# Optimal Polyline Simplification under the Local Fréchet Distance in 2D in (Near-)Quadratic Time

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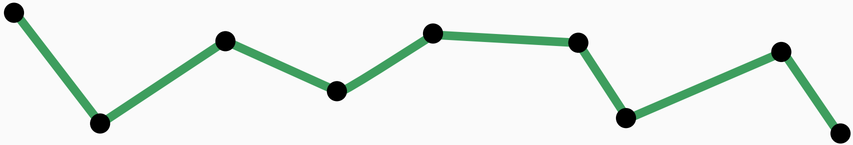
# Polyline Simplification



**Problem Setting**

- a sequence of $n$ vertices $p_1, \ldots, p_n$
- **straight** segments
- select a **minimal subset** of vertices (no interpolation!)
- such that a distance **measure** is within a given **threshold** $\delta$:

$$d(P, S) \leq \delta$$

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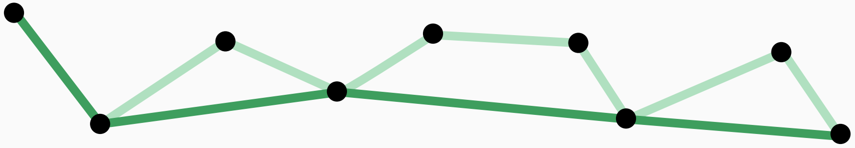

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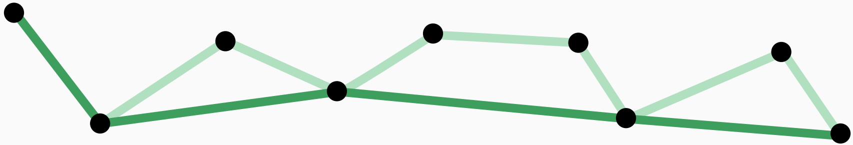

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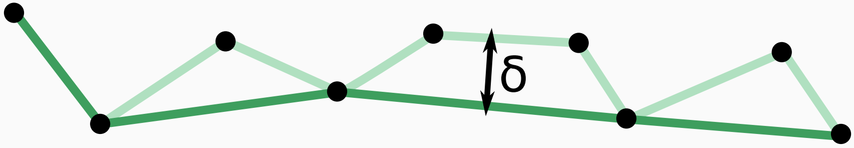


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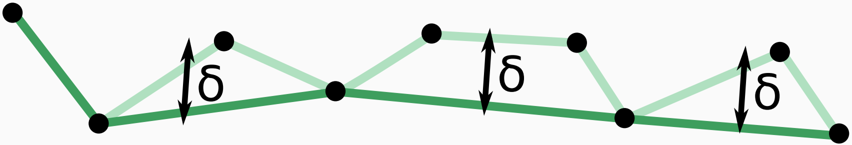
# Local Simplification




## Problem Setting

- **Local** Simplification = segment-wise
- distance measure applied to each **segment**
- select **minimal subset** of vertices such that

$$d(P_{[i,i+1]}, S_{[j,k]}) \leq \delta$$

# Distance Measures

## Hausdorff Distance

- minimize maximum distance between two vertices
- works for **any** set of points
- ignores their **order** ☹

## Fréchet Distance

- minimize maximum distance over all **mapping functions**
- recognizes the **course** of the trajectory ☺
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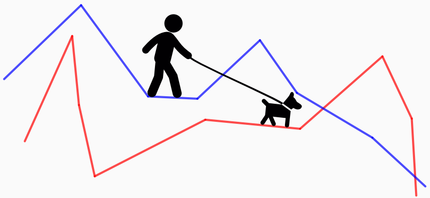
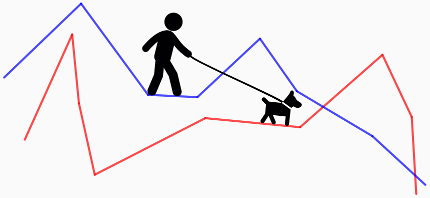
# Fréchet Distance



- it's the **dog-leash** distance
- man and dog walk the curves
- at variable speed (but never backwards)
- find the **minimum** required length of the leash

# Fréchet Distance



**Definition**

Given two parameterized curves $P, Q : [0,1] \longrightarrow \mathbb{R}^2$

$$d_F(P,Q) = \inf_{\sigma,\tau} \max_{\substack{s\in[0,1],\\ t\in[0,1]}} || P(\sigma(s)) - Q(\tau(t)) ||$$

the Fréchet distance is the infimum over all **continuous and increasing** bijections $\sigma, \tau : [0,1] \longrightarrow [0,1]$.

$|| \cdot ||$ is the underlying norm (Euclidean, or other)

# State of the Art

| local Hausdorff Distance | | local **Fréchet Distance** | |
|---|---|---|---|
| $O(n^3)$ | Imai,Iri '88 | $\mathbf{O(n^3)}$ | Godau '91 |
| $O(n^2 \log n)$ | Melkman,O'Rourke '88 | $O(n^{2.5})$ | Buchin et al.'22 |
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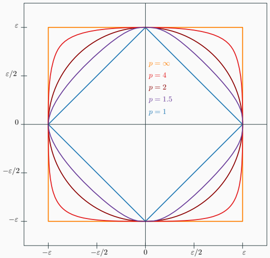


**NEW**

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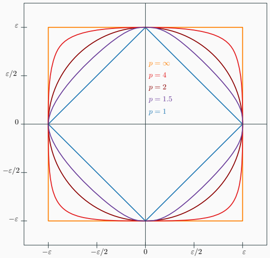


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| $\mathbf{O(n^2)}$ | $L_1$, $L_\infty$ |

