Lower Bounds for the Thickness and the Total Number of Edge Crossings of Euclidean Minimum Weight Laman Graphs and (2,2)-Tight Graphs

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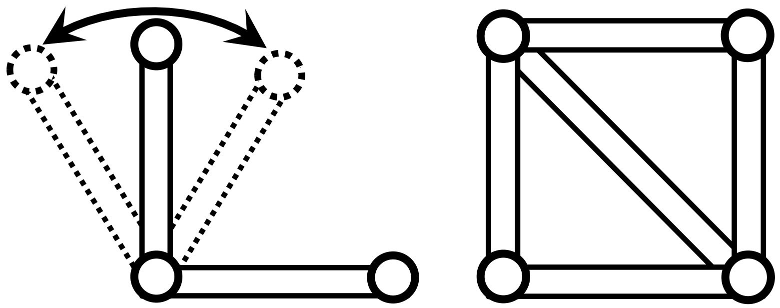
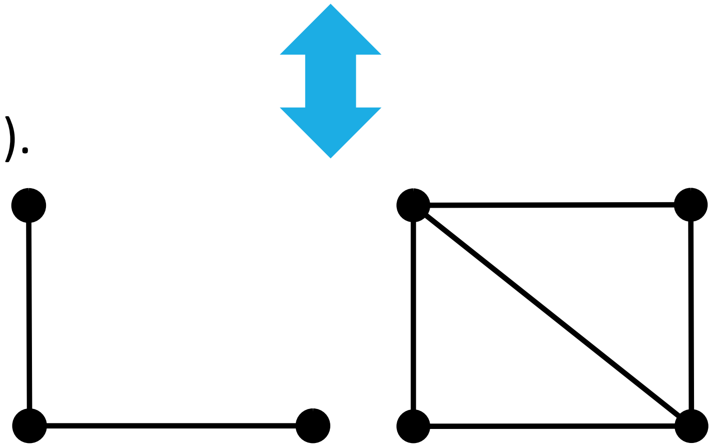
Bar-joint frameworks

- A bar-joint framework is studied in combinatorial rigidity theory.
  - It consists of rigid bars and rotatable joints.
  - It corresponds to the geometric graph
    by regarding each joint as vertex on the plane
    and each bar as straight-line edge.
- Our focus is whether the framework is rigid or flexible.
  It depends on only the combinatorial characterization
  if point set is generic (algebraically independent over the rational field).

  - We assume a <u>*semi-generic*</u> point set.
    - ◆ No three points are colinear
    - ◆ All interpoint distances are distinct



A bar-joint framework



Geometric graph

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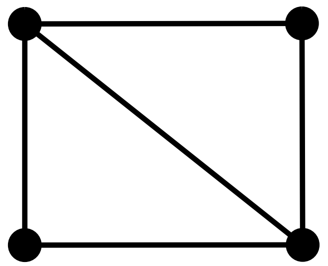
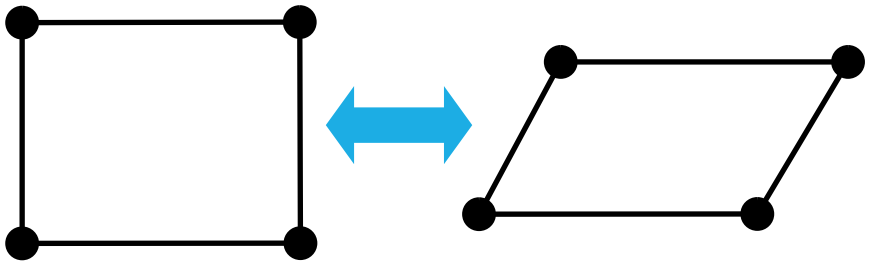


rigid



flexible

Laman Graph [Laman 1970]

- Laman graph $\Leftrightarrow$ minimally rigid on the plane.

Laman graph

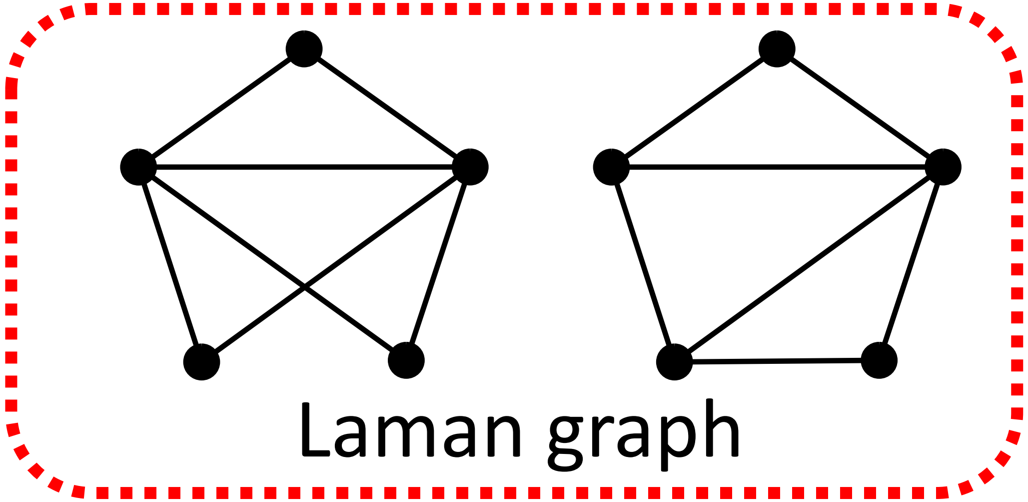
Graph G satisfies following conditions

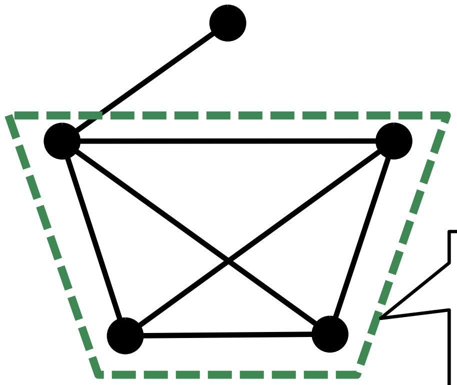
- For graph $G = (V, E)$

  $|E(G)| = 2|V(G)| - 3$

- For any subgraph H of G with $E(H) \neq \emptyset$

  $|E(H)| \leq 2|V(H)| - 3$





Violates condition for subgraph H

$|E(H)| > 2|V(H)| - 3$

$6 > 2 \cdot 4 - 3 = 5$

(k, ℓ)-tight graphs

Generalization

Laman graph

Graph G satisfies following conditions

- For graph $G = (V, E)$

  $|E(G)| = \mathbf{2}|V(G)| - \mathbf{3}$

- For any subgraph H of G with $E(H) \neq \emptyset$

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(k, ℓ)-tight graph

Graph G satisfies following conditions

- For graph $G = (V, E)$

  $|E(G)| = \boldsymbol{k}|V(G)| - \ell$

- For any subgraph H of G with $E(H) \neq \emptyset$

  $|E(H)| \leq \boldsymbol{k}|V(H)| - \ell$

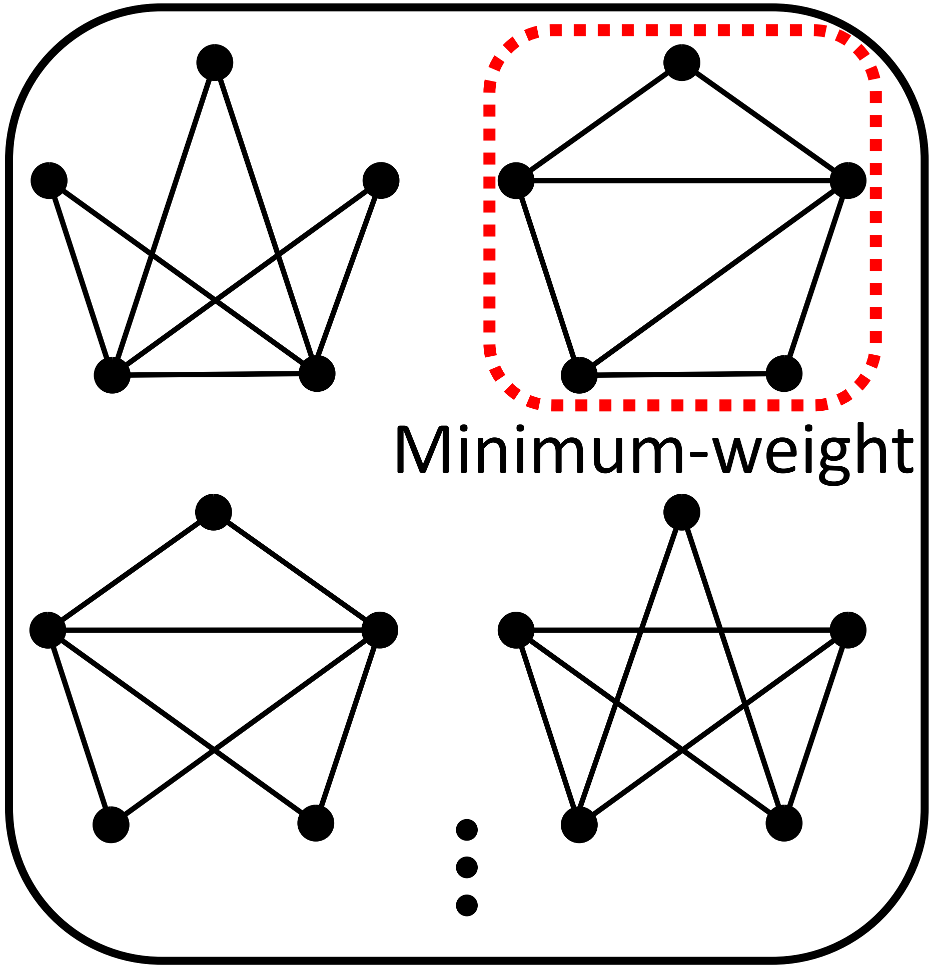
- Case $k = 2, \ell = 3$: (2,3)-tight graphs = Laman graphs.
- Case $k = 1, \ell = 1$: (1,1)-tight graphs = spanning trees.

Euclidean minimum-weight Laman graphs

■ **MLG(P): Minimum**-weight Laman Graph on P

- The Laman graph on P with the **minimum total edge-length** over all Laman graph on P.
- Given a semi-generic point set P, we can **uniquely** obtain MLG(P) by a greedy algorithm[1] (**Polynomial time**).
- Depending on the given point set P, MLG(P) may have **edge crossings**.

