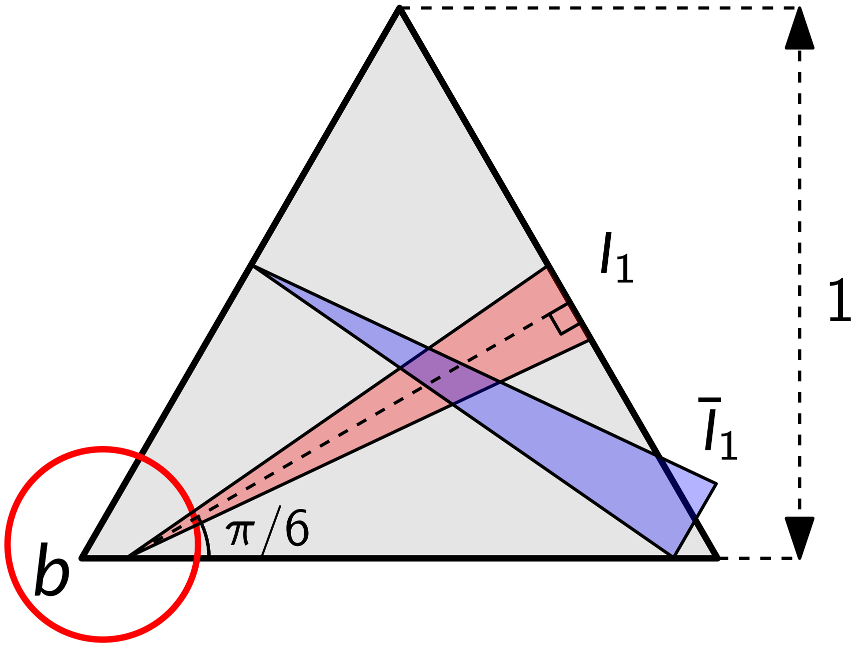
Maintaining the perimeter of the triangles 2.



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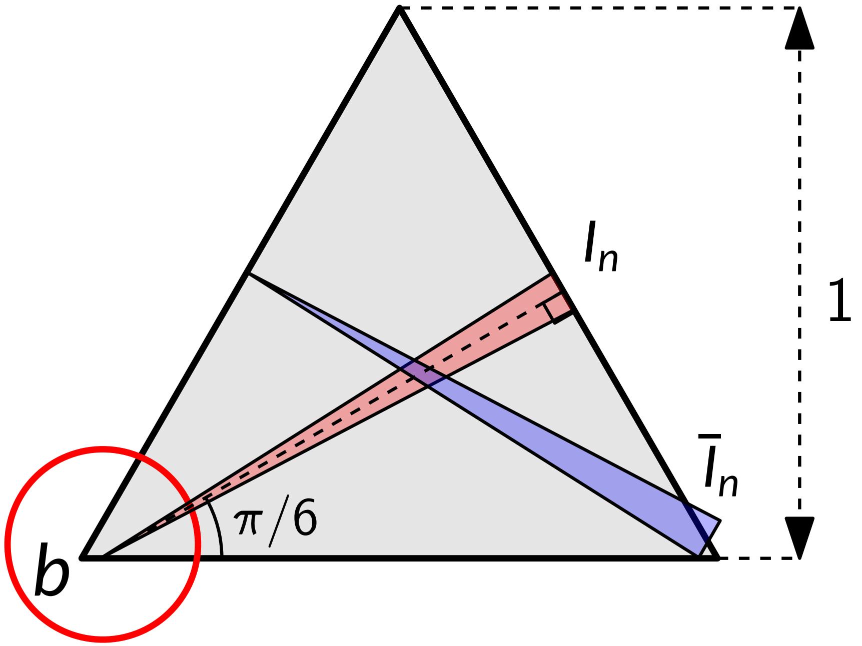

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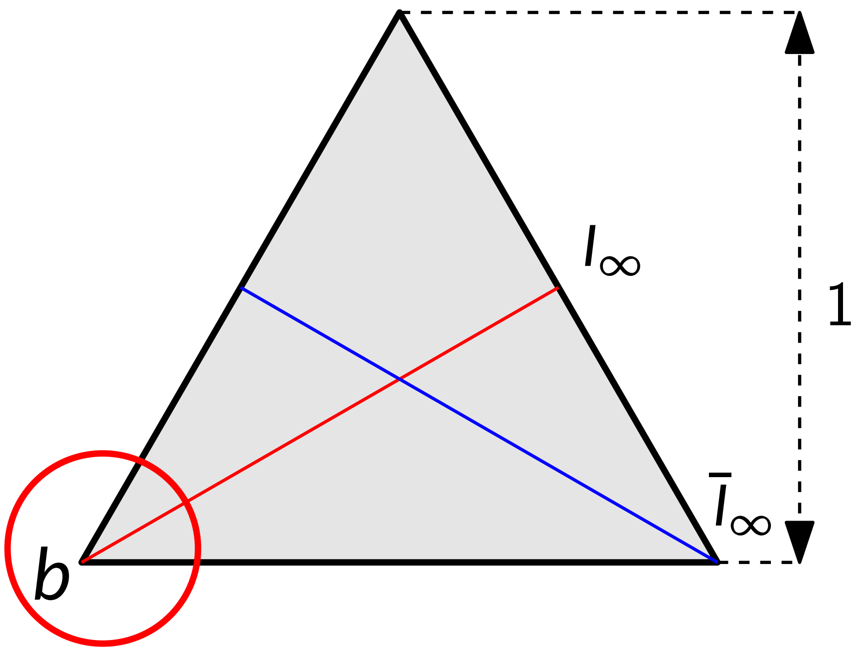

