Super Guarding and Dark Rays in Art Galleries

MIT CompGeom Group
Hugo A. Akitaya   Erik D. Demaine   Adam Hesterberg   Anna Lubiw
Jayson Lynch   Joseph O'Rourke   Frederick Stock

CCCG 2023

Overview

1. The art gallery problem and some variants

Overview

1. The art gallery problem and some variants
2. Introduce a new variant inspired by previous variants

OVERVIEW

1. The art gallery problem and some variants
2. Introduce a new variant inspired by previous variants
3. Prove necessary and sufficient bounds for convex polygons

Overview

1. The art gallery problem and some variants
2. Introduce a new variant inspired by previous variants
3. Prove necessary and sufficient bounds for convex polygons
4. Give a loose bound for simple polygons

Overview

1. The art gallery problem and some variants
2. Introduce a new variant inspired by previous variants
3. Prove necessary and sufficient bounds for convex polygons
4. Give a loose bound for simple polygons
5. Fun with wedges!

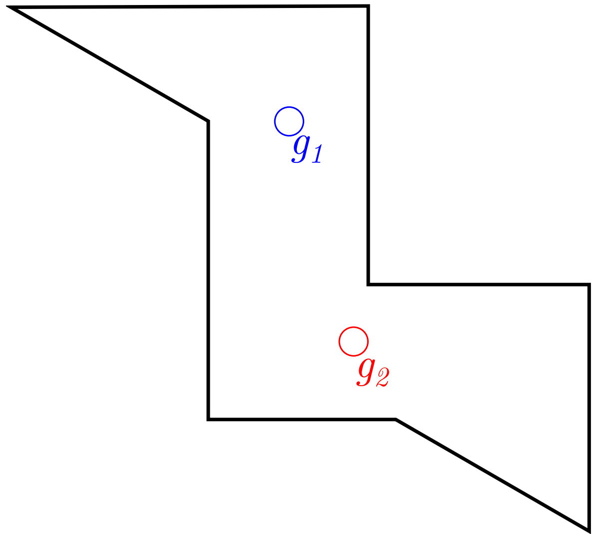
The Art Gallery Problem

- How do we place guards such that every point of our art gallery is visible to a guard?
- Given a polygon P, find a set of points G such that every point $p \in P$ is visible from a point $g \in G$.
- $\lfloor n/3 \rfloor$ guards are sometimes necessary and always sufficient.

Figure. https://www.louvre.fr/en

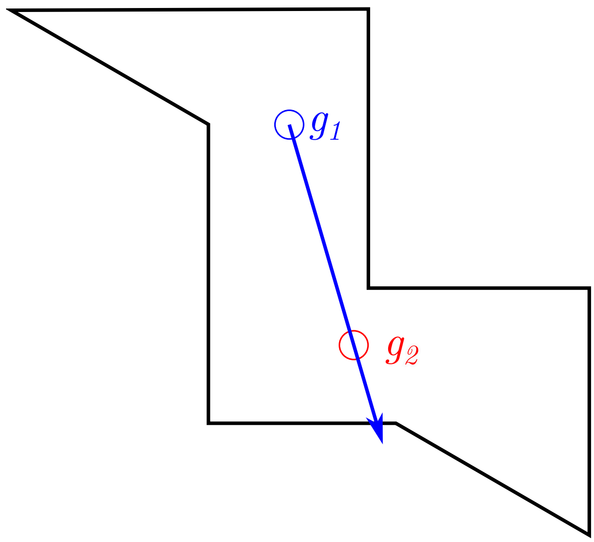
Guards Blocking Guards Variant

- Guards cannot see through other guards



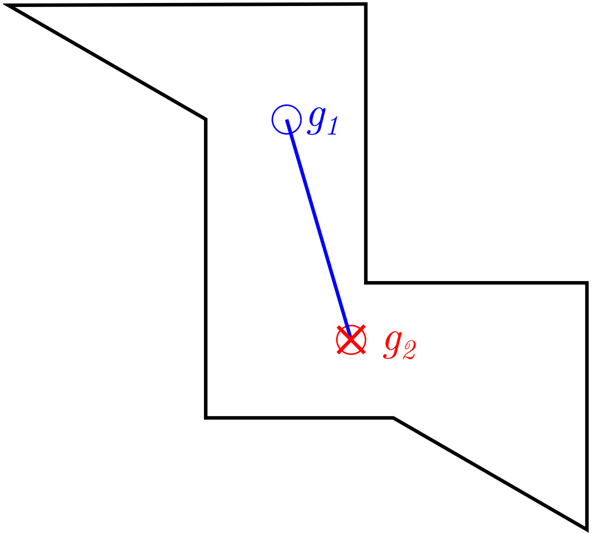
Guards Blocking Guards Variant

- Guards cannot see through other guards



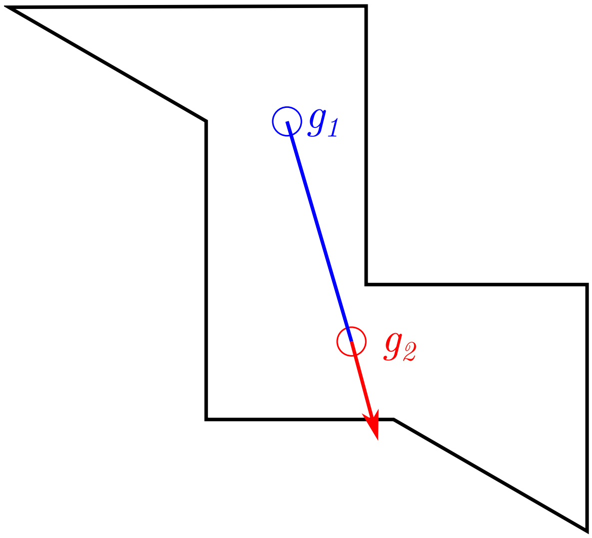
Guards Blocking Guards Variant

- Guards cannot see through other guards



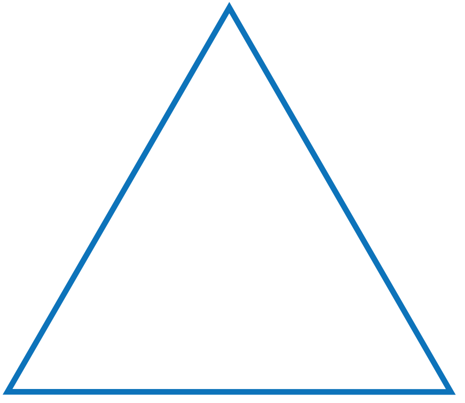
Guards Blocking Guards Variant

- Guards cannot see through other guards
- The bound of $\lfloor n/3 \rfloor$ still holds, a blocking guard can see what it is obscuring from another guard



