Geometric Graphs with Unbounded Flip-Width

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Graph width: a maze of equivalent definitions

Bounded treewidth $\Longleftrightarrow$

Hierarchical clustering of edges by vertex separators of size $O(1)$

No large grid minor

Subgraph of chordal graph with no large cliques

Tree decomposition with no large bags

$O(1)$ cops can win "cops with helicopters" pursuit-evasion game

No "bramble", touching subgraphs with high hitting number

No "haven" assigning "large component" to small vertex deletions

No "tangle" assigning "large side" to small vertex separators

Graph width: a maze of different parameters

Treedepth, shrubdepth, etc: star-like graphs

Bandwidth, cutwidth, pathwidth, etc: path-like graphs

Treewidth, branchwidth, carving width, etc: tree-like graphs

Bounded expansion, polynomial expansion: separator theorems

Row treewidth, row pathwidth, etc: grid-like product structure

clique-width, rank-width, matching-width, twin-width, flip-width, monadic dependence, etc:

generalizations to well-structured dense graphs

Motivation: Algorithms from logical descriptions

Logic of graphs:

- Variables are vertices, edges, or sets of vertices or edges
- Predicates are equality, membership, incidence, or adjacency

Example: Is there a universal vertex? $\exists v \forall w (v \neq w \Rightarrow v \sim w)$

Messier example: Is there a Hamiltonian cycle?

- Does there exist a set C of edges, such that
- Every proper subset X of vertices has an edge in C with exactly one endpoint in X ($\Rightarrow$ C connects the graph), and
- Every vertex has exactly two incident edges in C?

Motivation: Algorithms from logical descriptions

Model checking: Is this formula true of this graph?

Fixed-parameter tractable for many combinations of variable type and graph width:

- Formulas with sets of vertices or edges and bounded tree-width
- Sets of vertices (but not edges) and bounded clique-width
- Individual vertices (but not sets) and nowhere dense, bounded twin-width or beyond

Conjectured "or beyond": flip-width

Width from pursuit–evasion games

Cops with helicopters occupy vertices, trying to catch a robber moving on graph paths
Each turn:

- Cops announce where they will move
- Robber moves through unoccupied vertices
- Cops move as announced

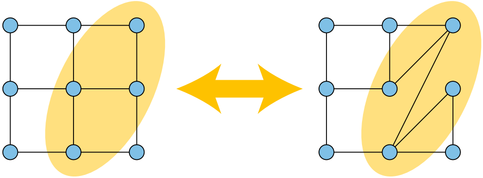
$O(1)$ cops catch unlimited-speed robber $\Rightarrow$ bounded treewidth

$O_s(1)$ cops catch speed-s robber (limited to paths of $\leq s$ vertices)
$\Rightarrow$ nowhere dense

Flip-width

Similar pursuit-evasion game with more powerful cops

Instead of occupying one vertex, a cop can flip a subset of vertices, replacing edges by non-edges and vice versa in that subset



Each turn:

- Cops announce which subsets they will flip next
- Robber moves in the current flipped graph
- Cops undo their current flips and perform the announced flips

$O_s(1)$ cops catch speed-s robber (by leaving robber at an isolated vertex) $\Rightarrow$ bounded flip-width

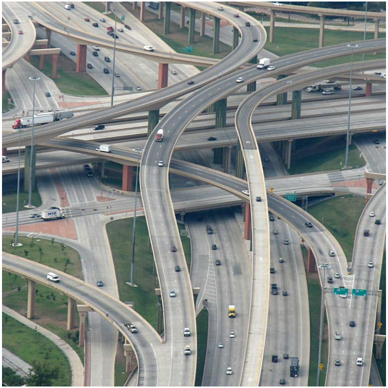
[Toruńczyk 2023]

How to prove unbounded flip-width

Use a special subgraph called an interchange to find a winning strategy for the robber in the flip-width game

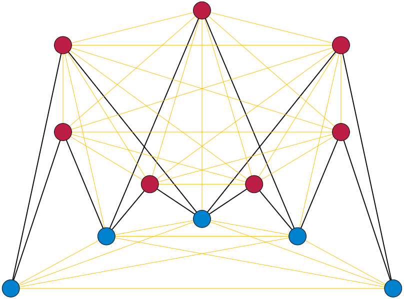
Our main result: Many dense geometric graphs contain interchanges