$\triangle_1$: Minimal H_1-covering

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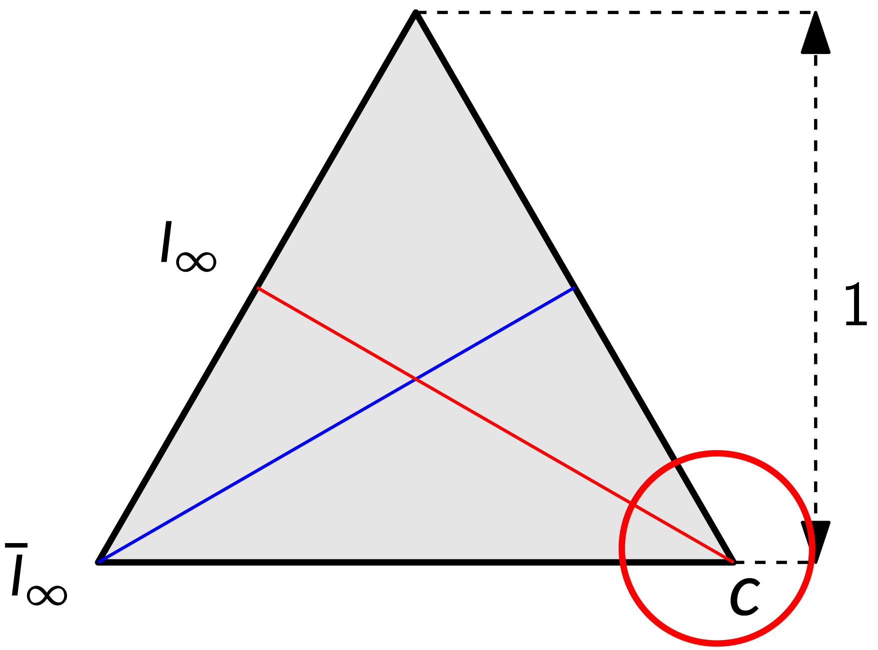
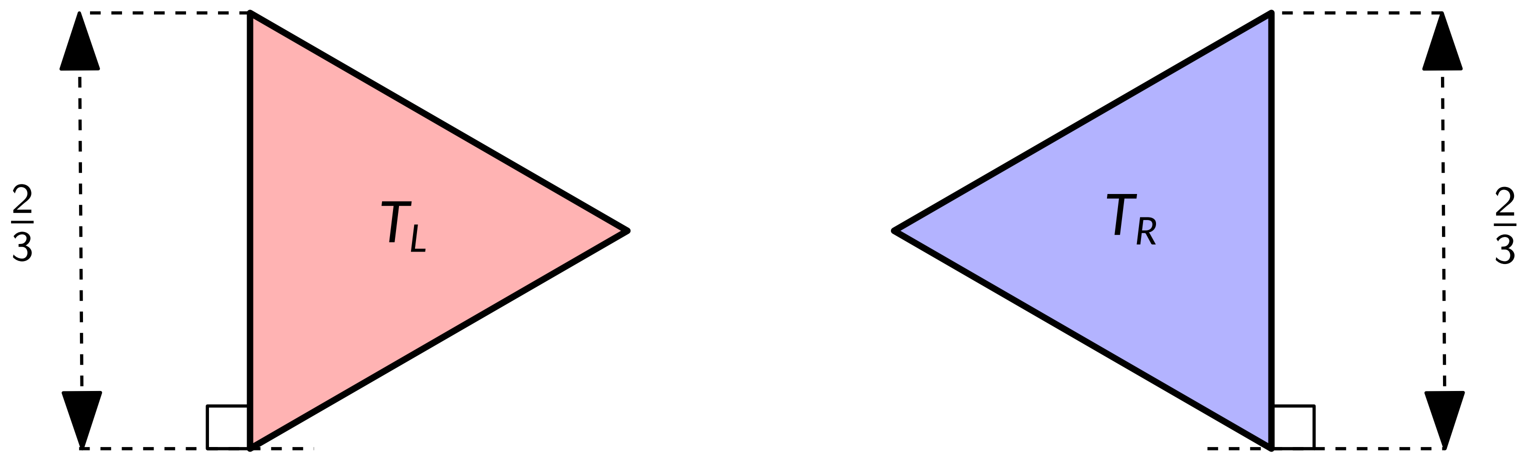


$\triangle_1$: The Smallest-area Triangle

Let T_L be an equilateral triangle of perimeter 2 such that it has a vertical side and its opposite corner lies to the right.

Let T_R be a copy of T_L rotated by π.