Laman graphs on P



[1] A. Lee and I. Streinu. Pebble game algorithms and sparse graphs. Discrete Mathematics , 2008

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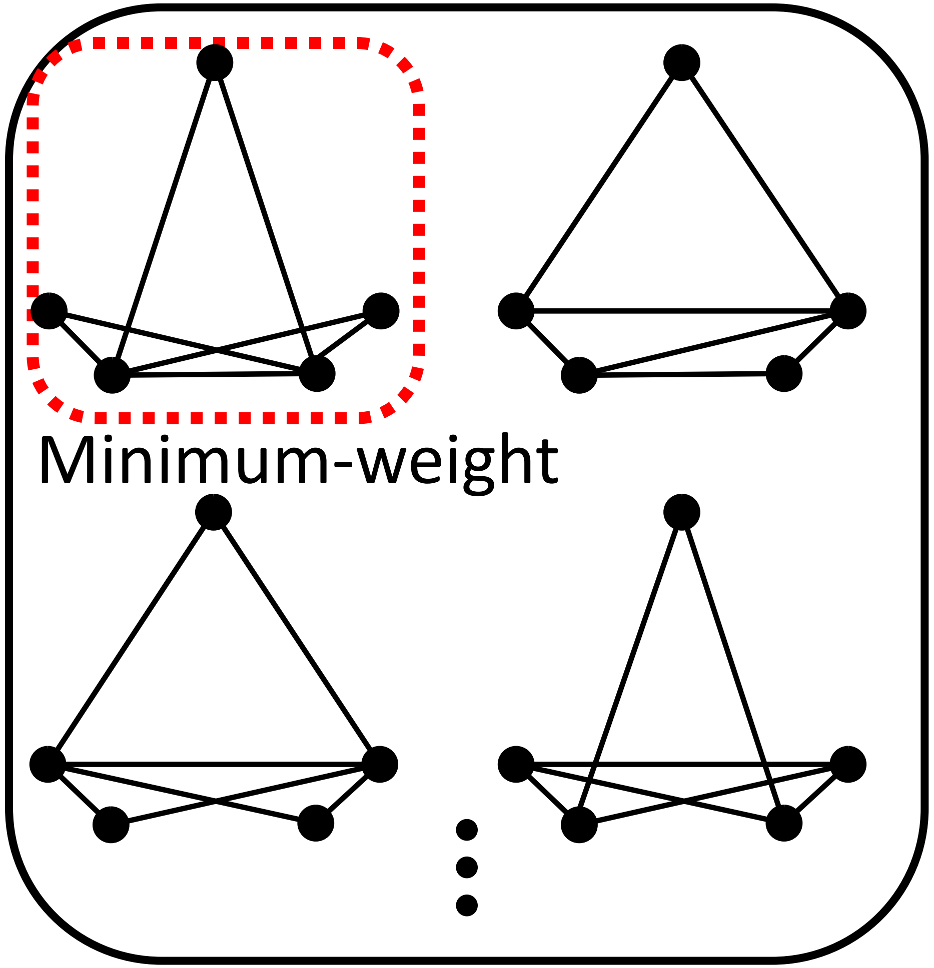

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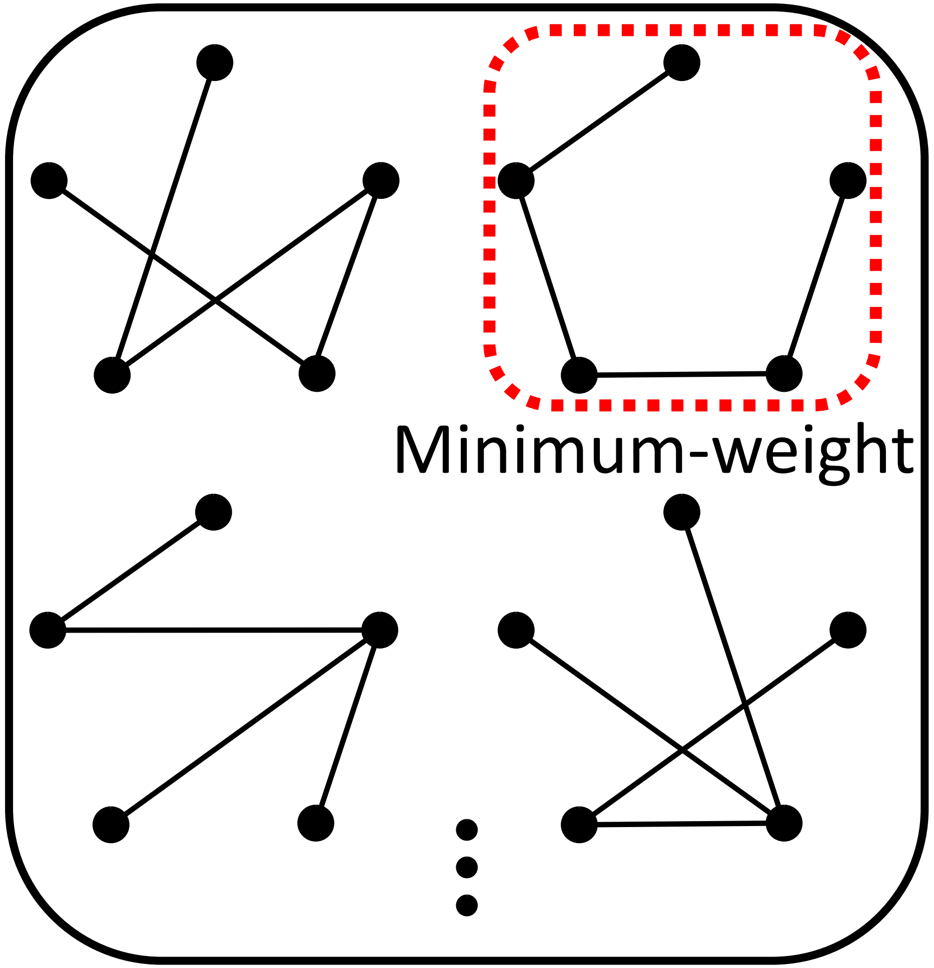
Euclidean minimum-weight (k, ℓ)-tight graph

- **(k, ℓ)-MTG(P): Minimum**-weight (k, ℓ)-tight graph on P.



- The (k, ℓ)-tight graph on P with the **minimum total edge-length** over all (k, ℓ)-tight graph on P.
- Given a semi-generic point set P, we can **uniquely** obtain (k, ℓ)-MTG(P) by a greedy algorithm[1] (**Polynomial time**).
- $(1,1)$-MTG(P) has **no edge crossings** for any P. (Minimum-weight spanning tree)
- Depending on the given point set P, $(2,2)$-MTG(P) may have **edge crossings**.

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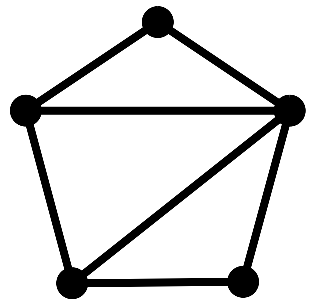
Our focus on edge crossings

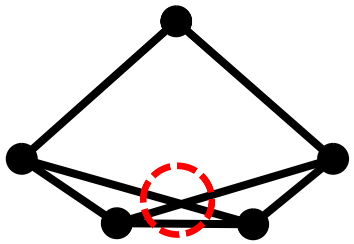
1. The number of total edge crossings (called **crossing number**)
   - We show lower bounds for the **maximum crossing number** of MLG(P) and $(2,2)$-MTG(P).
   (We explore a semi-generic point set P such that **maximize crossing number**.)

P_1



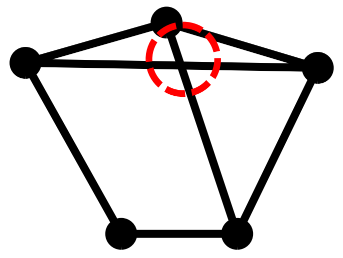
MLG(P_1):
Crossing number is 0

P_3



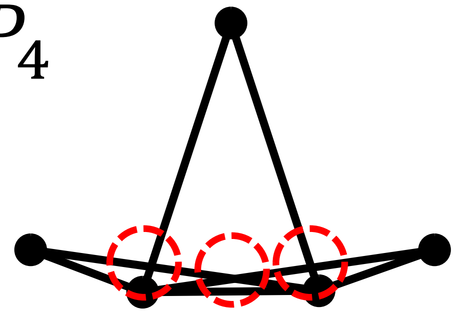
MLG(P_3):
Crossing number is 1

P_2



MLG(P_2):
Crossing number is 1

P_4



MLG(P_4):
Crossing number is 3

Our focus on edge crossings

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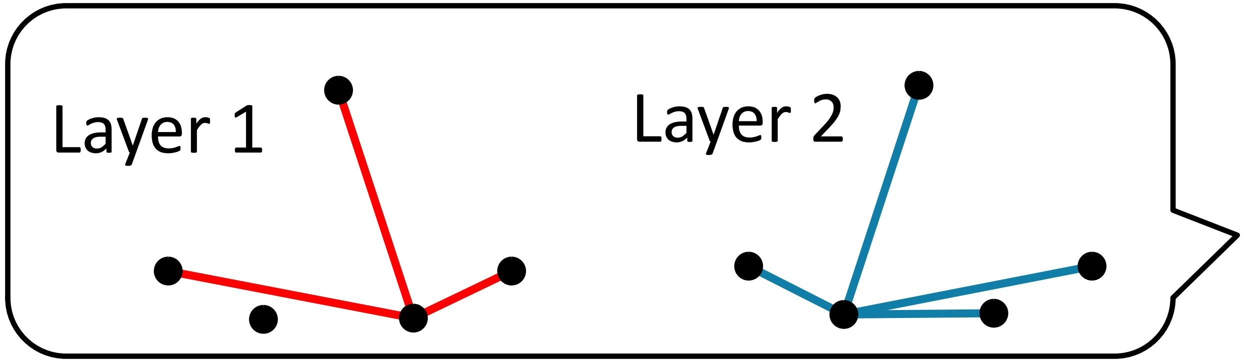
   - We show lower bounds for the **maximum crossing number** of MLG(P) and $(2,2)$-MTG(P).
   (We explore a semi-generic point set P such that **maximize crossing number**.)
2. The geometric thickness
   - We show lower bounds for the **maximum geometric thickness** of MLG(P) and $(2,2)$-MTG(P).
   (We explore a semi-generic point set P such that **maximize geometric thickness**.)