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- proceeds in two phases

- First phase
  - valid shortcuts (brute force)
  - build shortcut graph


- Second phase
  - shortest path in graph
  - optimal simplification

→ total running time $O(n^3)$

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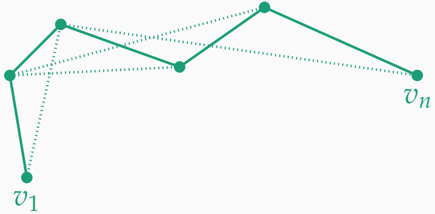

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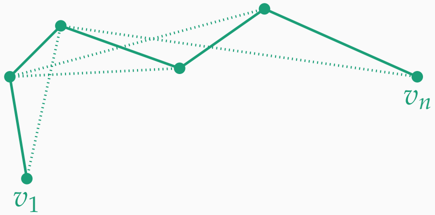


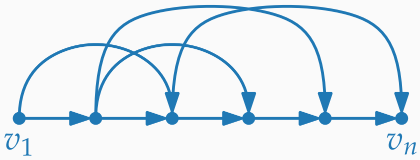

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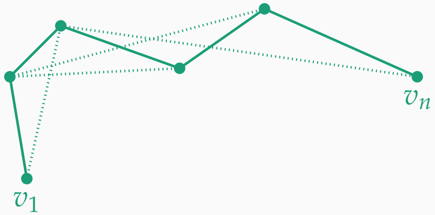


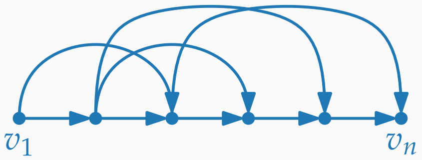

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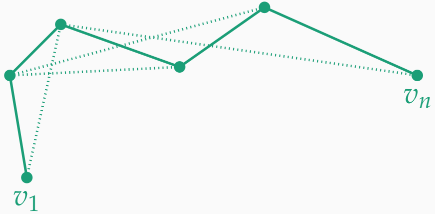


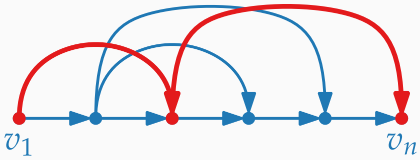

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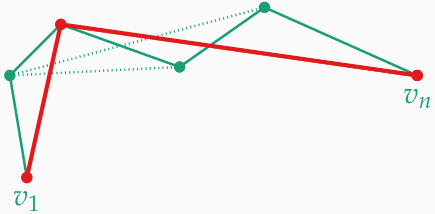


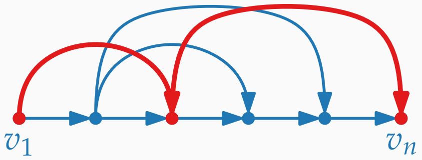

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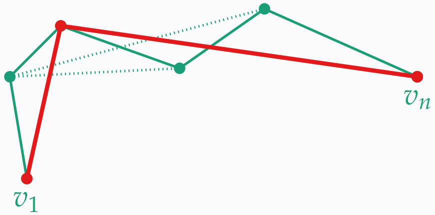


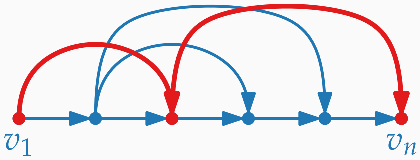

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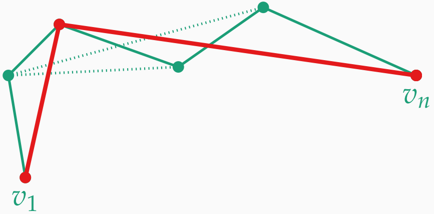


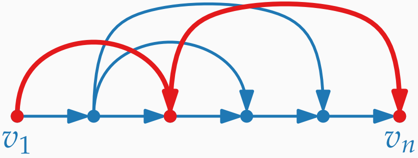

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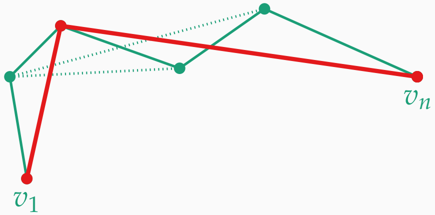


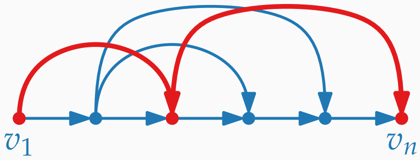

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- for each vertex $v_i,\ i \in \{1, \ldots, n\}$
  - traverse each subsequent vertex $v_j,\ j > i$
  - while maintaining a **cone**
  - and a **wave front**


- $\overline{v_i v_j}$ is a **valid** shortcut

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- can be updated **incrementally** ☺

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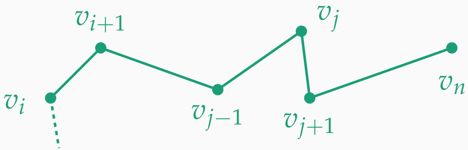


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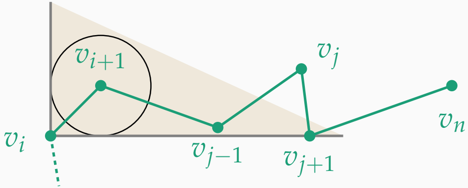


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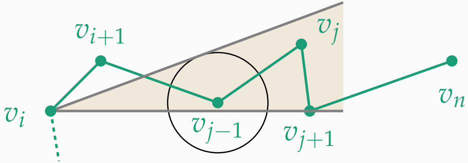


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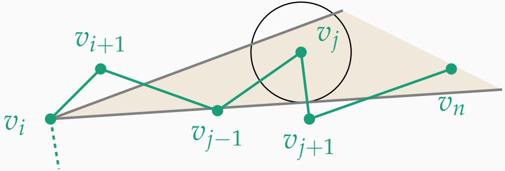