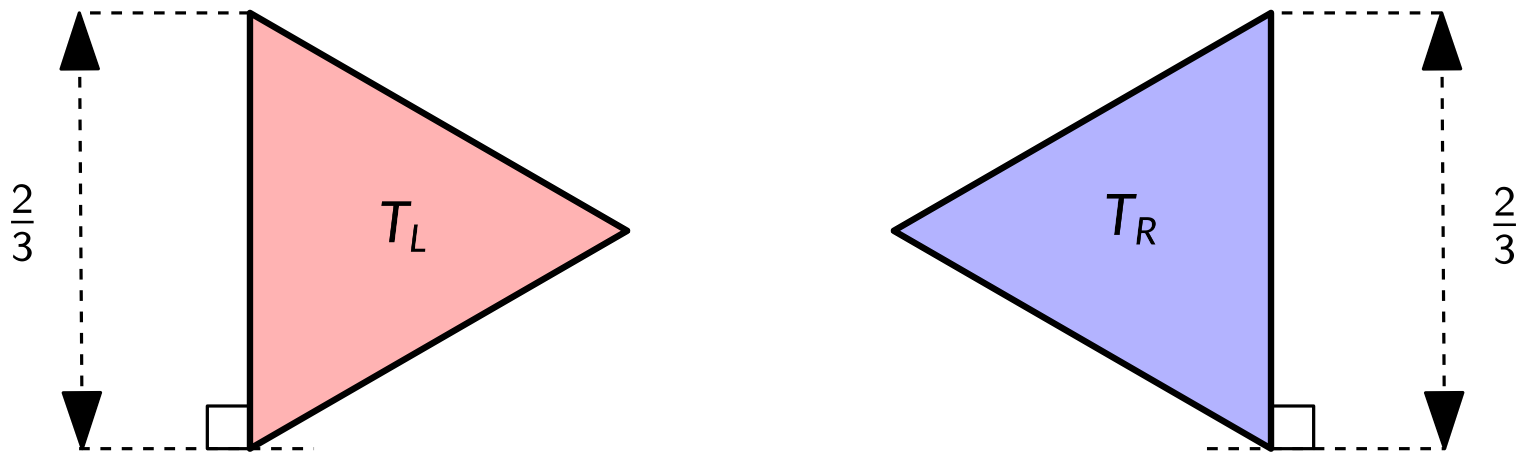


Let T_L be an equilateral triangle of perimeter 2 such that it has a vertical side and its opposite corner lies to the right.

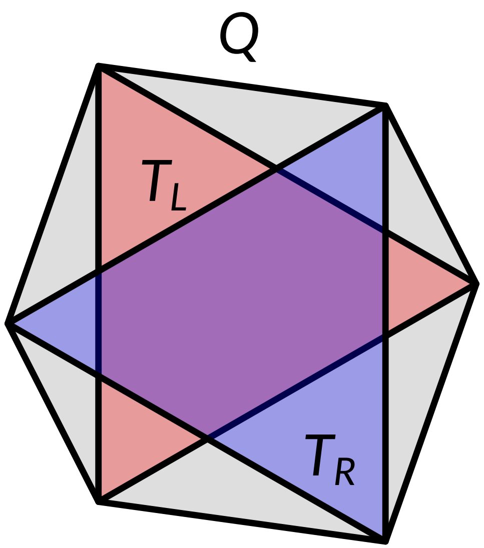
Let T_R be a copy of T_L rotated by π.

Observe that T_L and T_R are invariant under x-axis reflection.

$\Rightarrow$ What is the smallest-area triangle containing T_L and T_R?



Let Q be the convex hull of T_L and T_R



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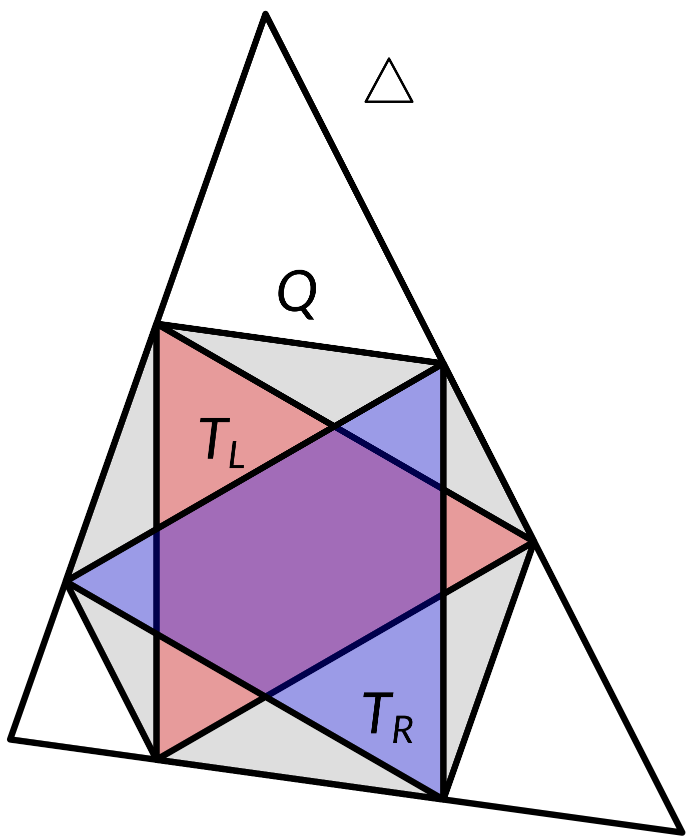
Let $\triangle$ be the smallest-area triangle containing Q