Multiple Coverage Variant

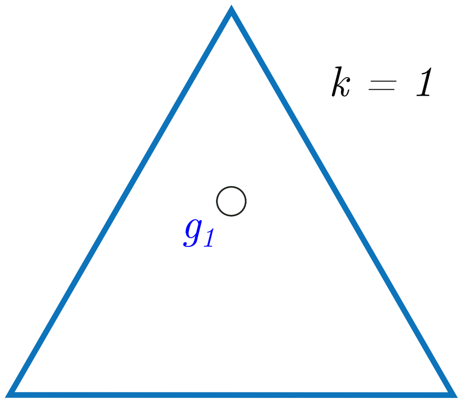
Suppose every point in the closed polygon must be seen by k guards (the guards $k-cover$ the polygon).



Multiple Coverage Variant

Suppose every point in the closed polygon must be seen by k guards (the guards $k-cover$ the polygon).

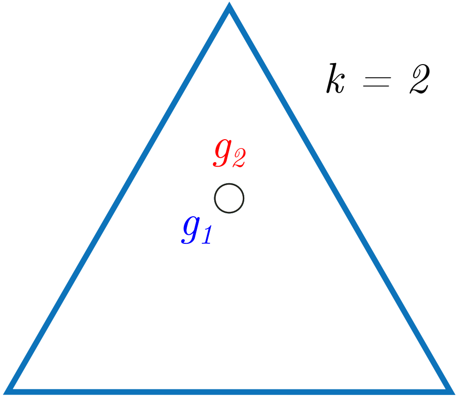
- If co-location of guards is allowed, this is trivial



Multiple Coverage Variant

Suppose every point in the closed polygon must be seen by k guards (the guards $k-$ *cover* the polygon).

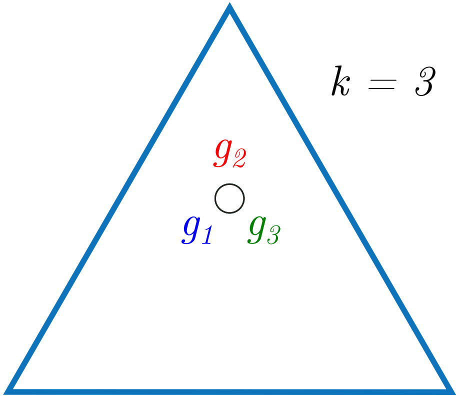
- If co-location of guards is allowed, this is trivial



Multiple Coverage Variant

Suppose every point in the closed polygon must be seen by k guards (the guards $k-$*cover* the polygon).

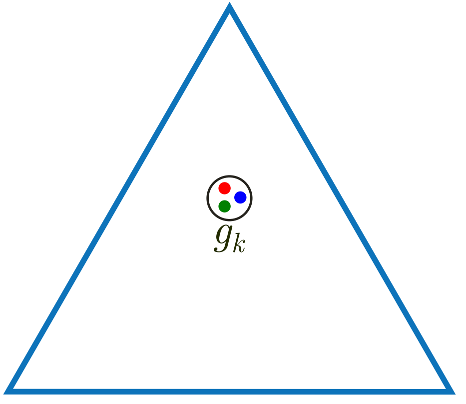
- If co-location of guards is allowed, this is trivial



Multiple Coverage Variant

Suppose every point in the closed polygon must be seen by k guards (the guards $k-cover$ the polygon).

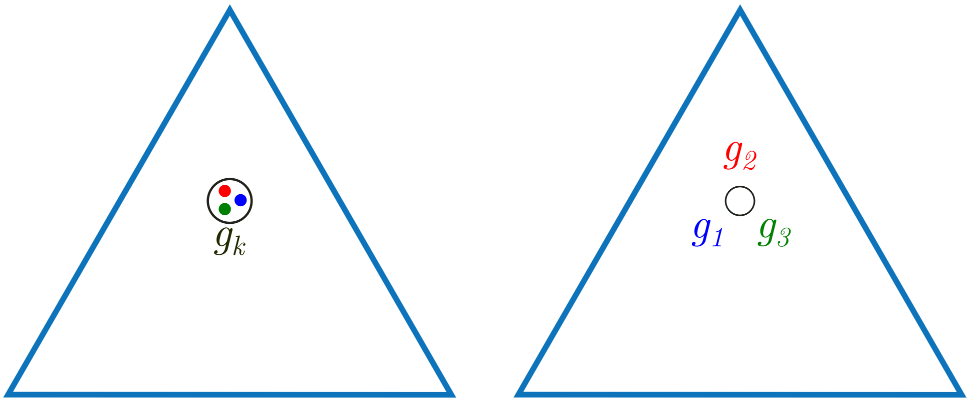
- If co-location is not allowed, but guards can see through each other, this is equivalent to $k = 1$ where we replace each guard with a cluster of k guards.



Multiple Coverage Variant

Suppose every point in the closed polygon must be seen by k guards (the guards $k-$ *cover* the polygon).

- If co-location is not allowed, but guards can see through each other, this is equivalent to $k = 1$ where we replace each guard with a cluster of k guards.
- hence we have a $k\lfloor n/3 \rfloor$ bound.



Super Guarding With Multiple Coverage (Dark Rays)

Let's combine these!

Super Guarding With Multiple Coverage (Dark Rays)

Let's combine these!

- ***k*-guard a polygon P with n vertices**

Super Guarding With Multiple Coverage (Dark Rays)

Let's combine these!

- *k-guard* a polygon P with n vertices
- **Guards cannot be co-located**

Super Guarding With Multiple Coverage (Dark Rays)

Let's combine these!