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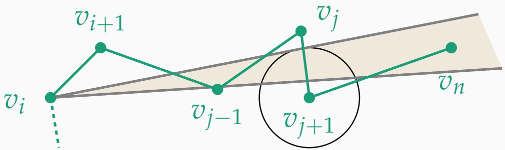


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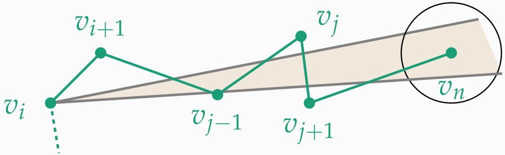


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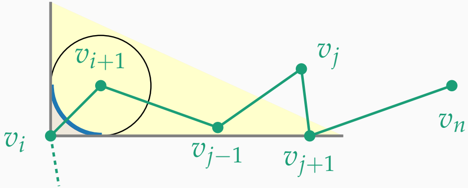


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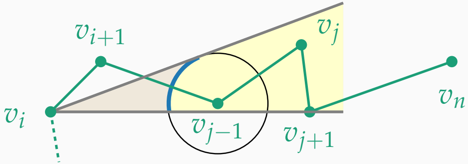


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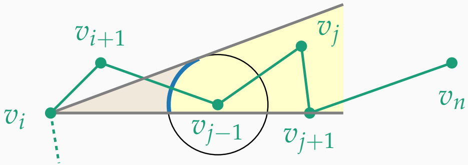


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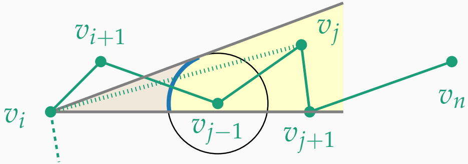


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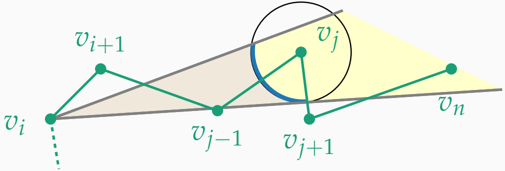


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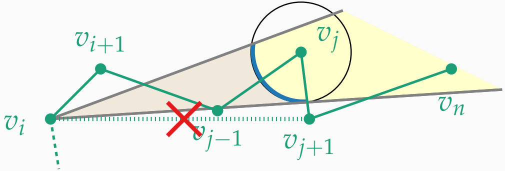


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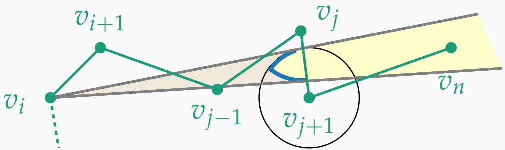


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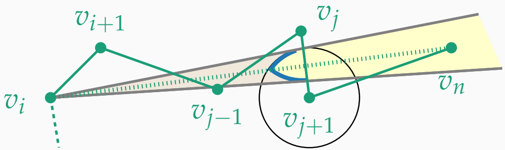


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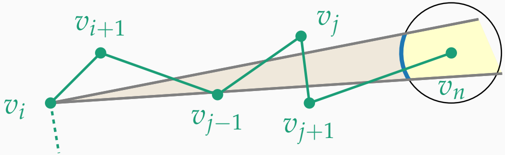


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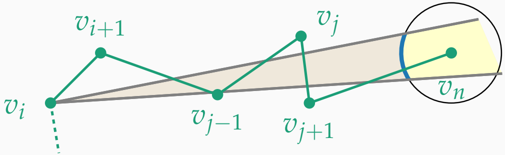


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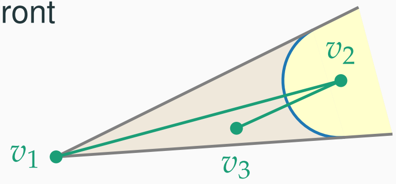


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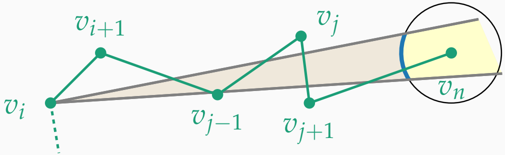


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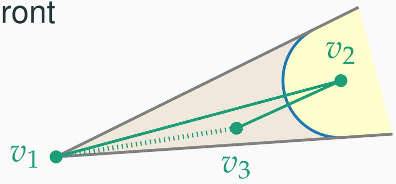


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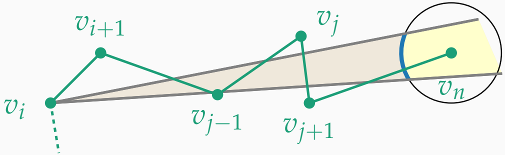


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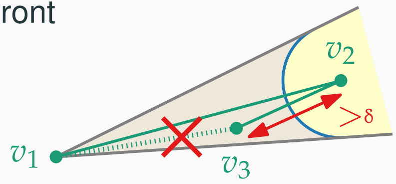


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# The Wave Front

- is a sequence of circular arcs
- how complex can it be?
- each vertex contributes $\leq 1$ arc → $O(n)$ arcs

→ stored in a binary tree

- find intersections: $O(\log n)$
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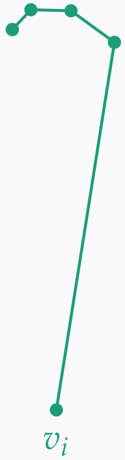
→ total time $\mathbf{O(n^2 \log n)}$