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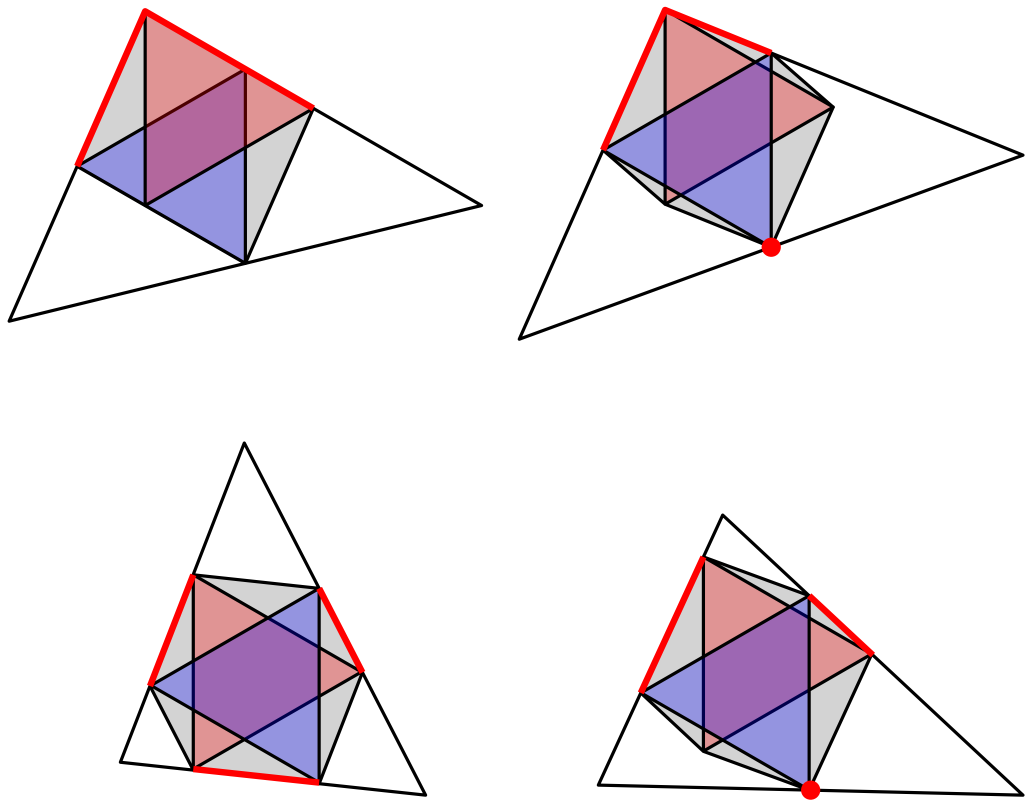

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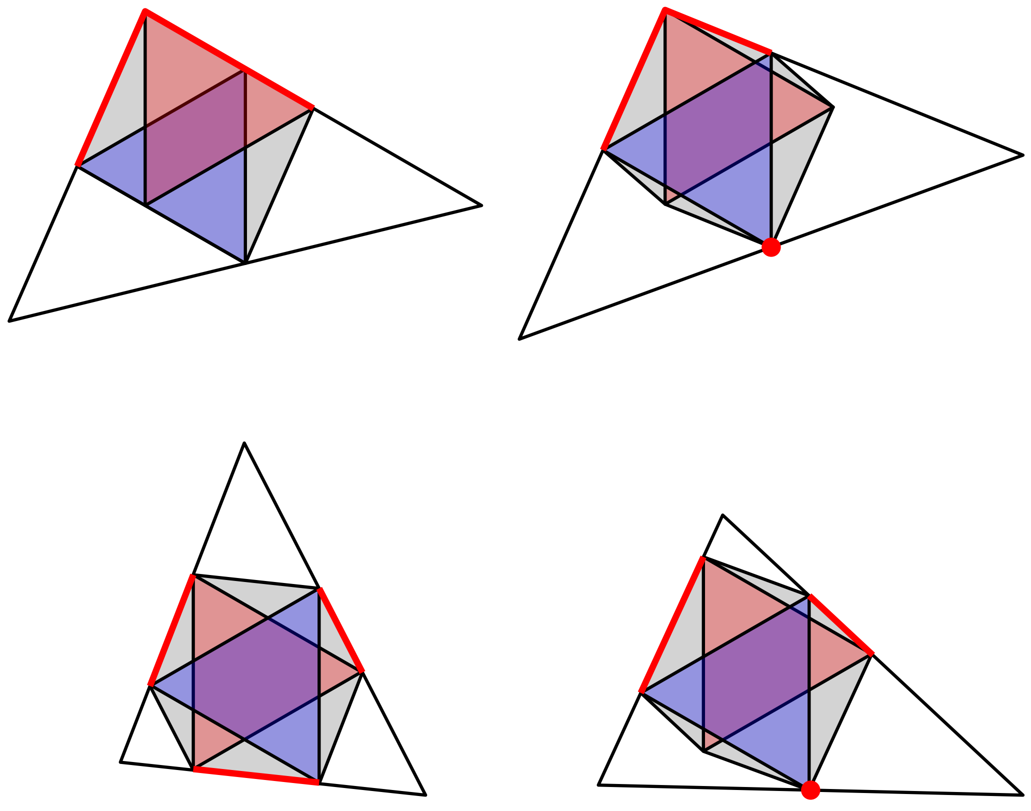
Let $\triangle$ be the smallest-area triangle containing Q