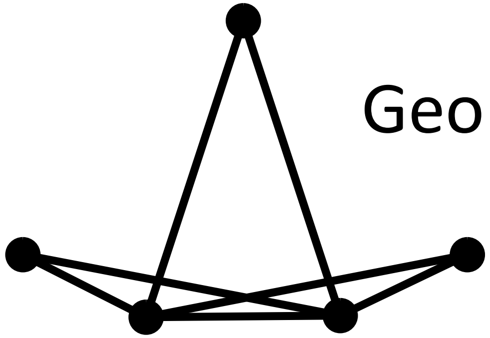
Our focus on edge crossings

1. The number of total edge crossings (called **crossing number**)

> The smallest number of layers necessary to partition the edge set of $G(P)$ into layers so that no layers have edge crossing.

2. The geometric thickness
   - We show lower bounds for the **maximum geometric thickness** of MLG(P) and $(2,2)$-MTG(P).
   (We explore a semi-generic point set P such that **maximize geometric thickness**.)





Geometric thickness is 2

Comparison of previous result and our result

The maximum number		$(2,3)$-MTG(P) (MLG(P))		$(2,2)$-MTG(P)	
		Previous result [1]	**Our result**	Previous result [2]	**Our result**
Crossing number	Upper bound	$2.5\|P\| - 5$	-	$22\|P\| - 22$	-
	Lower bound	$(1.25 - \epsilon)\|P\|$	$\boldsymbol{(1.42 - \epsilon)\|P\|}$	-	$\boldsymbol{(1.83 - \epsilon)\|P\|}$
Geometric thickness	upper bound	4	-	-	-
	Lower bound	2	**3**	-	**3**

[1] Y. Kobayashi et. al., Improving upper and lower bounds for the total number of edge crossings of Euclidean minimum weight Laman graphs., COCOON, 2021
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Improvement

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Geometric thickness					

This upper bound is obtained by substituting $k := 2, \ell := 2$ for the upper bound $(6k^2 + 4k - 10) \cdot \frac{|E|}{2}$ for general k and ℓ.

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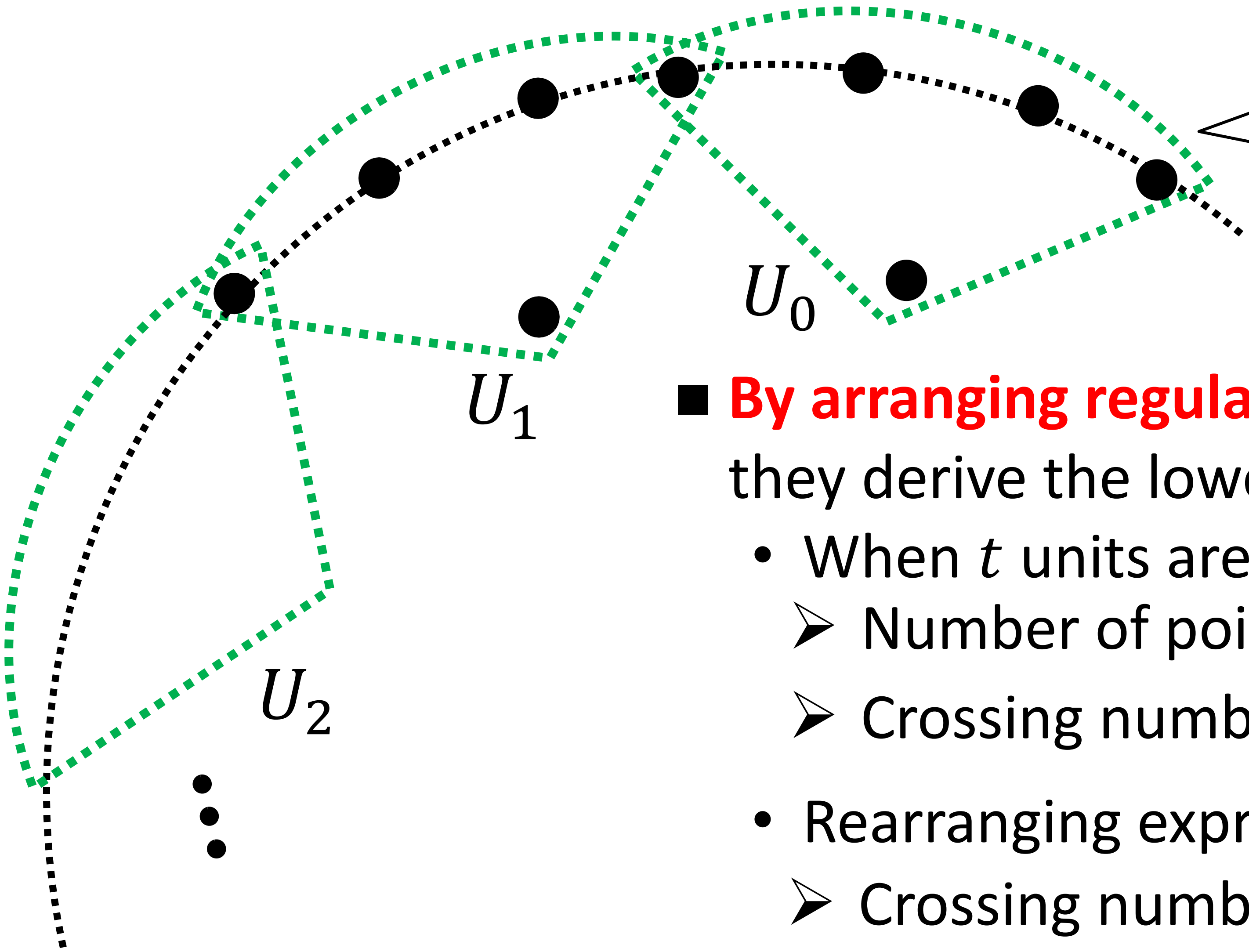
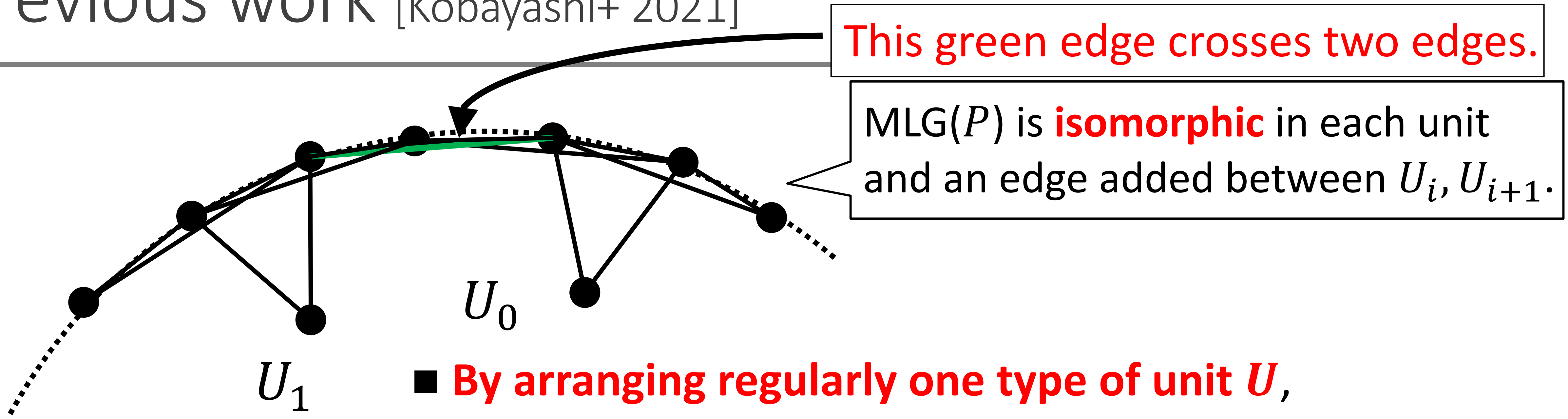
Previous work [Kobayashi+ 2021]



A unit U Consists of 5 points.
Their relative positions are same.

- **By arranging regularly one type of unit $\boldsymbol{U}$**, they derive the lower bound for the crossing number.
  - When t units are arranged:
    - Number of points set: $|\mathrm{P}| = 4t + 1$
    - Crossing number: $5t - 2$
  - Rearranging expression of crossing number
    - Crossing number: $\left(\frac{5}{4} - \frac{13}{16t+4}\right)|P| \geq (1.25 - \epsilon)|P|$

Previous work [Kobayashi+ 2021]