- *k-guard* a polygon P with n vertices
- Guards cannot be co-located
- **Guards block each other's line-of-sight**

Super Guarding With Multiple Coverage (Dark Rays)

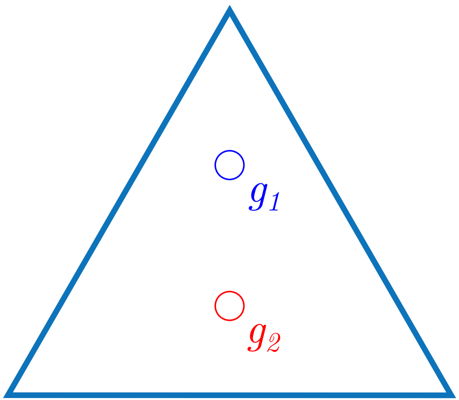
Let's combine these!

- *k-guard* a polygon P with n vertices
- Guards cannot be co-located
- Guards block each other's line-of-sight

How many guards are necessary and sufficient?

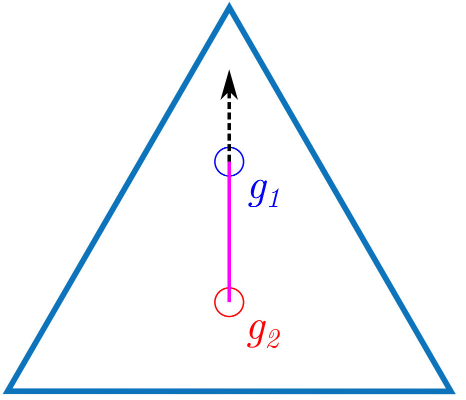
Dark Rays and Dark Points

- Let g_1 and g_2 be two guards visible to each other



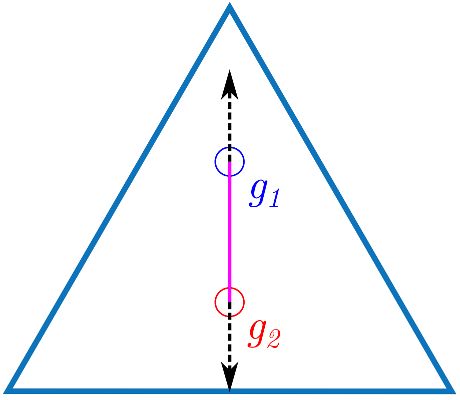
Dark Rays and Dark Points

- g_2 *generates* a *dark ray* at g_1, and g_1 generates a dark ray at g_2



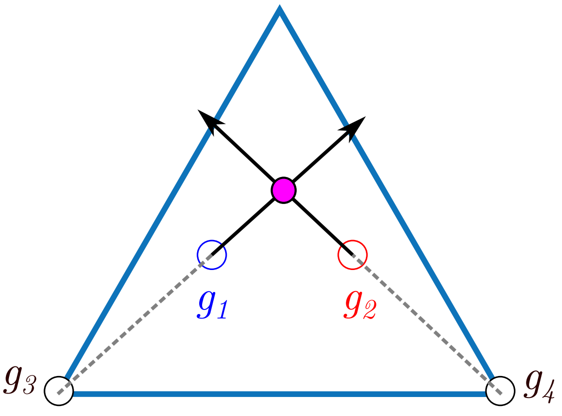
Dark Rays and Dark Points

- g_2 *generates* a *dark ray* at g_1, and g_1 generates a dark ray at g_2



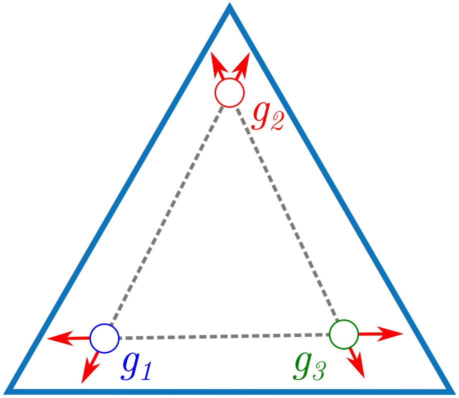
Dark Rays and Dark Points

- A point p is *dark* if it is contained in a dark ray, and *d-dark* if it is contained in d dark rays



Convex Polygons

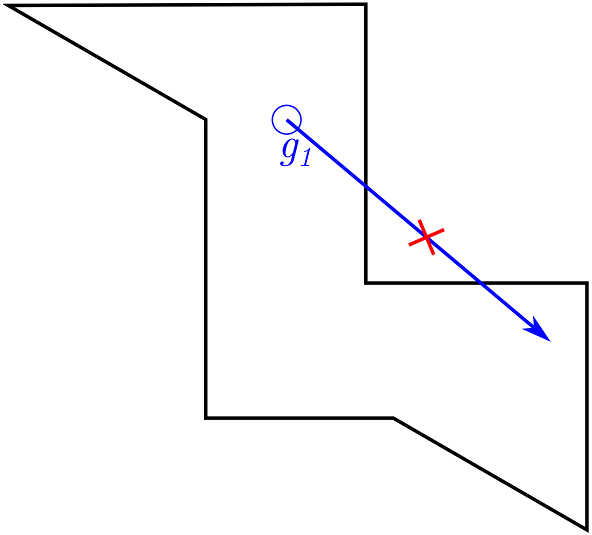
In a convex polygon, guard visibility reduces to dark rays and points



Convex Polygons

In a convex polygon, guard visibility reduces to dark rays and points

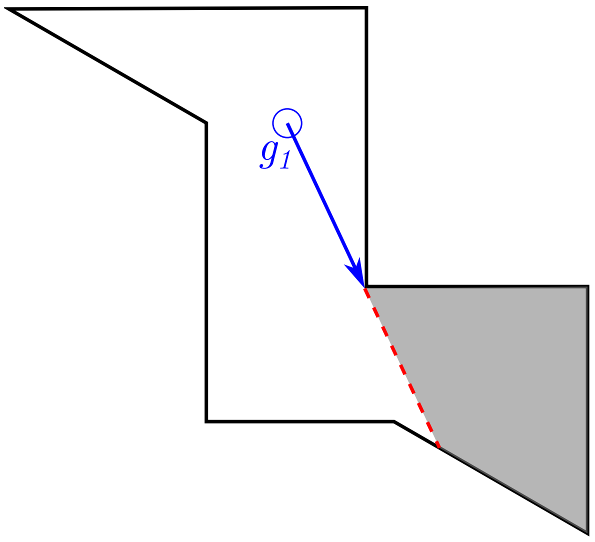
- Reflex angles can block line-of-sight



Convex Polygons

In a convex polygon, guard visibility reduces to dark rays and points

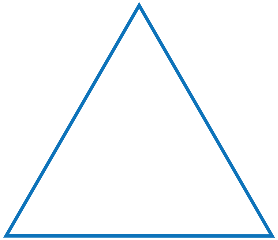
- Reflex angles can block line-of-sight **creating dark regions**