1. $|Q| \geq |T_L| + |T_R| = 2|T_L|$



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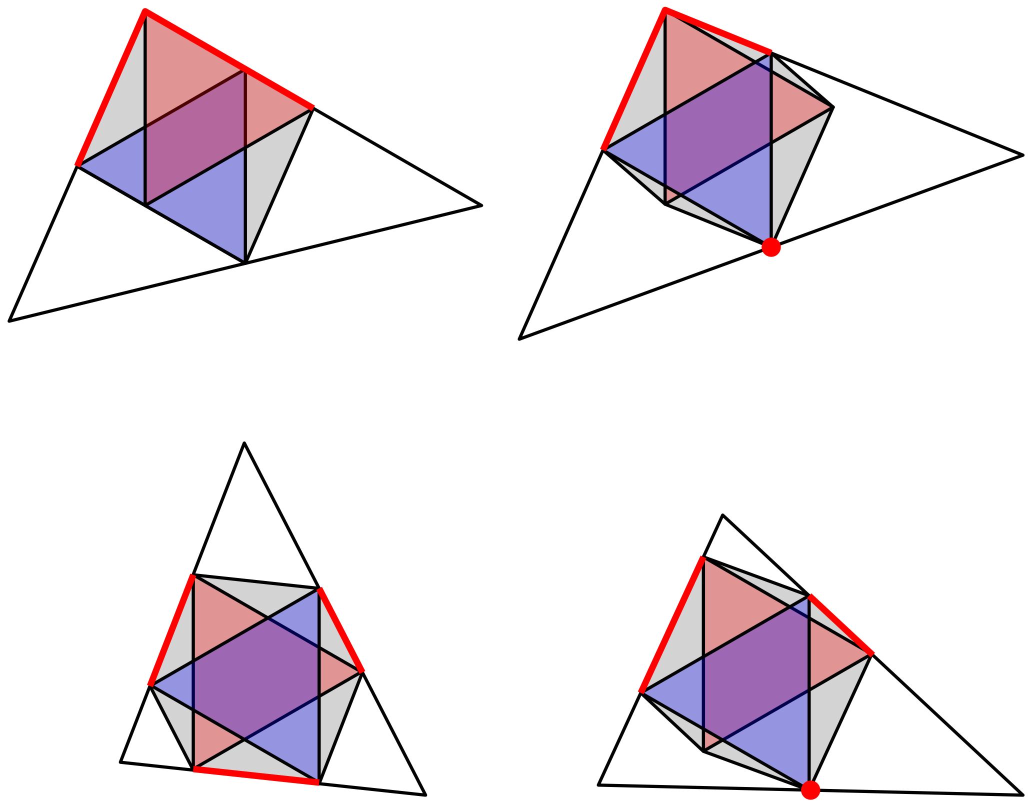
1. $|Q| \geq |T_L| + |T_R| = 2|T_L|$

2. $|\triangle| \geq \frac{3}{2}|Q|$



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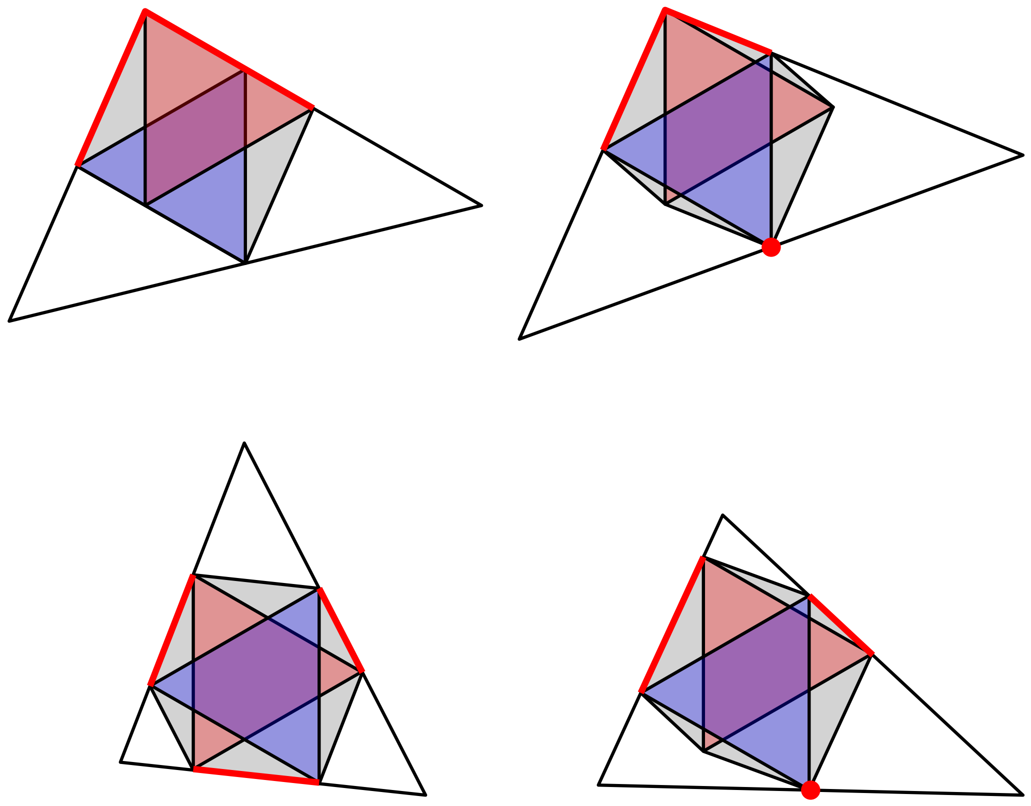
Let $\triangle$ be the smallest-area triangle containing Q