- **By arranging regularly one type of unit U**, they derive the lower bound for the crossing number.
  - When t units are arranged:
    - Number of points set: $|\mathrm{P}| = \mathbf{4}t + 1$
    - Crossing number: $\mathbf{5}t - 2$
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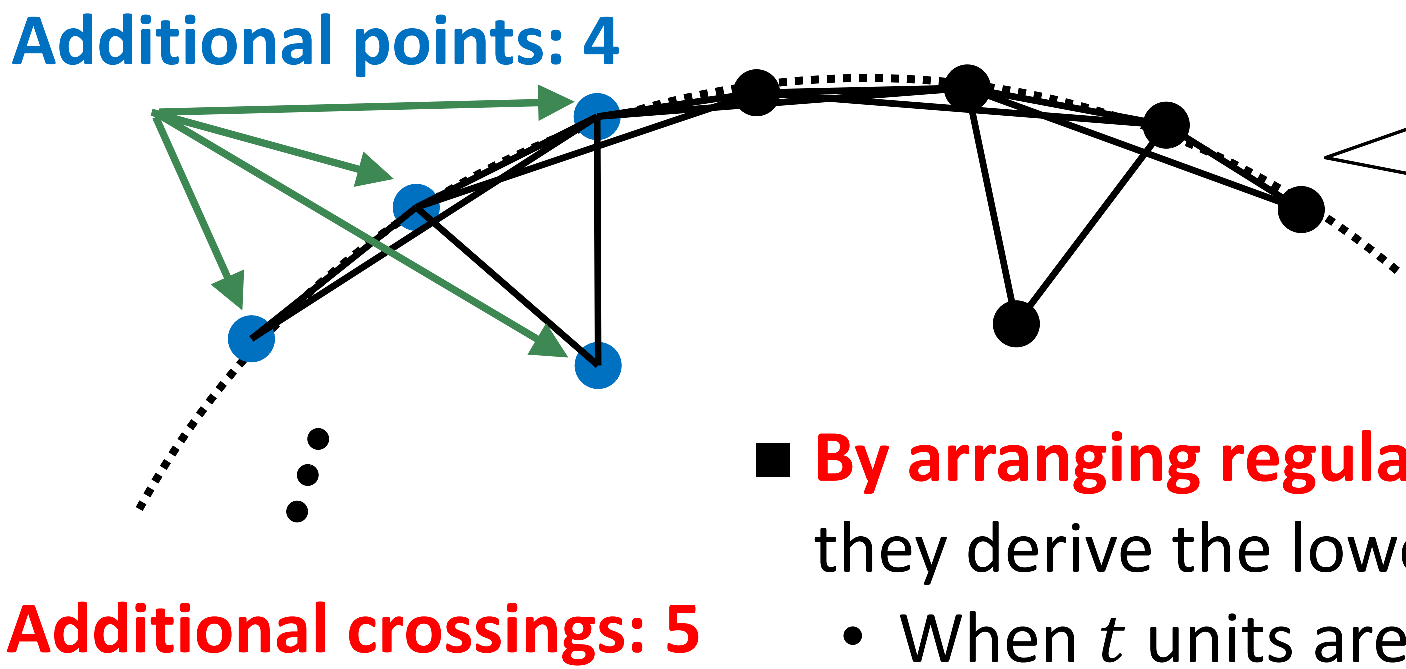

MLG(P) is **isomorphic** in each unit and an edge added between U_i, U_{i+1}.

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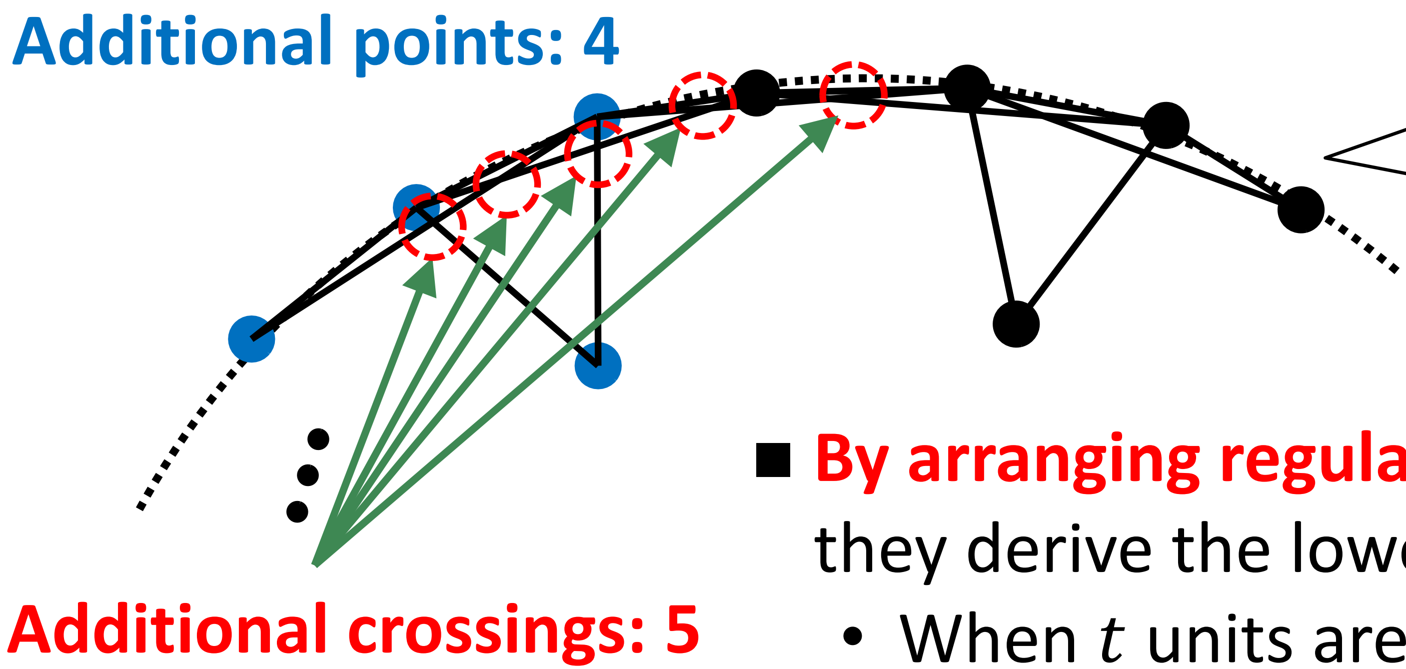

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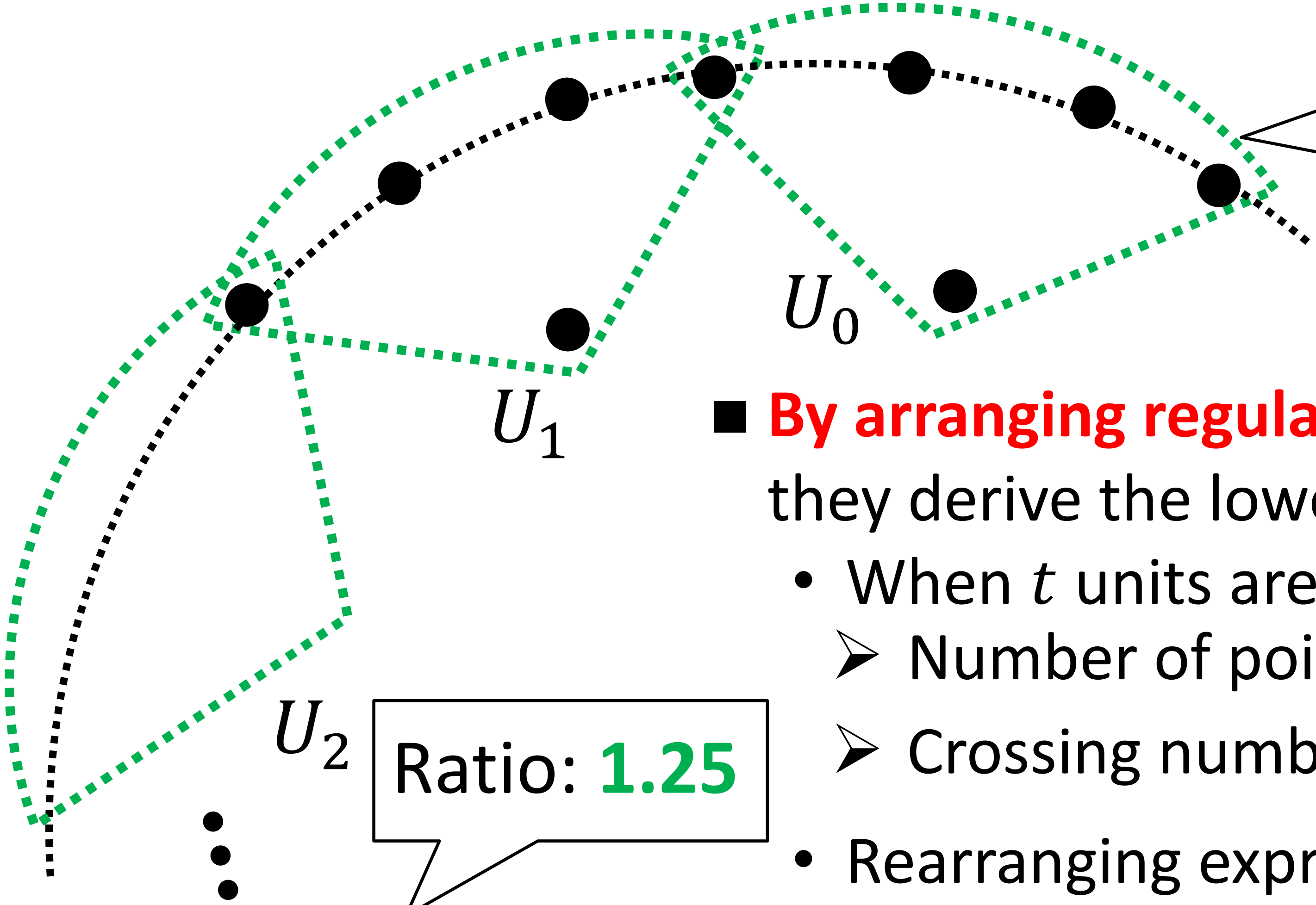


MLG(P) is **isomorphic** in each unit and an edge added between U_i, U_{i+1}.

Ratio: **1.25**

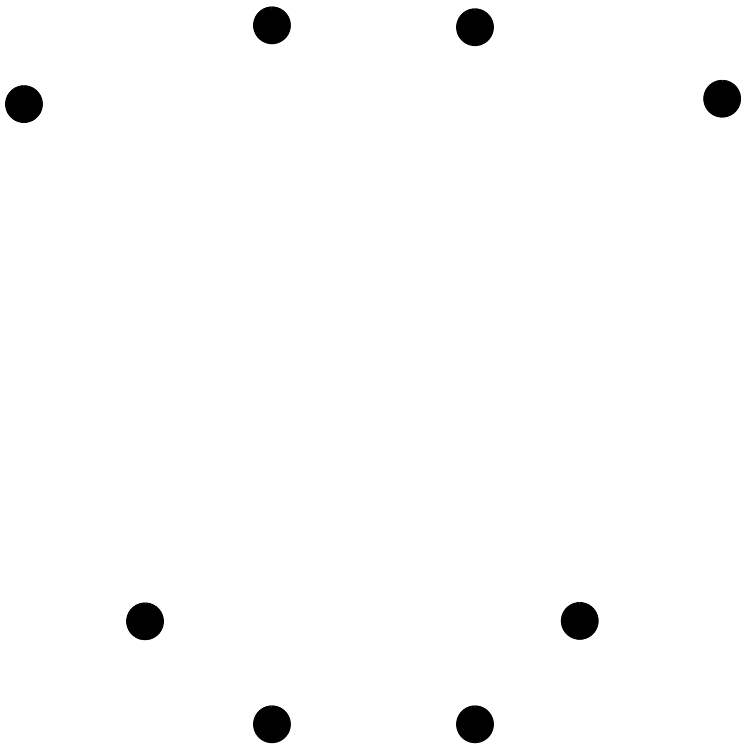
Additional points: 4

Additional crossings: 5

■ **By arranging regularly one type of unit U**, they derive the lower bound for the crossing number.