Theorem 1

For a closed convex n-gon, coverage to depth k requires $g \in \{k, k+1, k+2\}$ guards:

1. For $k \leq n$: $g = k$ guards are necessary and sufficient.



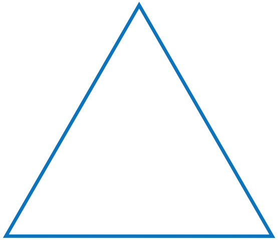
k	1	2	3
g	1	2	3

Table. Number of guards g to k-guard a triangle

THEOREM 1

For a closed convex n-gon, coverage to depth k requires $g \in \{k, k+1, k+2\}$ guards:

1. For $k \leq n$: $g = k$ guards are necessary and sufficient.
2. For $n < k < 4n - 2$: $g = k + 1$ guards are necessary and sufficient.



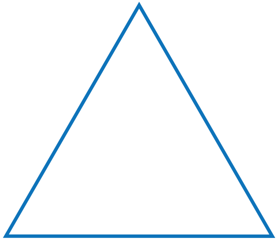
k	1	2	3	4	5	6	7	8	9
g	1	2	3	5	6	7	8	9	10

Table. Number of guards g to k-guard a triangle

Theorem 1

For a closed convex n-gon, coverage to depth k requires $g \in \{k, k+1, k+2\}$ guards:

1. For $k \leq n$: $g = k$ guards are necessary and sufficient.
2. For $n < k < 4n - 2$: $g = k + 1$ guards are necessary and sufficient.
3. For $4n - 2 \leq k$: $g = k + 2$ guards are necessary and sufficient.



k	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
g	1	2	3	5	6	7	8	9	10	12	13	$\cdots$

Table. Number of guards g to k-guard a triangle

OBSERVATION

1. k**-guarding with** $g = k$ **guards is possible if and only if there is no dark point inside** P**.**

Observation

1. k-guarding with $g = k$ guards is possible if and only if there is no dark point inside P.
2. **k-guarding with $g = k + 1$ guards is possible if and only if there is no 2-dark point inside P.**

Observation

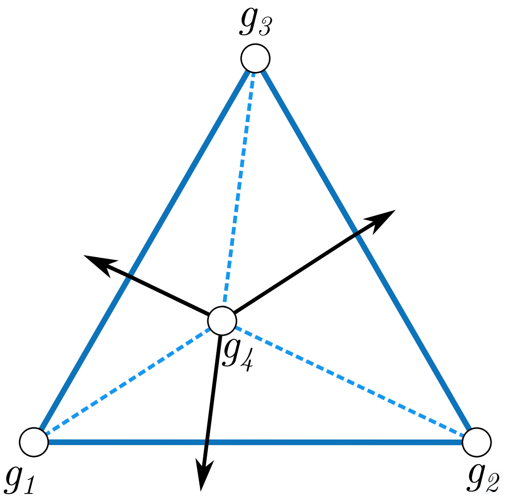
1. k-guarding with $g = k$ guards is possible if and only if there is no dark point inside P.
2. k-guarding with $g = k + 1$ guards is possible if and only if there is no 2-dark point inside P.
3. k**-guarding with** $g = k + 2$ **guards is always possible because we can perturb the guards to remove** 3**-dark points**

Theorem 2

The maximum number of guards that can be placed in a convex n-gon P without creating 2-dark points in P is $4n - 2$.

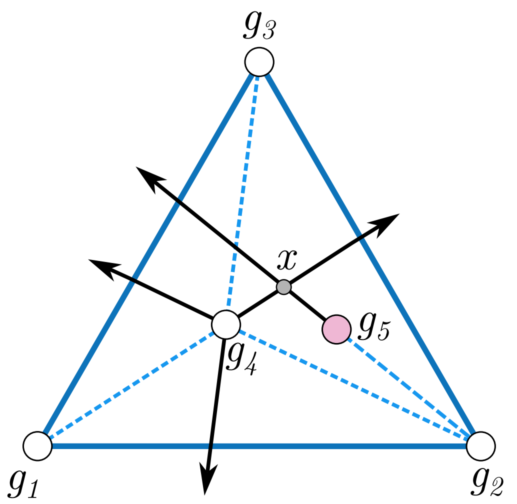
Triangle Lemma

- Suppose some guards are placed in P without creating 2-dark points.
- Let T be a closed triangle in P with guards g_1, g_2, g_3 at its corners.



Triangle Lemma

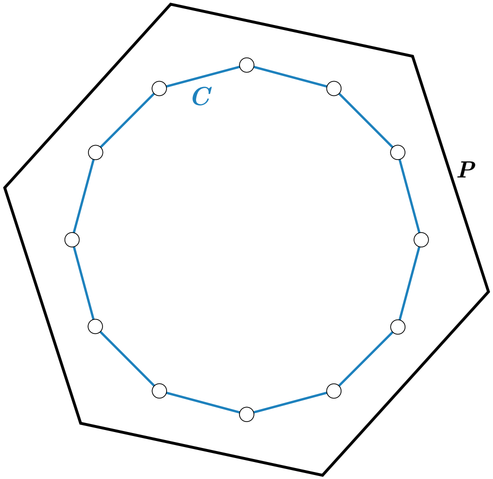
- Suppose some guards are placed in P without creating 2-dark points.
- Let T be a closed triangle in P with guards g_1, g_2, g_3 at its corners.
- Then, T contains at most one more guard.



Theorem 5

The number of guards g that can be placed in a convex n-gon P so that no two dark rays intersect inside is at most $g = 4n - 2$.

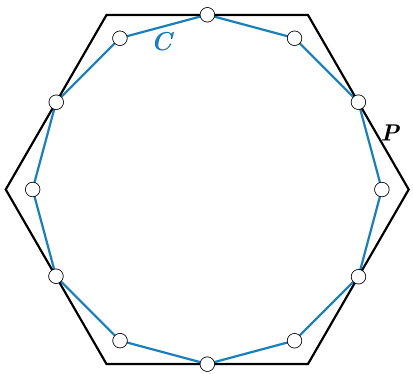
- Take the convex hull C of the guards in P.