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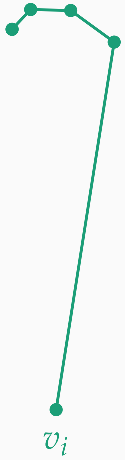

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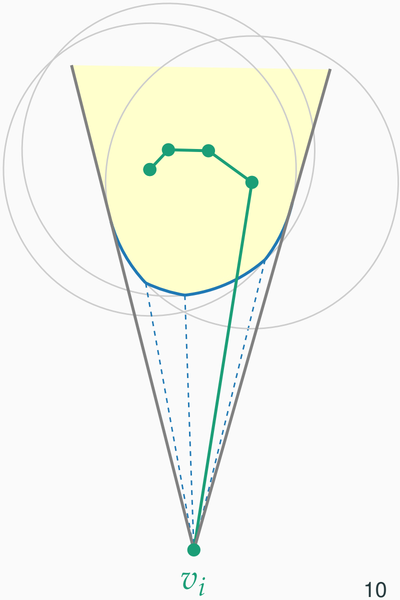

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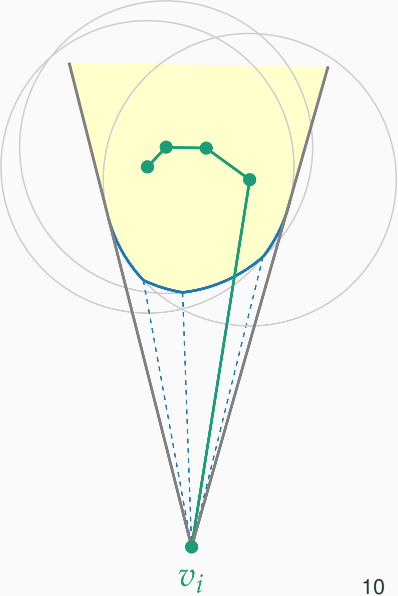

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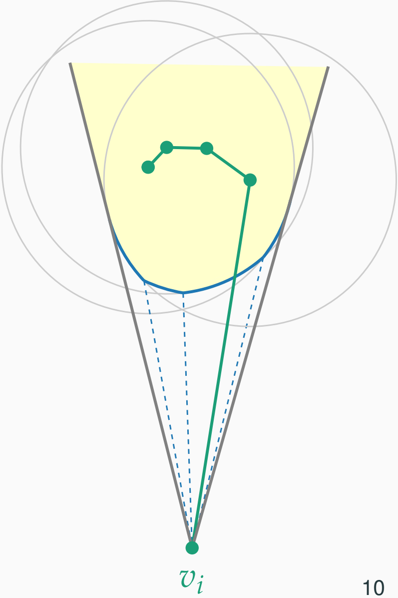

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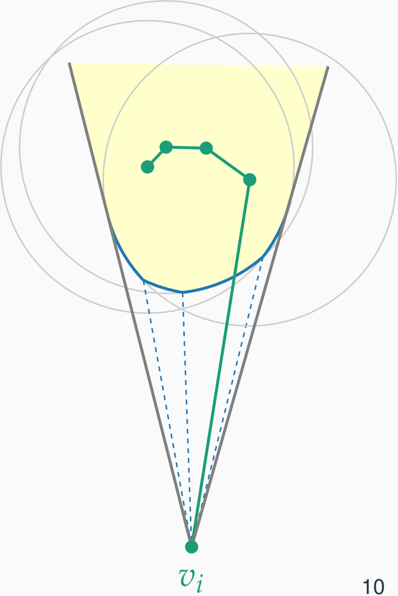

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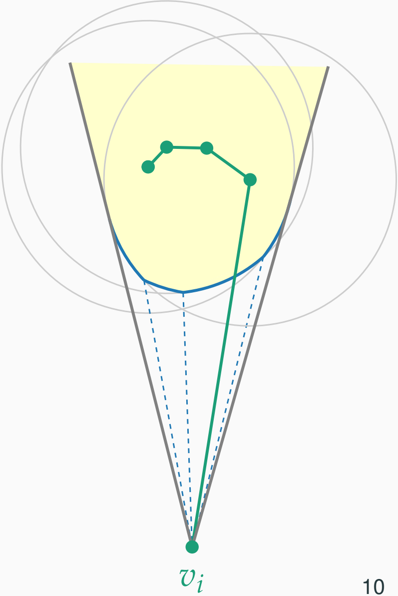

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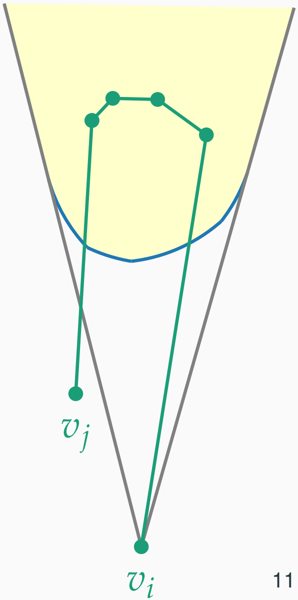

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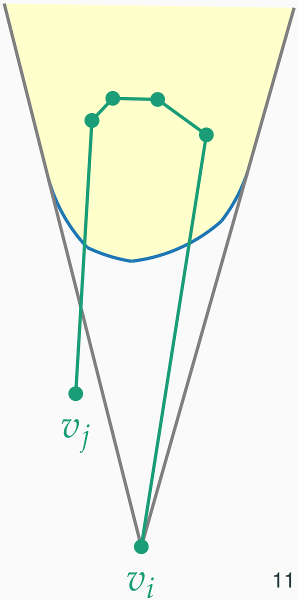