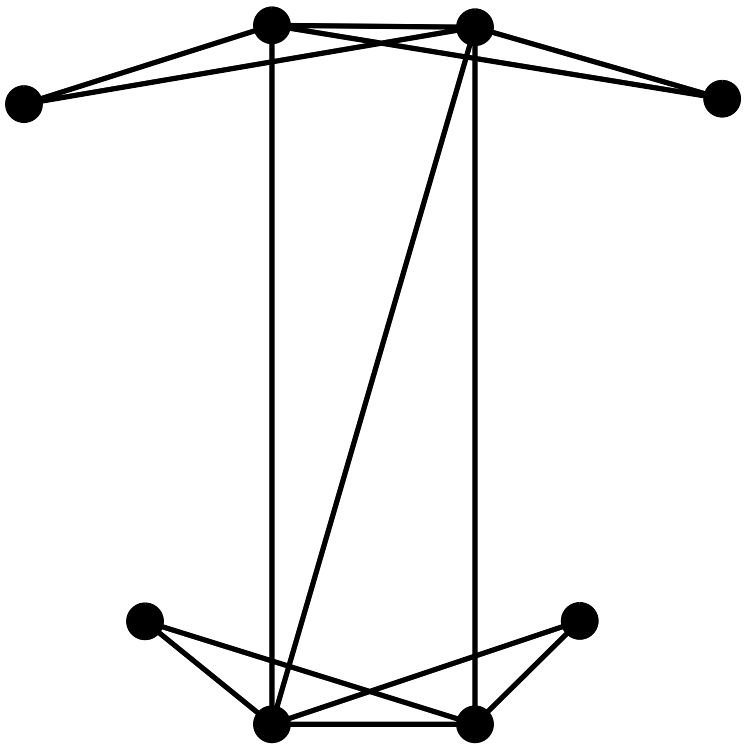
- When t units are arranged:
  - ➢ Number of points set: $|\mathrm{P}| = \mathbf{4}t + 1$
  - ➢ Crossing number: $\mathbf{5}t - 2$
- Rearranging expression of crossing number
  - ➢ Crossing number: $\left(\frac{\mathbf{5}}{\mathbf{4}} - \frac{13}{16t+4}\right)|P| \geq (\mathbf{1.25} - \epsilon)|P|$

Our method for MLG(P)



- We Propose a new unit consisting of 8 points.

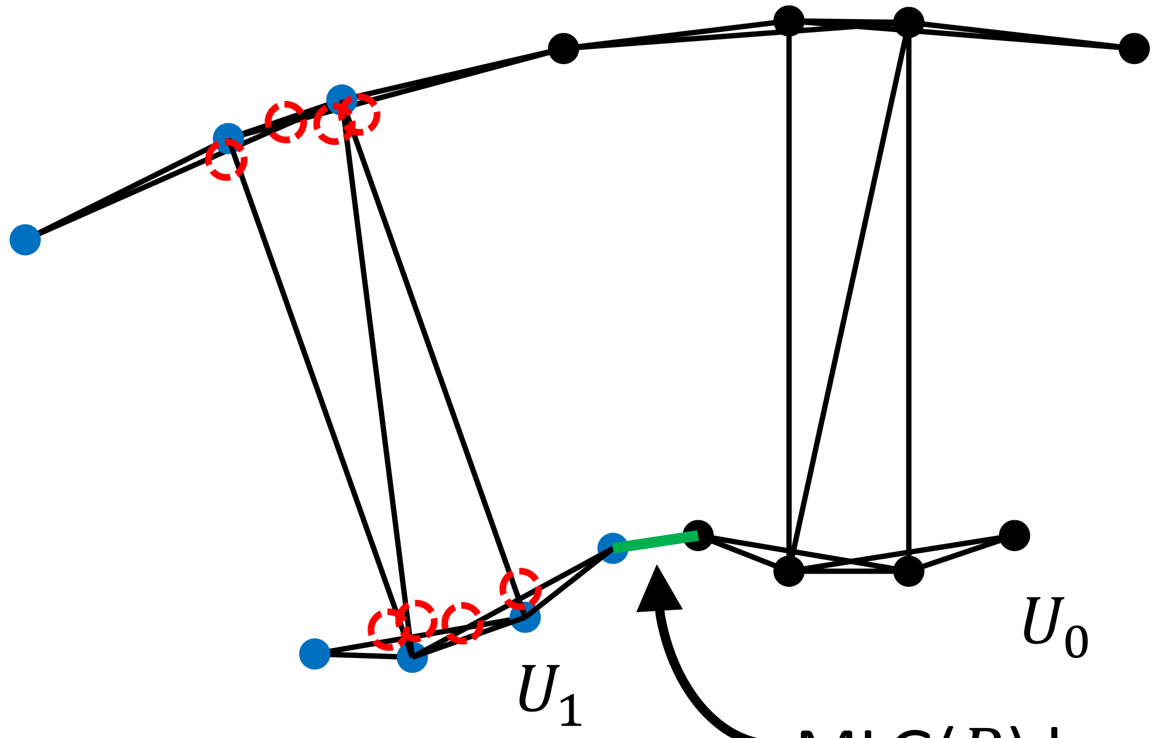
Our method for MLG(P)



MLG(P) of new unit

- We Propose a new unit consisting of 8 points.
  - MLG(P) has 8 edge crossings.

Our method for MLG(P)



- If we apply Kobayashi's method: (Arranging new units **regularly**)
  - There is **no edge crossings** between the neighboring units U_i, U_{i+1}.

MLG(P) has this green edge.
(It doesn’t cross other edges.)

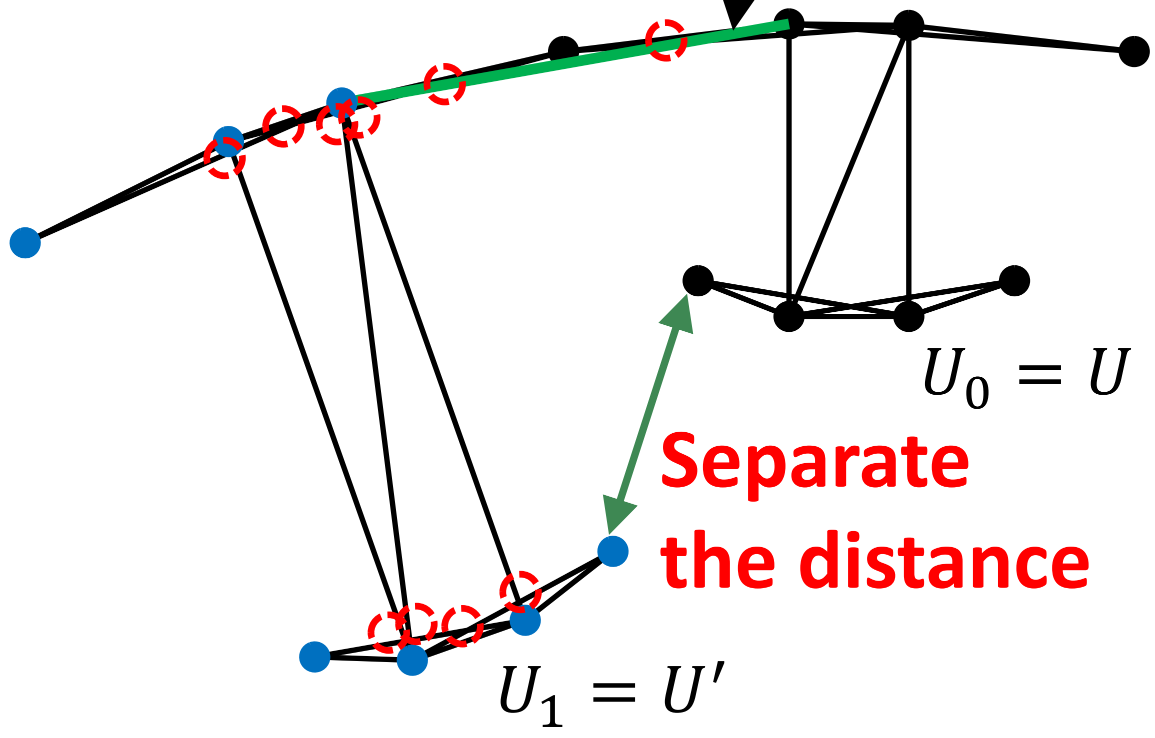
Additional points: 7

Additional crossings: 8

Ratio: $\mathbf{1.14\cdots}$

Our method for MLG(P)

MLG has this green edge.
(It crosses two edges.)



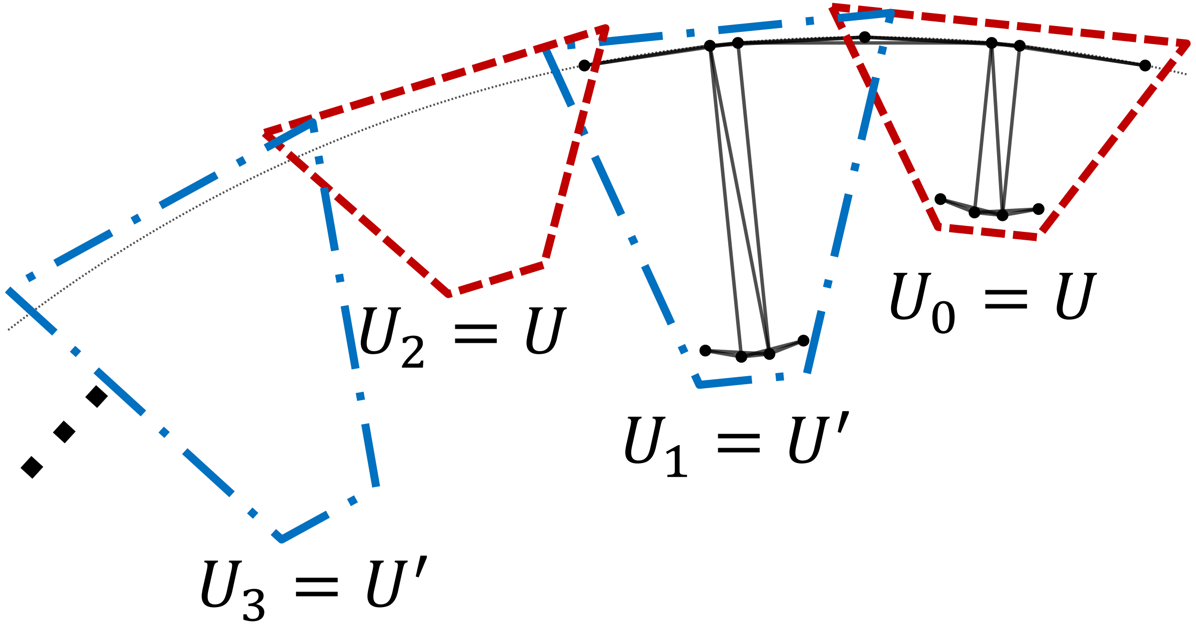
Additional points: 7

Additional crossings: 10

Ratio: $\mathbf{1.42\cdots}$

- We extend the Kobayashi's method to arrange **two types of units U, U'**.
  - Both of these two units U, U' derive the isomorphic MLG.
  - This method will give **two edge crossings** between the neighboring units U_i, U_{i+1}.