1. $|Q| \geq |T_L| + |T_R| = 2|T_L|$

2. $|\triangle| \geq \frac{3}{2}|Q| = 3|T_L| = |\triangle_1|$



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1. $|Q| \geq |T_L| + |T_R| = 2|T_L|$
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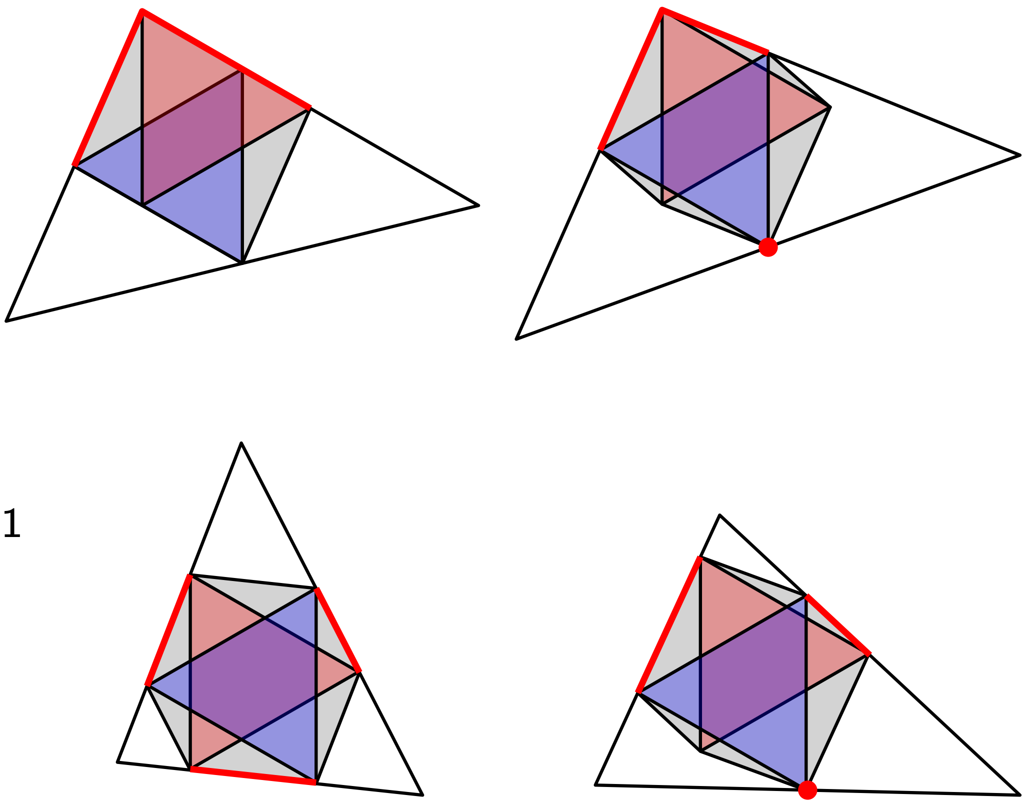
3. The equality holds iff $\triangle = \triangle_1$



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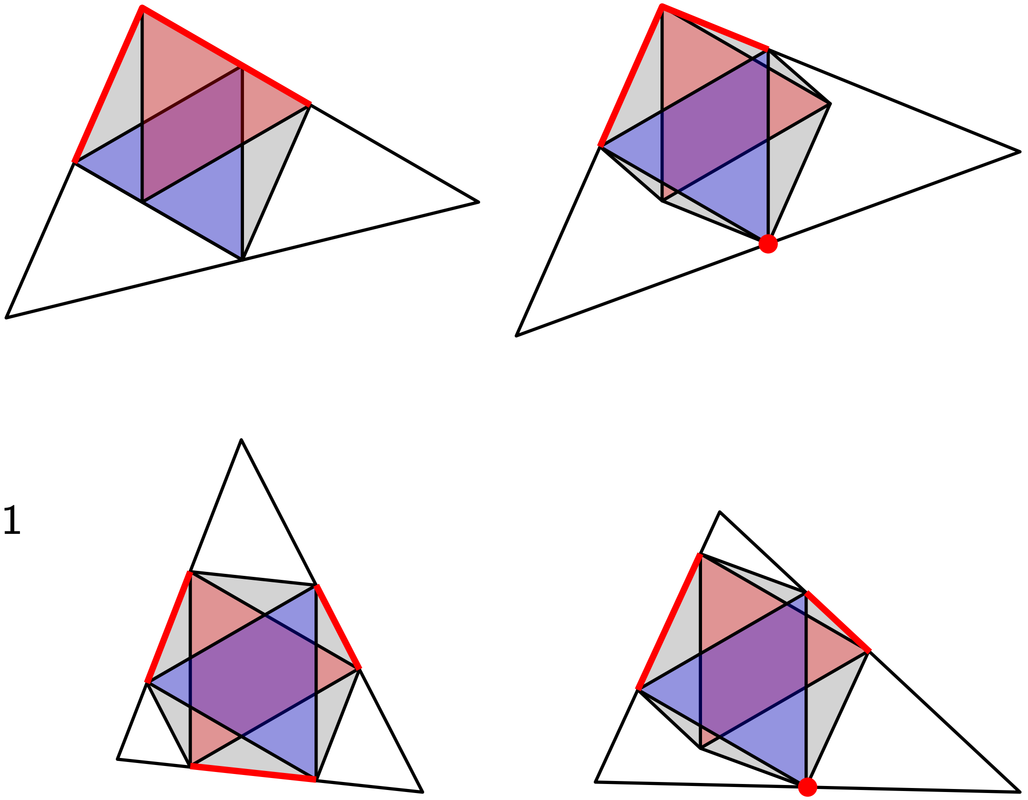
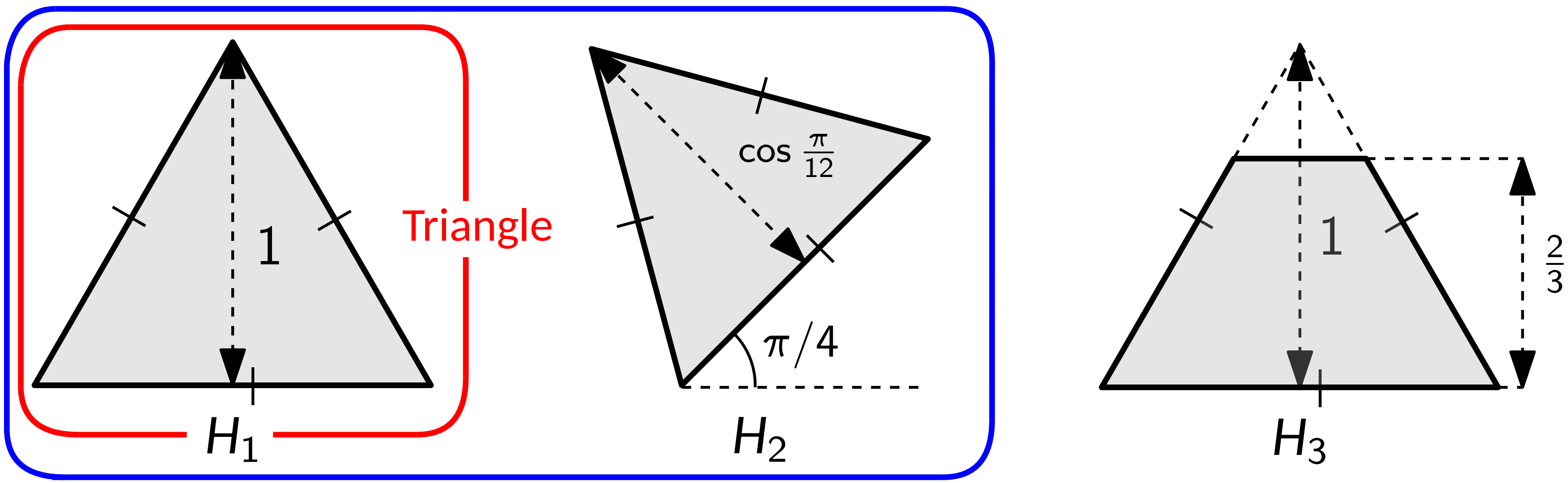
$\Rightarrow \triangle_1$ is the smallest triangle