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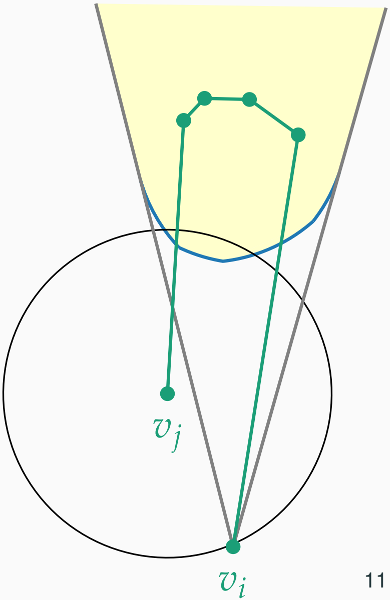

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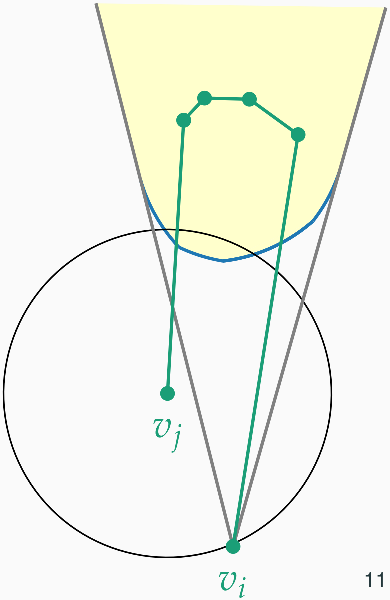

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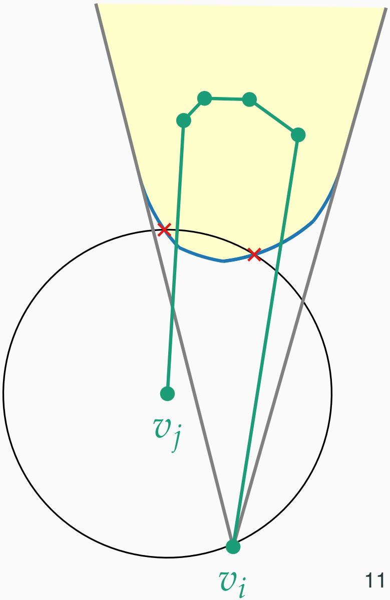

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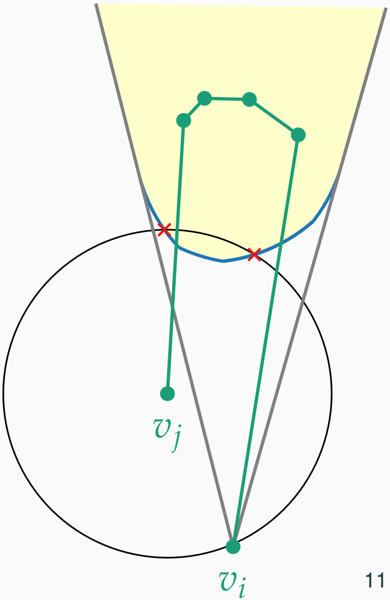

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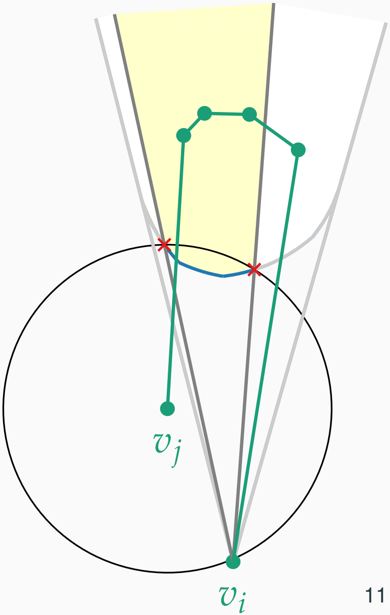

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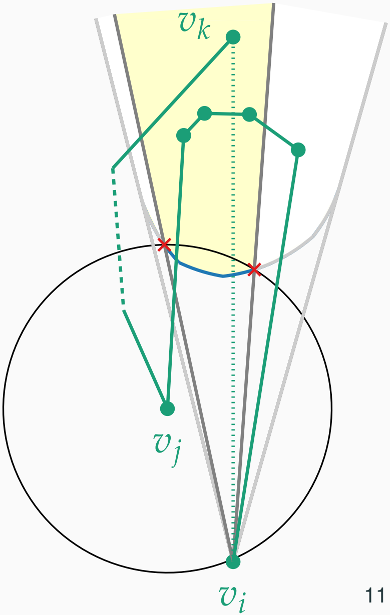

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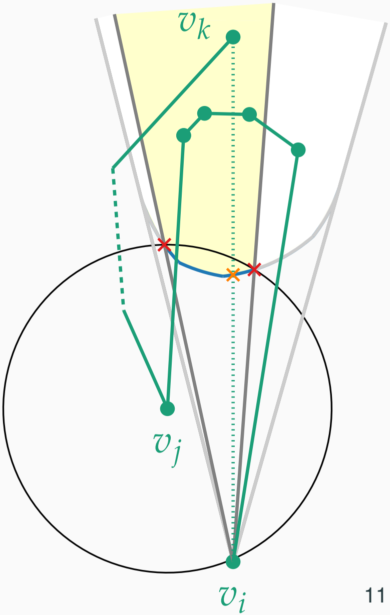

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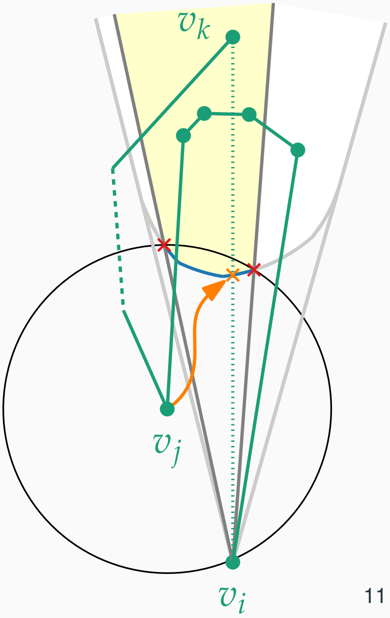