Our method for MLG(P)



- We extend the Kobayashi's method to arrange **two types of units U, U'**.
  - When t units are arranged:
    - Number of points set: $|\mathrm{P}| = \mathbf{7}t + 1$
    - Crossing number: $\mathbf{10}t - 2$
  - Rearranging expression of crossing number

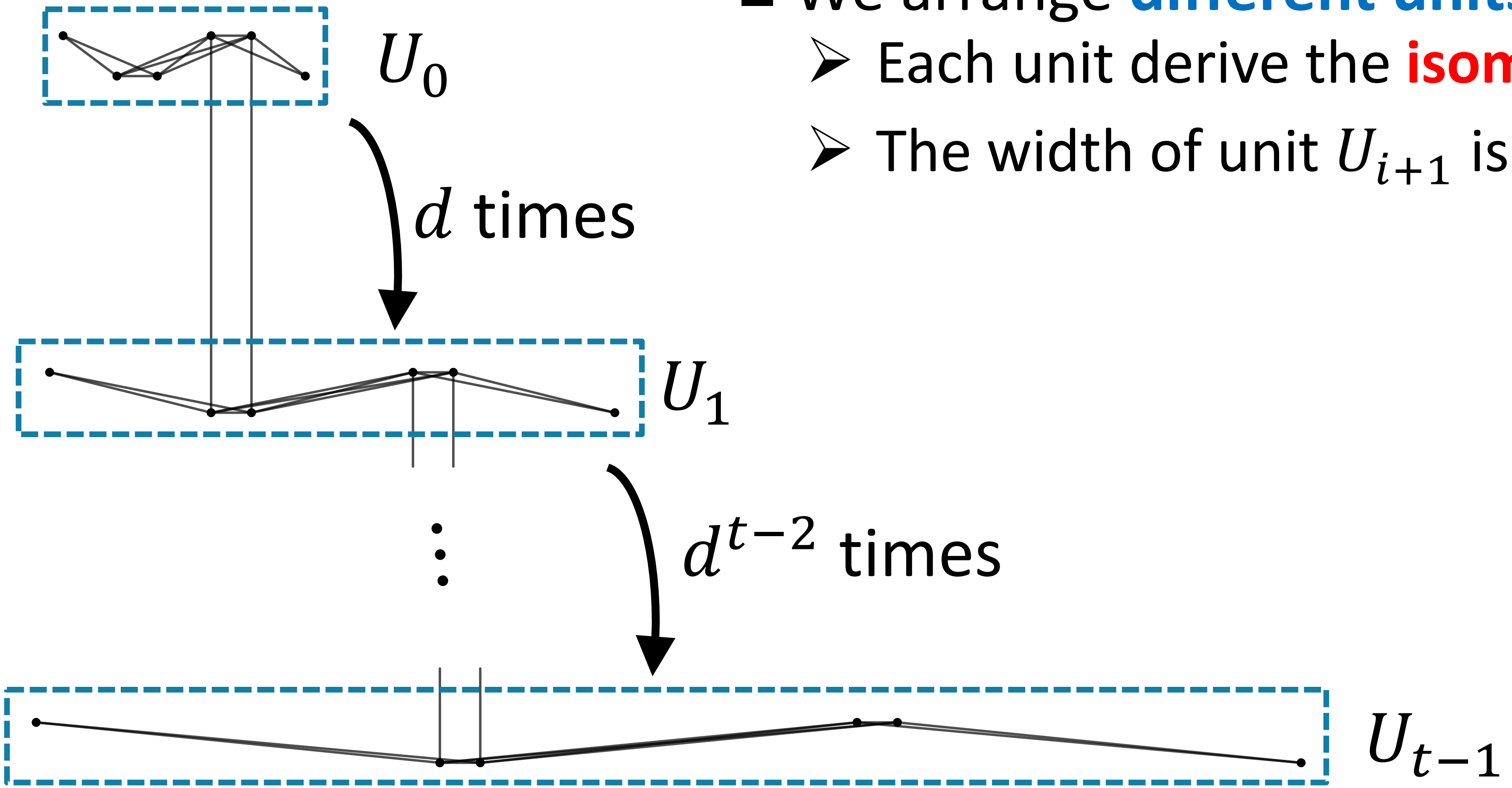
Additional points: 7

Additional crossings: 10

- Crossing number: $\left(\frac{\mathbf{10}}{\mathbf{7}} - \frac{24}{49t+7}\right)|P| \geq (\mathbf{1.42} - \epsilon)|P|$

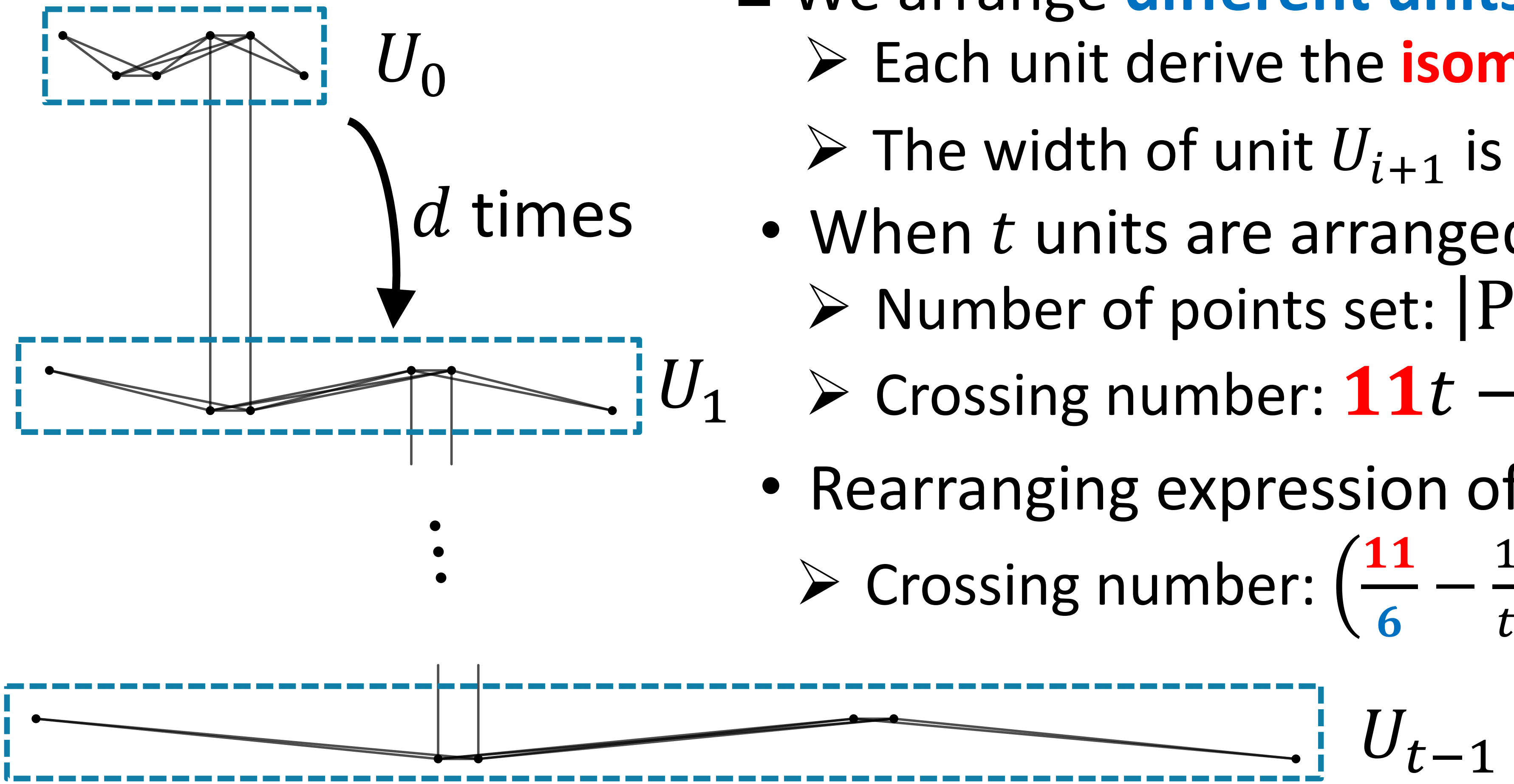
Ratio: $\mathbf{1.42}\cdots$

Our method for $(2,2)$-MTG(P)



- We arrange **different units** consisting of 6 points.
  - Each unit derive the **isomorphic** $(2,2)$-MTG.
  - The width of unit U_{i+1} is d times that of unit U_i.