THEOREM 5

The number of guards g that can be placed in a convex n-gon P so that no two dark rays intersect inside is at most $g = 4n - 2$.

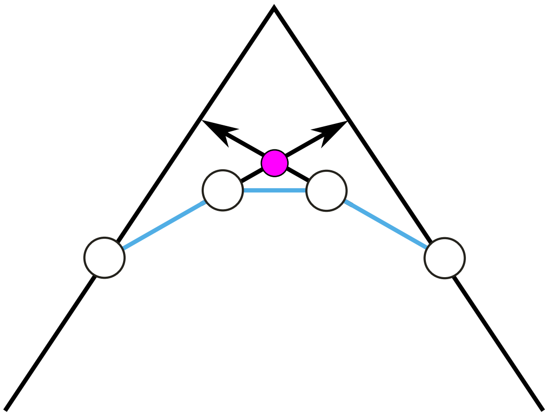
- Shrink and rotate P such that every edge has one or more guards in its interior or on its endpoint(s)



THEOREM 5

The number of guards g that can be placed in a convex n-gon P so that no two dark rays intersect inside is at most $g = 4n - 2$.

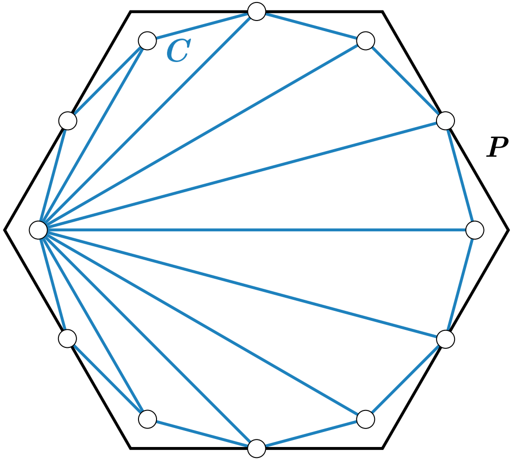
- There can be at most $2n$ guards on the boundary of C.



THEOREM 5

The number of guards g that can be placed in a convex n-gon P so that no two dark rays intersect inside is at most $g = 4n - 2$.

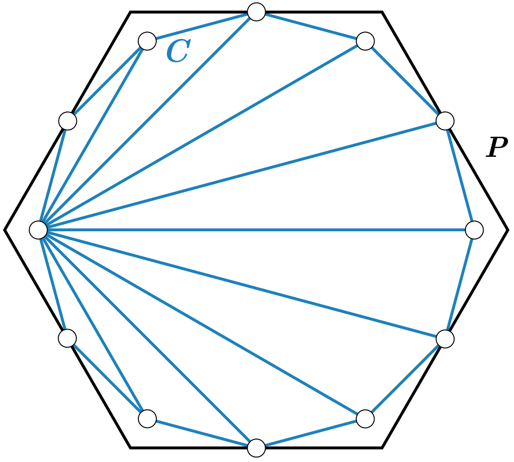
- There can be at most $2n$ guards on the boundary of C.
- A triangulation of C creates $2n - 2$ triangles, in which there can only be one guard (triangle lemma)



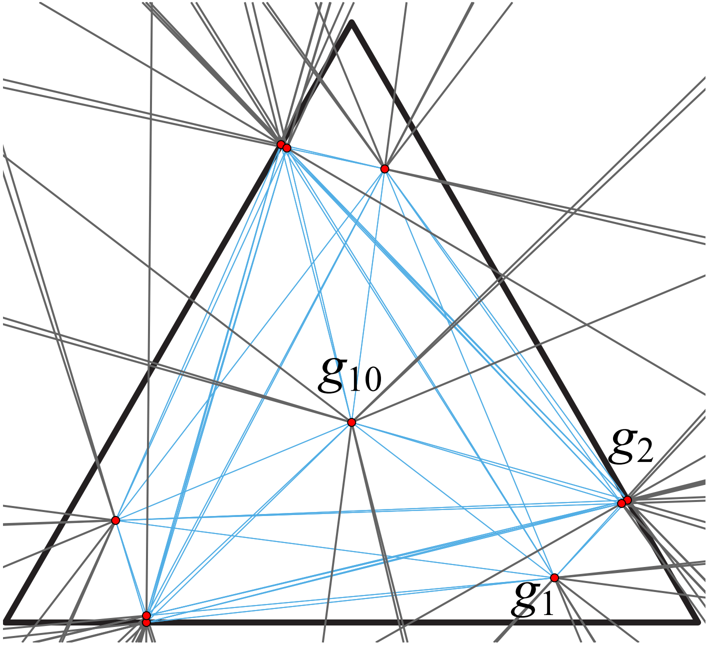
THEOREM 5

The number of guards g that can be placed in a convex n-gon P so that no two dark rays intersect inside is at most $g = 4n - 2$.

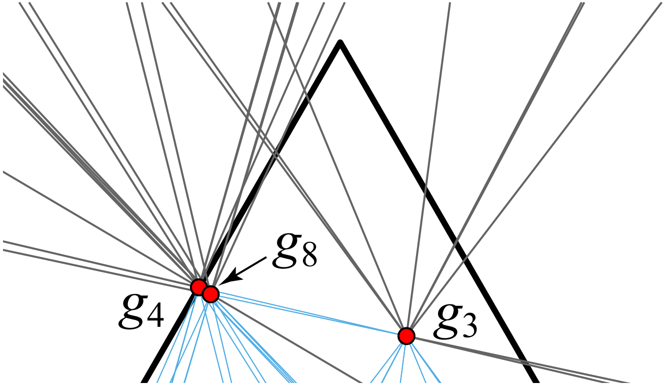
- C gives us $2n$ guards, and its triangulation gives us another $2n - 2$ guards
  - $\Rightarrow 2n + (2n - 2) = 4n - 2$