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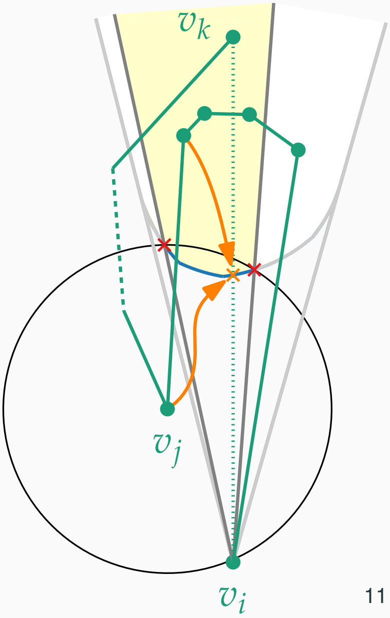

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- sum-norm $L_1$, max-norm $L_\infty$
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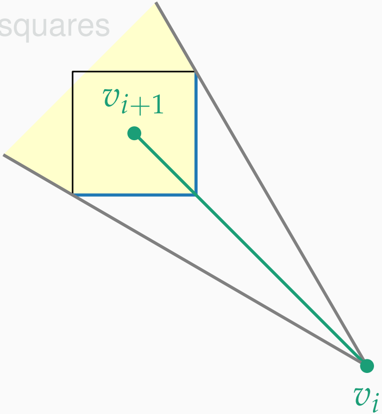
→ total time $\mathbf{O(n^2)}$



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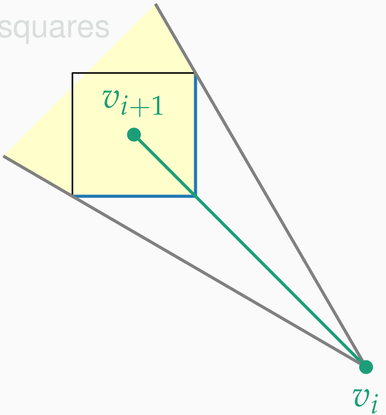

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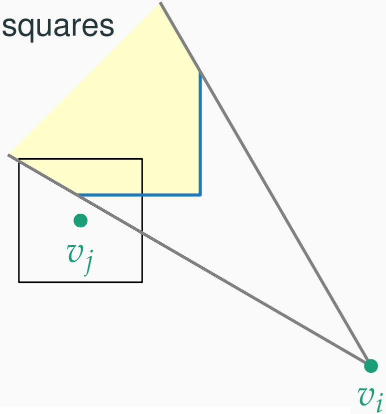

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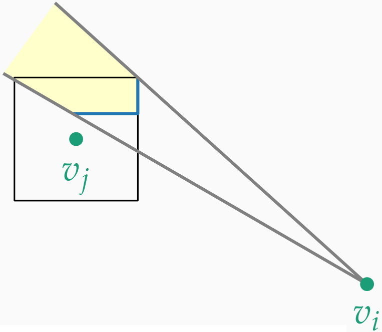

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- worst case = wavefront size $O(n)$
- we can build worst-case examples ✓

→ $O(n^2 \log n)$ is **tight**

- **but**: worst cases are **contrived**, unlikely to appear in the wild

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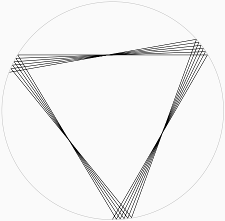

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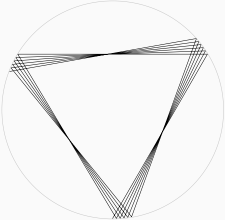

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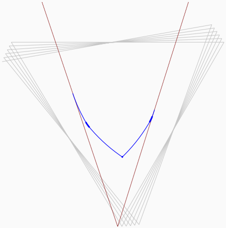

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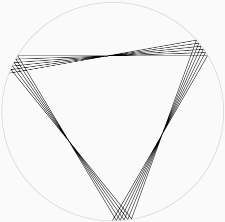

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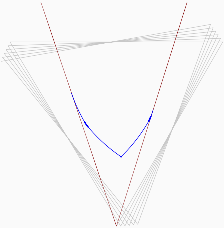

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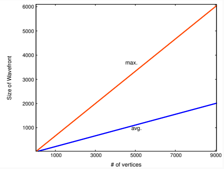


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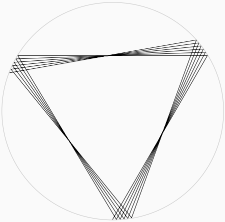

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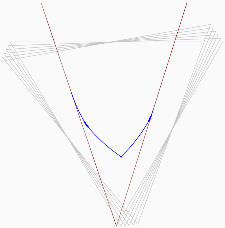

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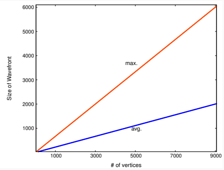


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**Conjecture**

- worst-case is rare
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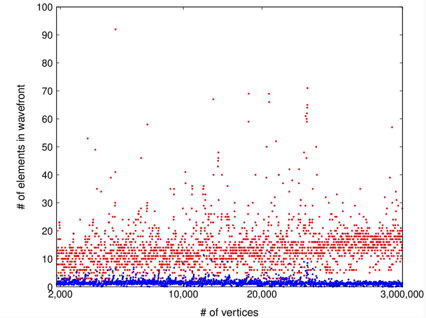
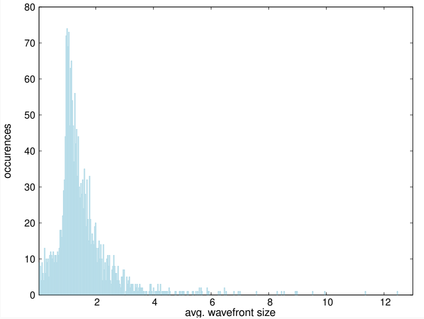
  evaluate on real-world data