Our method for $(2,2)$-MTG(P)



- We arrange **different units** consisting of 6 points.
  - Each unit derive the **isomorphic** (2,2)-MTG.
  - The width of unit U_{i+1} is d times that of unit U_i.
- When t units are arranged:
  - Number of points set: $|\mathrm{P}| = \mathbf{6}t$
  - Crossing number: $\mathbf{11}t - 6$
- Rearranging expression of crossing number
  - Crossing number: $\left(\frac{\mathbf{11}}{\mathbf{6}} - \frac{1}{t}\right)|P| \geq (\mathbf{1.83} - \epsilon)|P|$

Additional points: 6

Additional crossings: 11

Ratio: $\mathbf{1.83}\cdots$

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The maximum number		(2,3)-MTG(P) (MLG(P))		(2,2)-MTG(P)	
		Previous result [1]	**Our result**	Previous result [2]	**Our result**
Crossing number	Upper bound	$2.5\|P\| - 5$	-	$22\|P\| - 22$	-
	Lower bound	$(1.25 - \epsilon)\|P\|$	$(\mathbf{1.42} - \boldsymbol{\epsilon})\|\boldsymbol{P}\|$	-	$(\mathbf{1.83} - \boldsymbol{\epsilon})\|\boldsymbol{P}\|$
Geometric thickness	upper bound	4	-	-	-
	Lower bound	2	**3**	-	**3**

[1] Y. Kobayashi et. al., Improving upper and lower bounds for the total number of edge crossings of Euclidean minimum weight Laman graphs., COCOON, 2021
[2] S. Bereg et. al., On the edge crossing properties of Euclidean minimum weight Laman graphs., Computational Geometry, 2016

Comparison of previous result and our result

The maximum number		$(2,3)$-MTG(P) (MLG(P))		$(2,2)$-MTG(P)	
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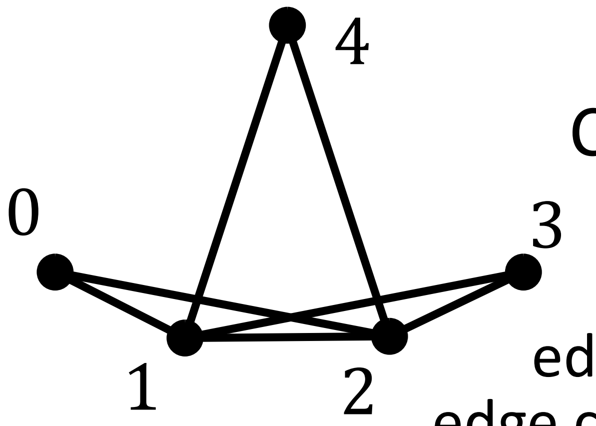
Geometric thickness and edge-crossing graph

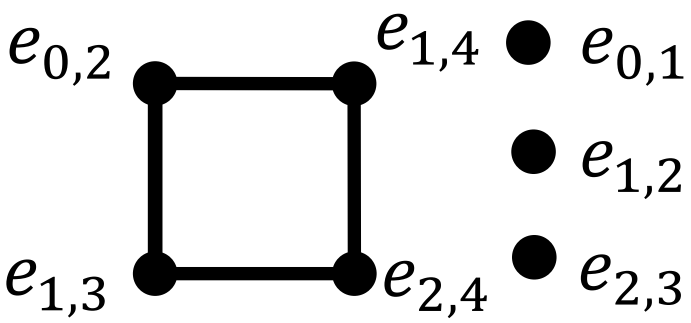
Edge-crossing graph (W, F) for geometric graph $G(P) = (P, E)$

- Each vertex $e \in W$ corresponds to edge $e \in E$.
- Each edge $(e, e') \in F$ corresponds to edge crossing of two edges e and e' of $G(P)$.

Geometric graph $G(P)$



Edge-crossing graph (W, F)



Correspondence

edge (i, j) : vertex $e_{i,j}$

edge crossing (i, j) and (x, y) : edge $(e_{i,j}, e_{x,y})$

Geometric thickness of $G(P)$ = Chromatic number of (W, F)

Geometric thickness and edge-crossing graph

Edge-crossing graph (W, F) for geometric graph $G(P) = (P, E)$

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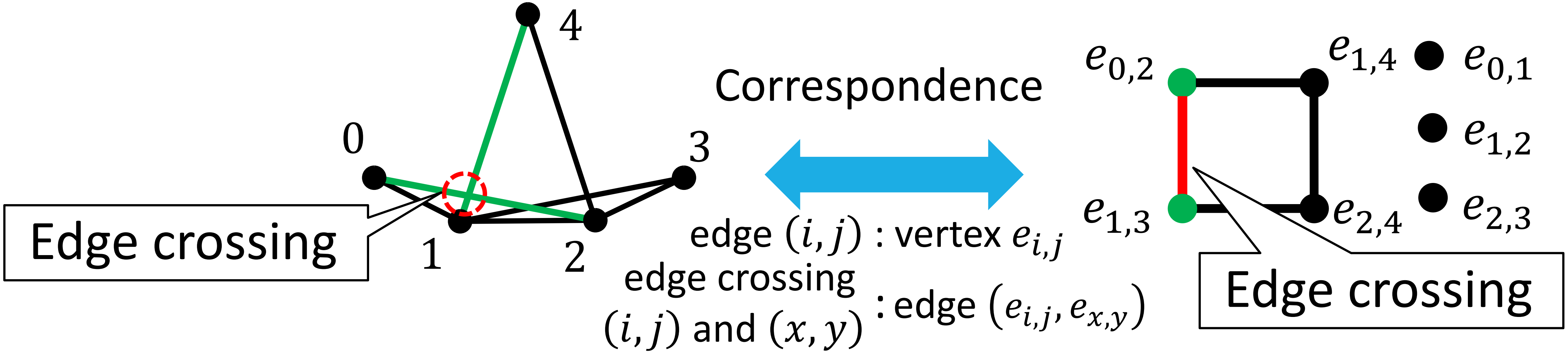
Geometric graph $G(P)$ Edge-crossing graph (W, F)



Geometric thickness of $G(P)$ = Chromatic number of (W, F)

Geometric thickness and edge-crossing graph

Edge-crossing graph (W, F) for geometric graph $G(P) = (P, E)$

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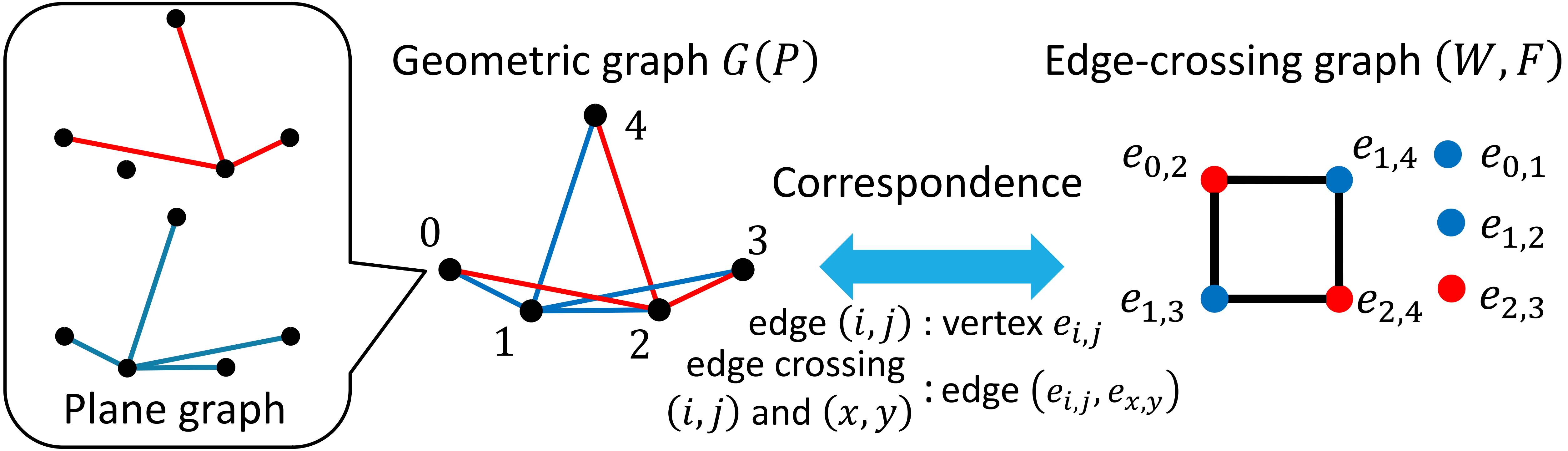
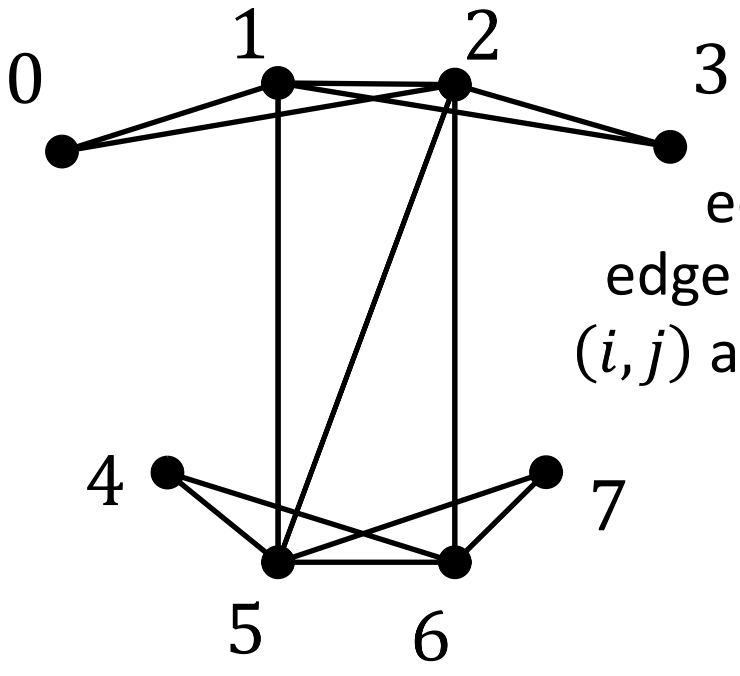


Geometric thickness of $G(P)$ = Chromatic number of (W, F)

Geometric thickness of MLG(P)

MLG(P)