- Set S_c of closed curves with length 2.

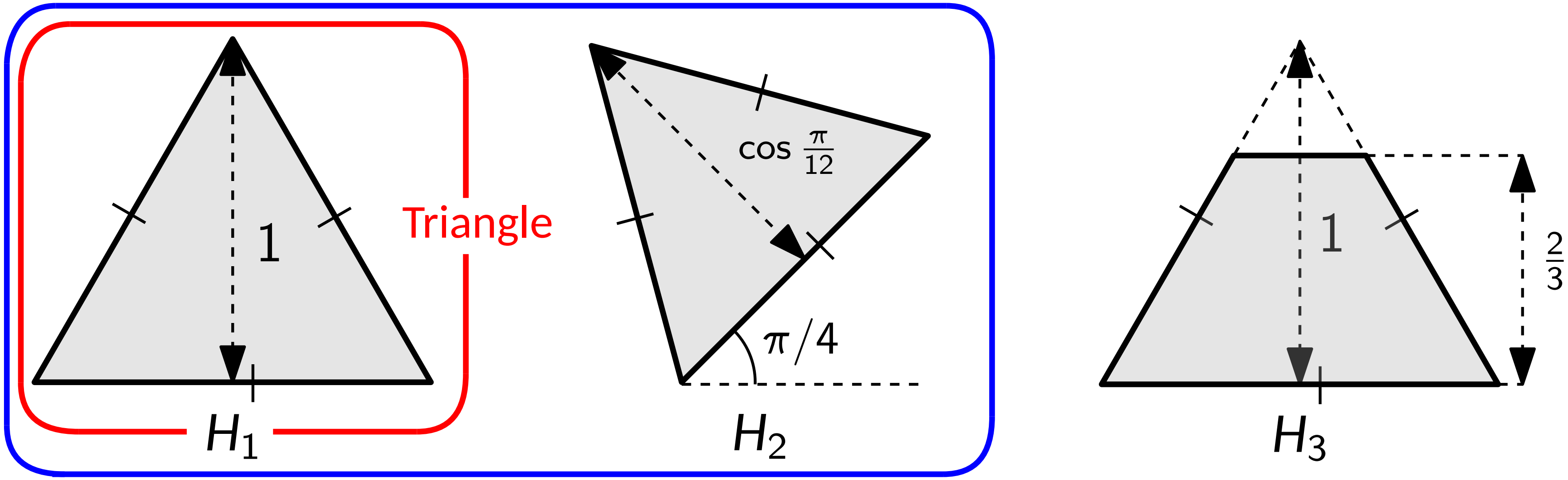


— : Minimal

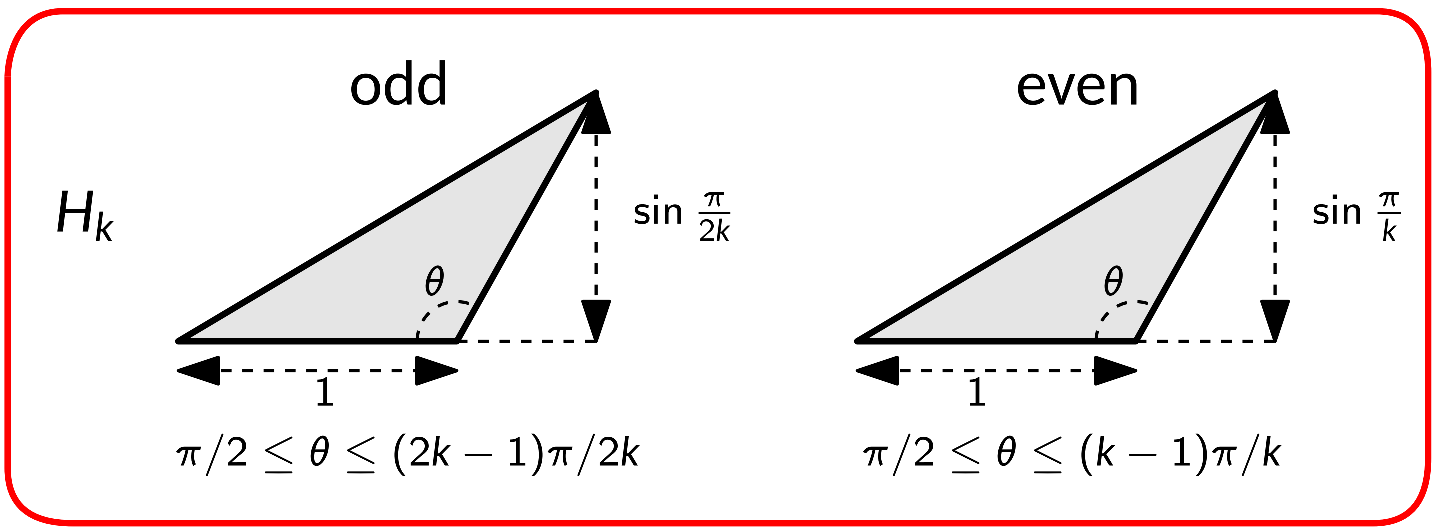
— : Smallest

Summary & Conclusion

- Set S_c of closed curves with length 2.



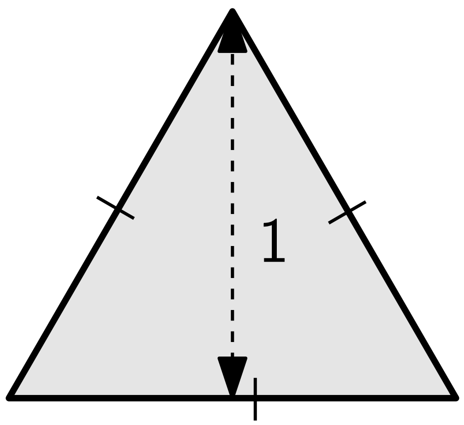
- Set of all unit line segments.



— : Minimal

— : Smallest

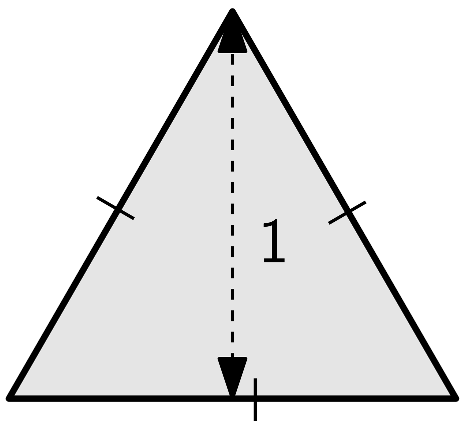
Future Work



: The smallest-area G_2-covering of S_c

The smallest-area triangle H_1-covering of S_c

Future Work



: The smallest-area G_2-covering of S_c

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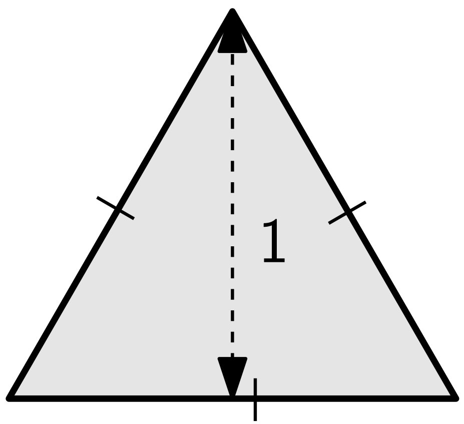