Therefore they do not have bounded flip-width, or any other width encompassing twin-width, clique-width, tree-width, nowhere density, etc



Definition of an interchange

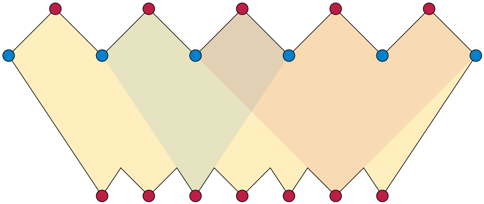
Intuitively: like a subdivision of a complete graph



Interchange of order *n* contains:

- Ordered sequence of *n* "lane" vertices (blue, left-to-right)
- "Ramp" vertices connecting pairs of lanes (red)
- Each ramp cannot be adjacent to lanes outside the interval between the two lanes it connects
- All other edges are optional (yellow)

Example: Visibility graphs of simple polygons



Place blue lane vertices on a horizontal line

Place red ramps on two parallel lines:
ramps for consecutive lanes above, others below

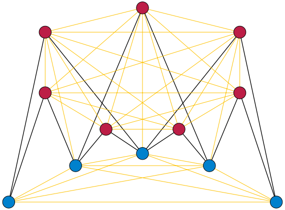
Form a triangle connecting each ramp to its two lanes, with sides
that block visibility to other lanes outside the triangle

Polygon = union of triangles with holes filled in

(Some vertices are neither ramps nor lanes: not a problem.)

How interchanges allow robber to escape

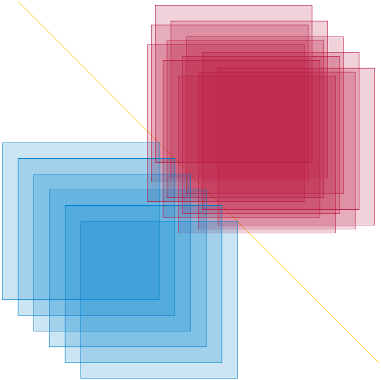
Main idea: move to a lane that, after the cops move, will have many two-edge paths to other lanes



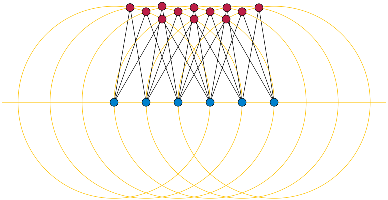
More details:

- c cops $\Rightarrow$ 2^c equivalence classes of lanes flipped the same way
- Each two triples of equivalent lanes have ≥ 1 two-edge path
- Many lanes $\Rightarrow$ many triples, each with many two-edge paths
- Enough two-edge paths from the current vertex $\Rightarrow$ some triple of equivalent vertices can all be reached $\Rightarrow$ one of them will have many paths in the next move

More geometric graphs with unbounded flip-width



Intersection graphs of
axis-aligned unit squares



Unit distance graphs

Unit disk graphs

More geometric graphs with unbounded flip-width

Interval graphs

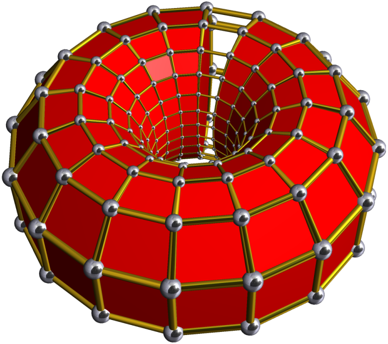
Permutation graphs

Circle graphs

Intersection graphs of axis-aligned line segments

More geometric graphs with unbounded flip-width

3d Delaunay triangulations and 4d convex polytopes



Augment $n \times n$ toroidal grid by
n points on central axis
n points on center circle of torus

Conclusions

Most sparse geometric graphs are known
to have bounded width of some form

(E.g. all planar geometric graphs are nowhere dense
and have bounded twin-width.)

But many standard families of dense geometric graphs
have unbounded flip-width

This is more general than other standard dense width parameters
(clique-width and twin-width) $\Rightarrow$ these widths also unbounded

We must look beyond these width parameters to apply graph
structure in geometric algorithms involving these graphs

References and image credits

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