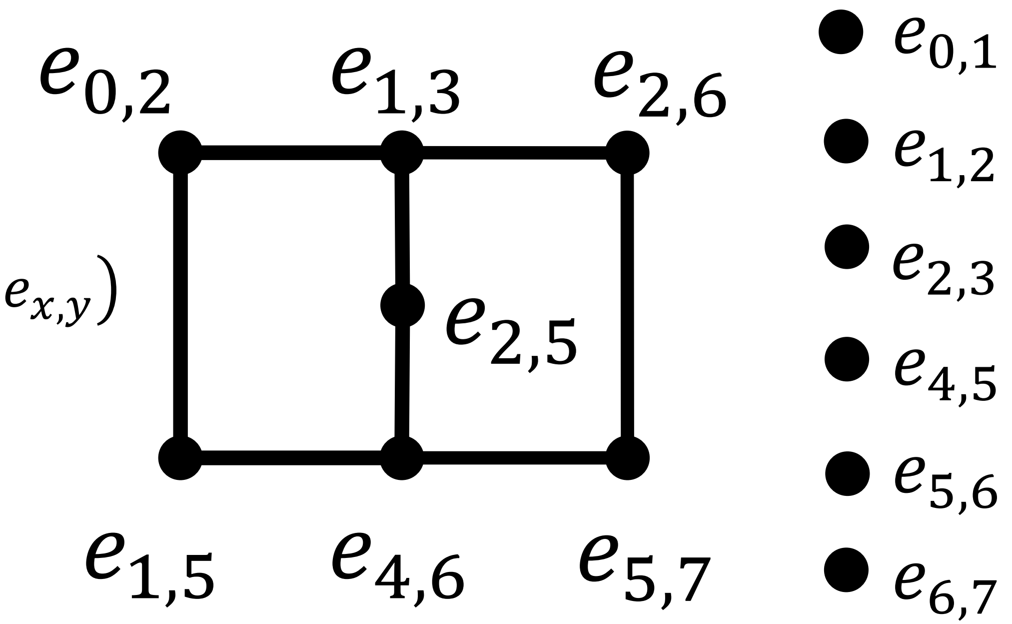


Edge-crossing graph for MLG(P)

edge (i,j) : vertex $e_{i,j}$

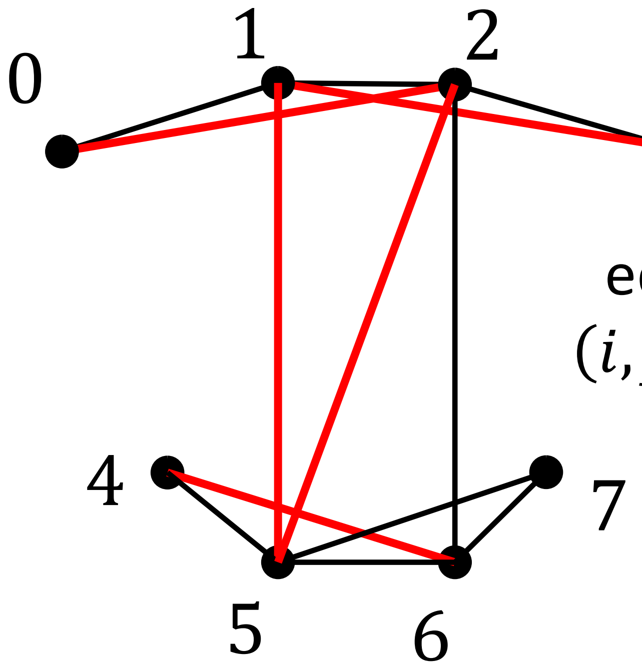
edge crossing (i,j) and (x,y) : edge $(e_{i,j}, e_{x,y})$

Correspondence



Geometric thickness of MLG(P)

MLG(P)



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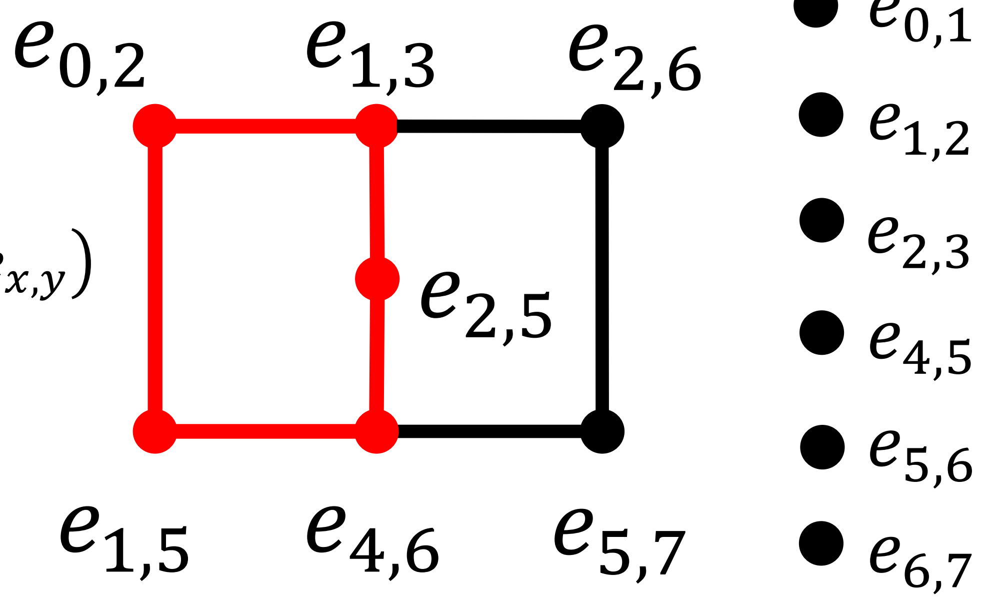
edge crossing (i, j) and (x, y) : edge $(e_{i,j}, e_{x,y})$

Correspondence

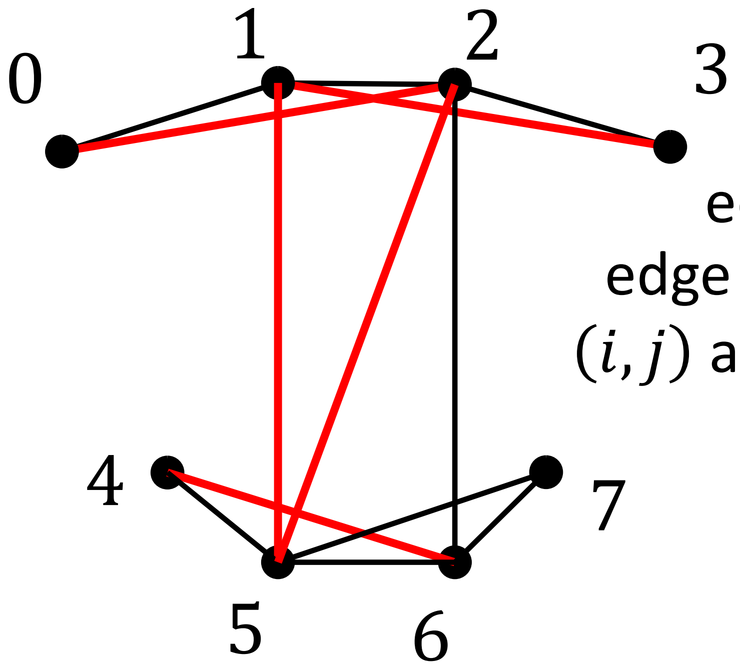
Edge-crossing graph for MLG(P)



Geometric thickness of MLG(P)

MLG(P)

Edge-crossing graph for MLG(P)



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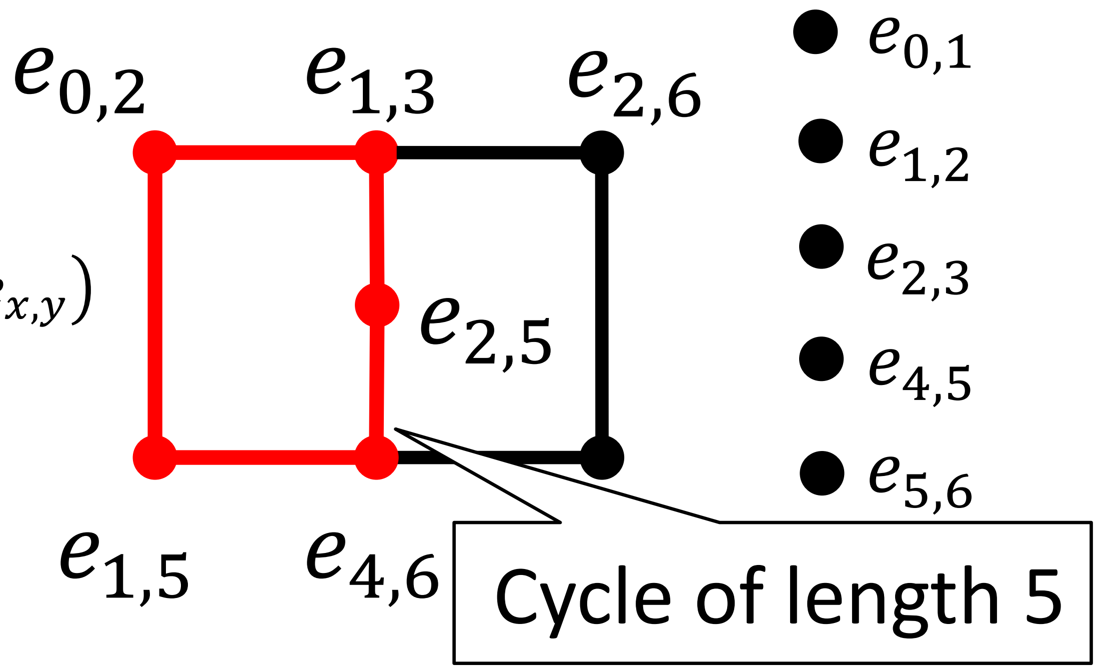
Correspondence



Geometric thickness is 3 or more

Chromatic number is 3 or more

Geometric thickness of $G(P)$ = Chromatic number of (W, F)

Comparison of previous result and our result

The maximum number		$(2,3)$-MTG(P) (MLG(P))		$(2,2)$-MTG(P)	
		Previous result [1]	**Our result**	Previous result [2]	**Our result**
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Geometric thickness	upper bound	4	-	-	-
	Lower bound	2	**3**	-	**3**

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[2] S. Bereg et. al., On the edge crossing properties of euclidean minimum weight laman graphs., Computational Geometry, 2016

Conclusion

- We improve lower bounds for the total number of edge crossings and geometric thickness of MLG(P).
- We extend lower bounds for them of $(2,2)$-MTG(P).

Future works

- There exists gap between lower and upper bounds for the total number of edge crossings and geometric thickness of MLG(P) and (k, ℓ)-MTG(P).