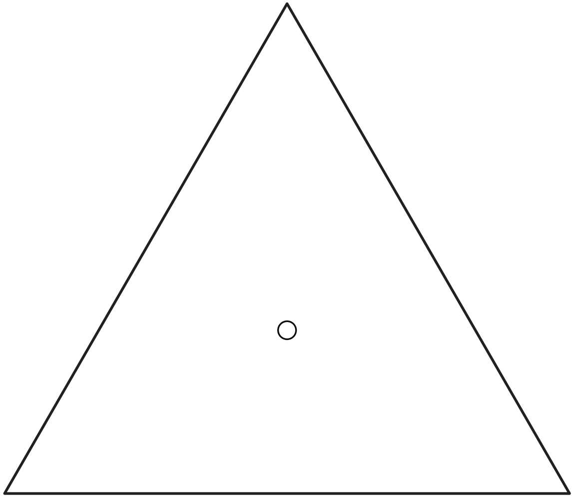
$4n - 2$ Guards in a Triangle



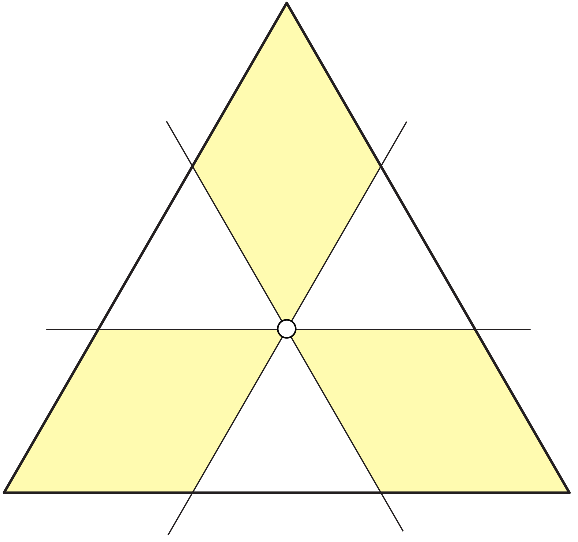
$4n - 2$ Guards in a Triangle



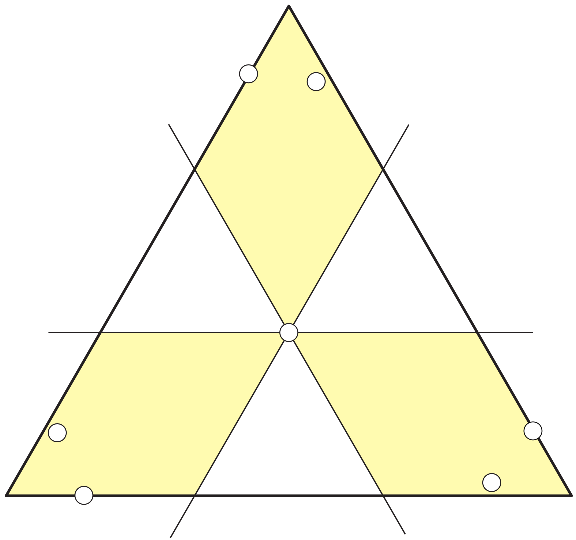
$4n - 2$ Guard Strategy - Triangles



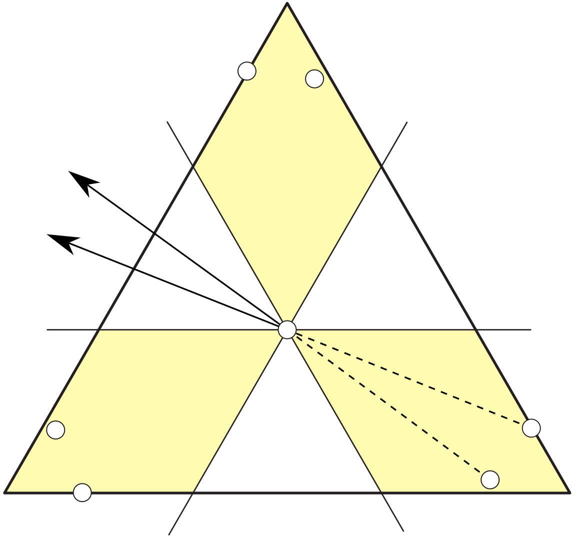
$4n - 2$ Guard Strategy - Triangles



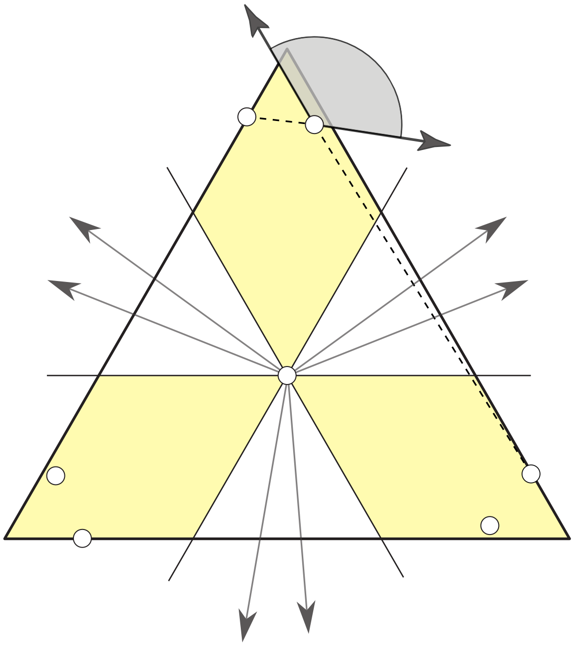
$4n - 2$ Guard Strategy - Triangles



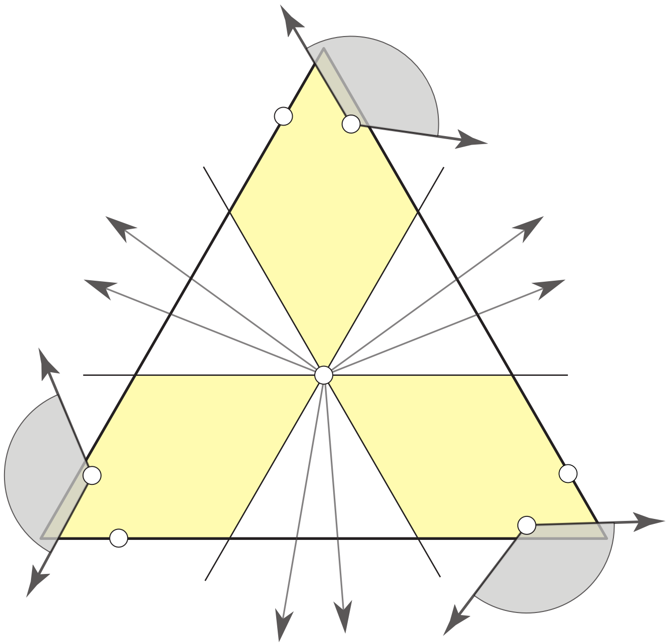
$4n - 2$ Guard Strategy - Triangles



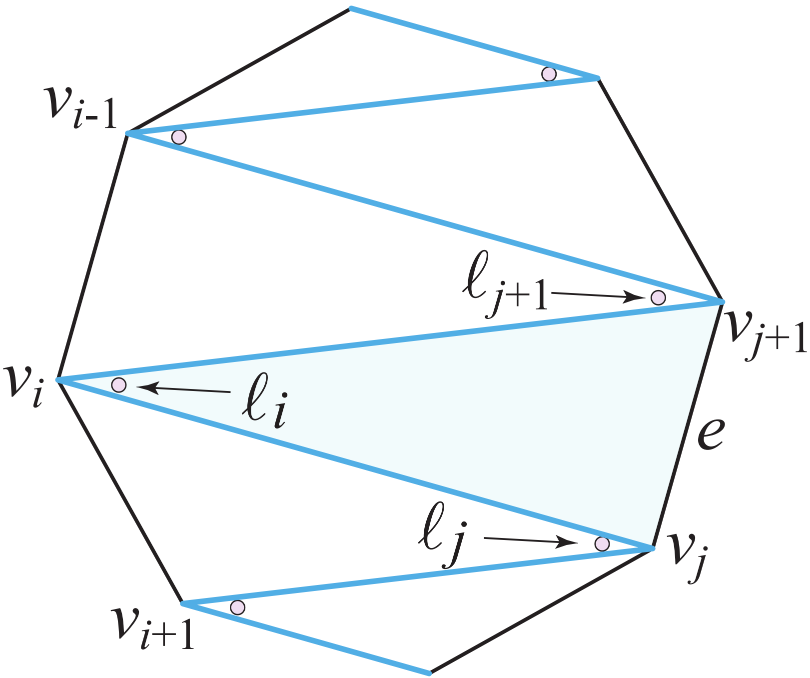
$4n - 2$ Guard Strategy - Triangles