# Real-world Instances

- **real-world** data: trajectories from OSM, up to 3 Mio. vertices
- Wave-fronts are always **small**: avg. $\leq 6$, max. $\leq 90$

→ **practical** running-time close to $O(n^2)$
(confirming our conjecture)



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- **real-world** data: trajectories from OSM, up to 3 Mio. vertices
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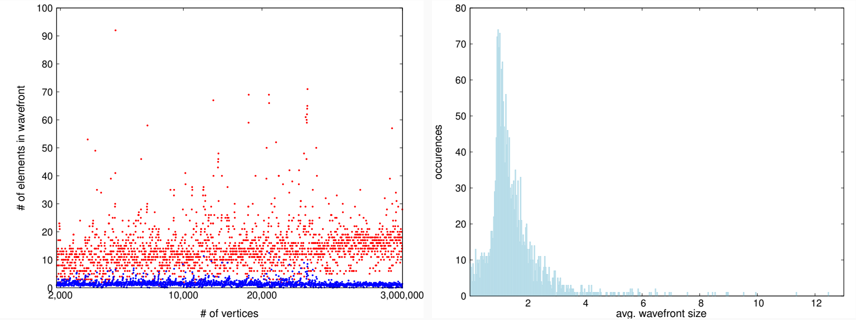

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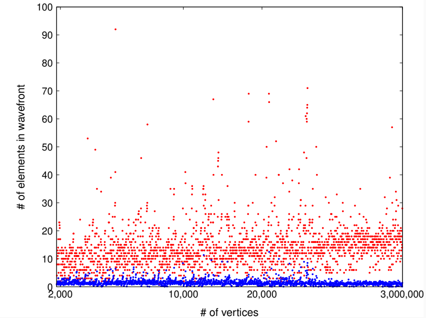


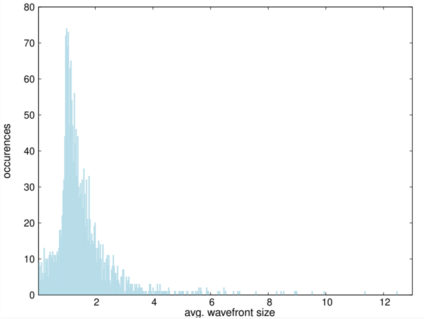

# Container Data Structure

- default: binary **tree**
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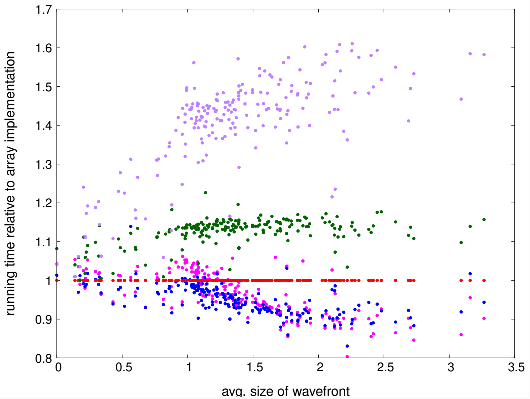


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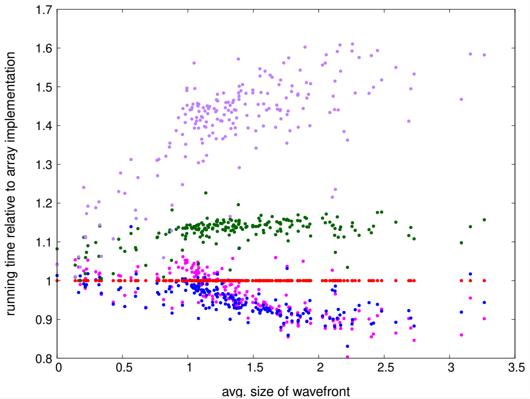


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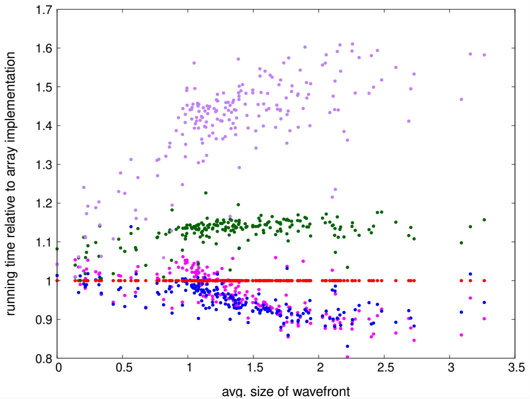


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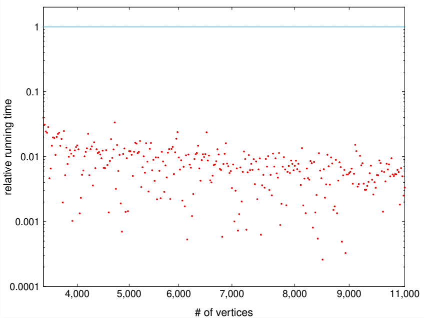

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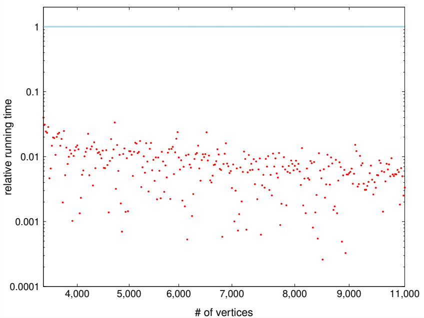