$\Rightarrow$ Is it the smallest-area H_1-covering of S_c?

Future Work



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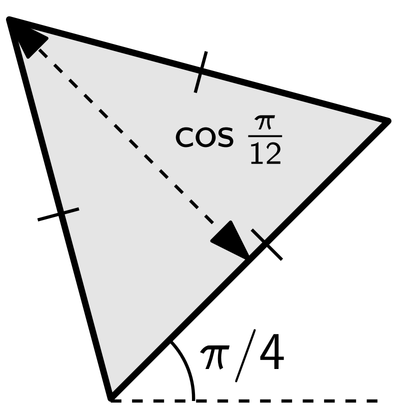
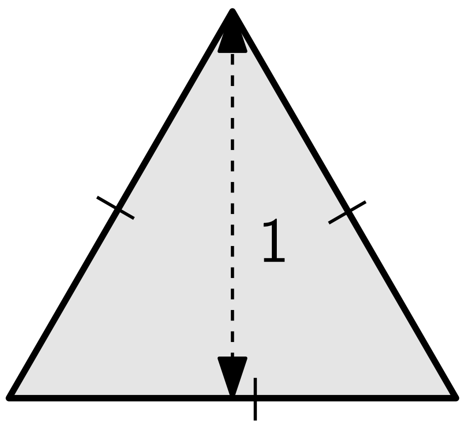


: A minimal G_4-covering of S_c

A minimal H_2-covering of S_c

Future Work



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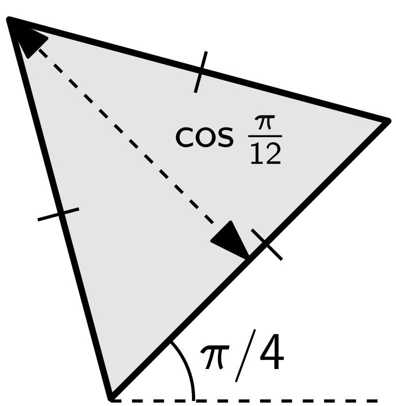

: A minimal G_4-covering of S_c

A minimal H_2-covering of S_c

$\Rightarrow$ Relation between G_{2k}-covering and H_{2k-1}-covering

Thank You!!