$4n - 2$ Guard Strategy - Triangles



$4n - 2$ Guard Strategy

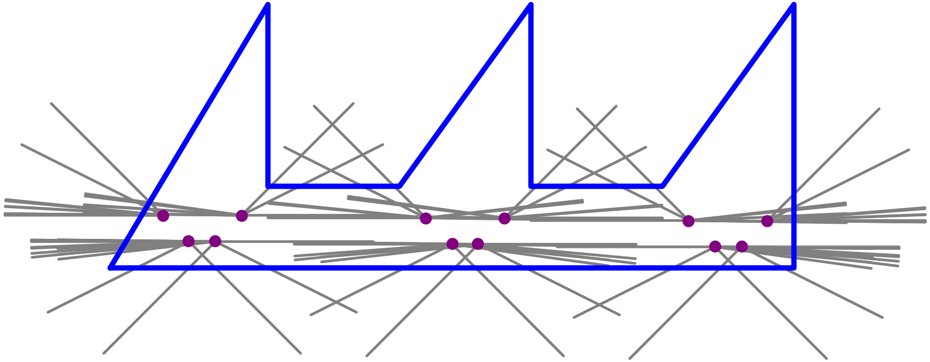


Theorem 6

It is possible to place $4n - 2$ guards in a convex n-gon P so that all dark-ray intersections lie strictly exterior to P.

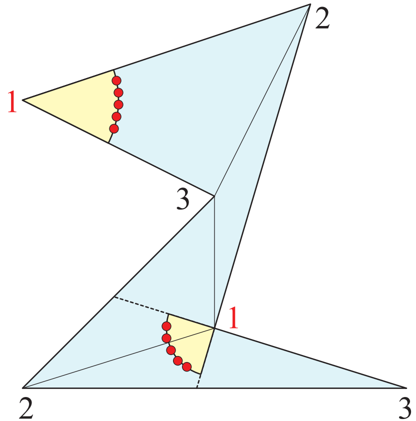
Simple Polygons

- To cover a simple polygon of n vertices to depth k, $g = k\lfloor n/3 \rfloor$ guards are sometimes necessary

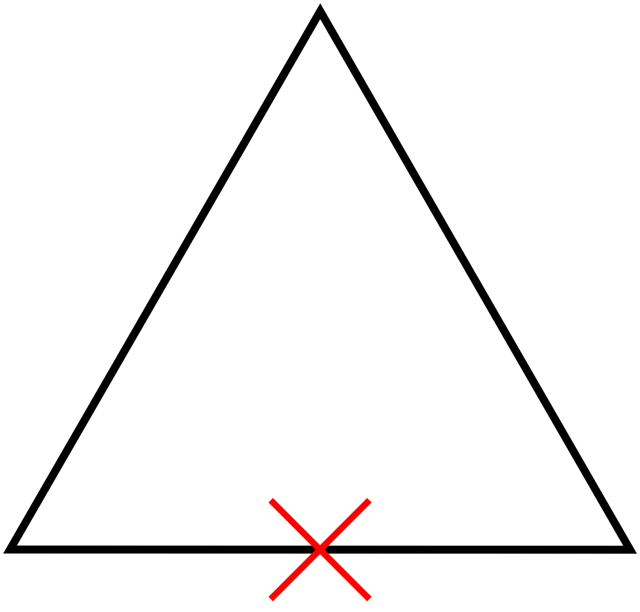


SIMPLE POLYGONS

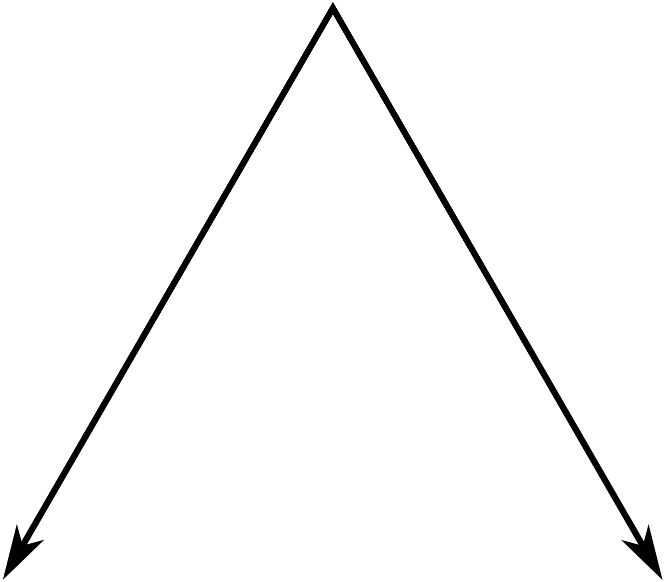
- $g = (k+2)\lfloor n/3 \rfloor$ guards always suffice.