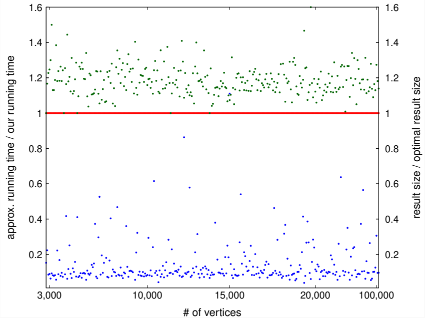

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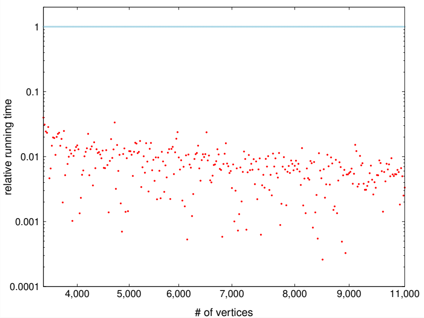


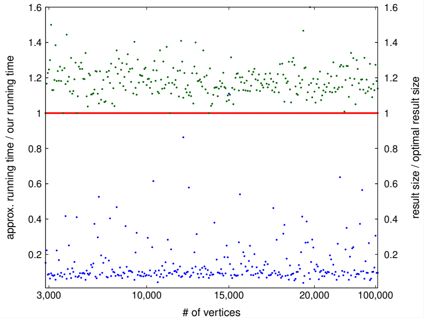

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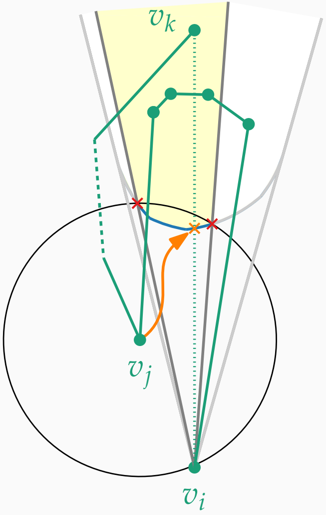
- **new** algorithm (using some old ideas)
- improves state-of-the-art to $\mathbf{O(n^2 \log n)}$
- even $\mathbf{O(n^2)}$ for $L_1$, $L_\infty$
- bounds are tight, but...
- worst-case is **unlikely** on real-world data

→ practical running time ≈ $\mathbf{O(n^2)}$

- allows for simpler implementation



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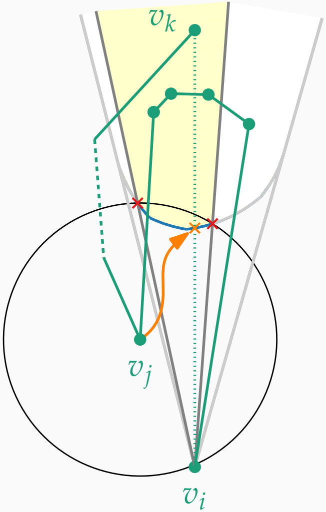

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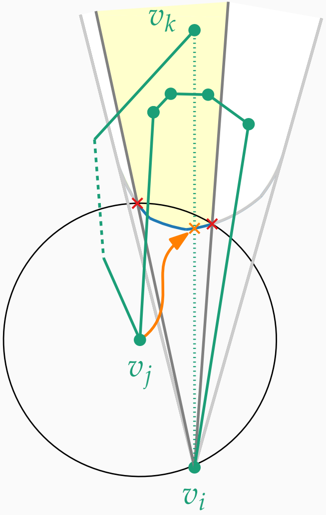

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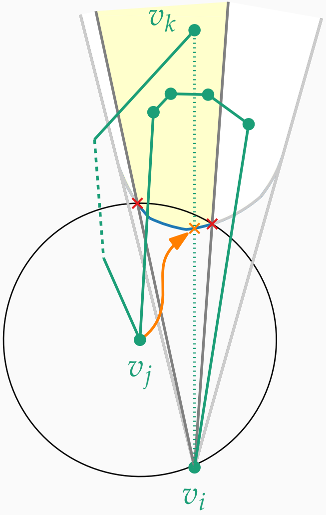

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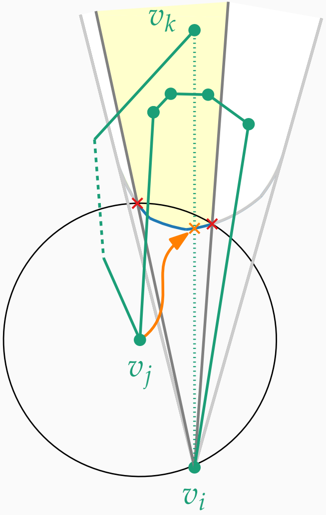

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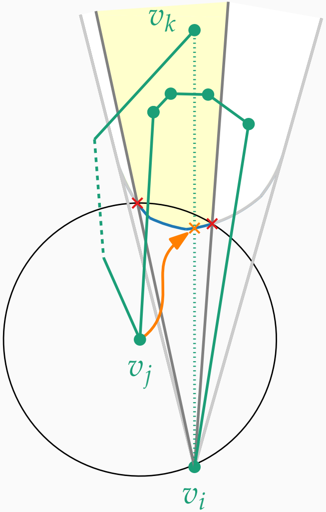

# Open Questions

- define worst-case
- lower bounds $< O(n^2)$
- higher dimensions
- norms $L_{p \in (1,\infty)}$
- spherical geometry
- drop endpoints