10 Guards in a Wedge

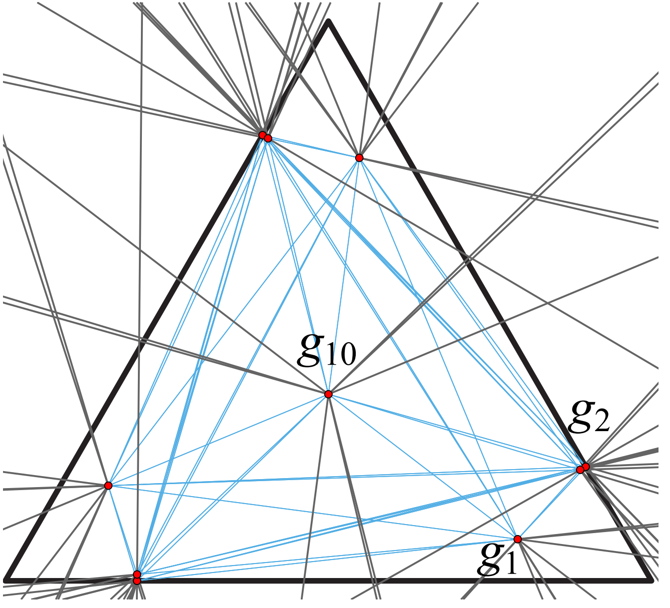


10 Guards in a Wedge



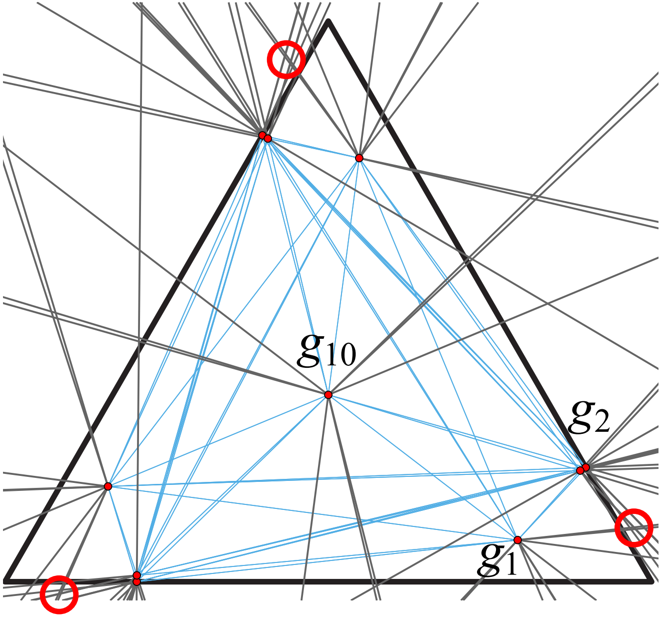
10 Guards in a Wedge

- Our triangle construction does not work in a wedge



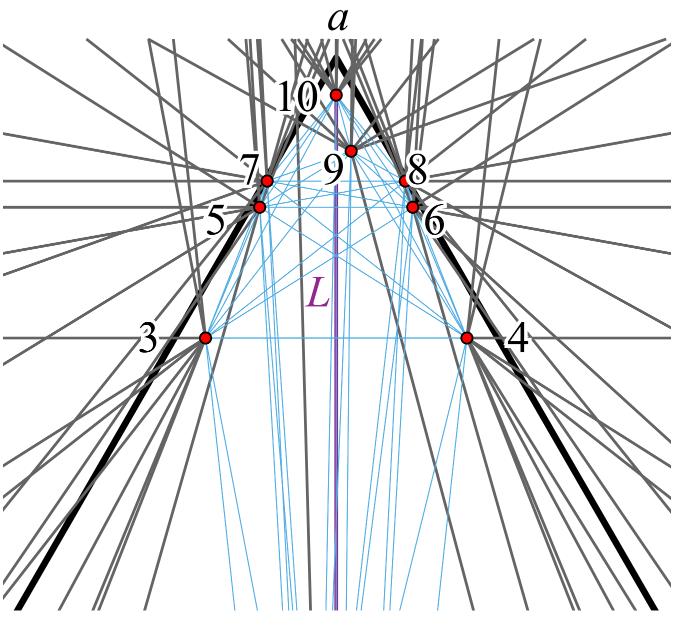
10 GUARDS IN A WEDGE

- Our triangle construction does not work in a wedge



10 Guards in a Wedge

- We need something else



Open Problems

1. **Investigate bounds or the complexity (NP-hard?) of placing points in a simple polygon so that no two dark rays intersect.**

Open Problems

1. Investigate bounds or the complexity (NP-hard?) of placing points in a simple polygon so that no two dark rays intersect.
2. **Close the simple polygon gap. (**$k\lfloor n/3 \rfloor$ **are necessary but** $(k+2)\lfloor n/3 \rfloor$ **are sufficient.)**

Open Problems

1. Investigate bounds or the complexity (NP-hard?) of placing points in a simple polygon so that no two dark rays intersect.
2. Close the simple polygon gap. ($k\lfloor n/3 \rfloor$ are necessary but $(k+2)\lfloor n/3 \rfloor$ are sufficient.)
3. **Can the tight bound for a wedge be generalized to tight bounds for unbounded convex polygons with two rays joined by a chain of $n-1$ vertices and $n-2$ edges?**

Thank you!

Any questions?