Every Combinatorial Convex Polyhedron Can Unfold with Overlap

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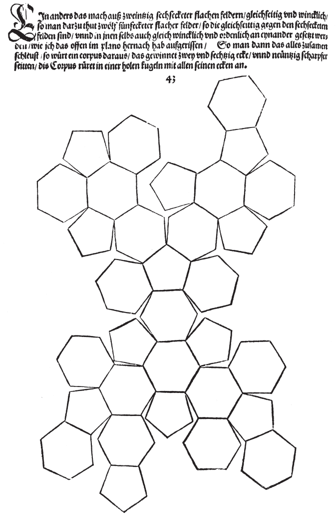
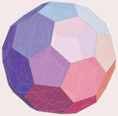
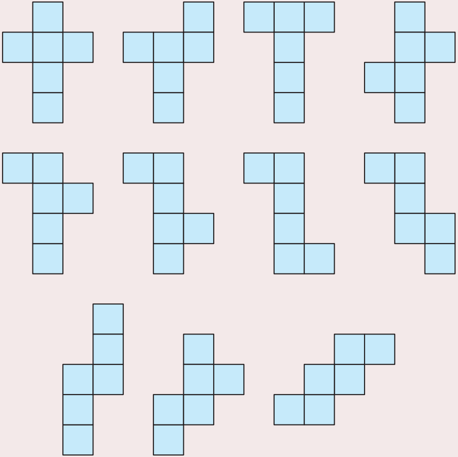
CCCG 2023

Dürer's Problem: 1525





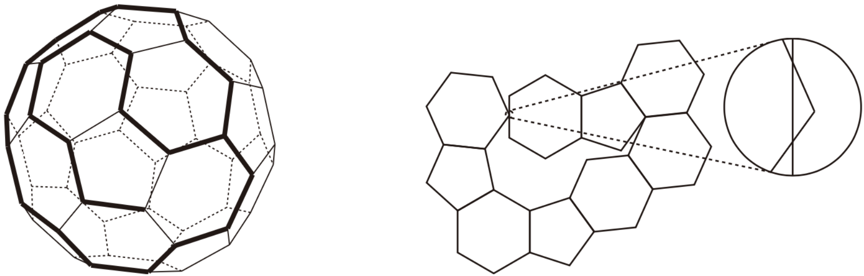
11 Cube Unfoldings



Horiyama and Shoji, *CCCG* 2011:

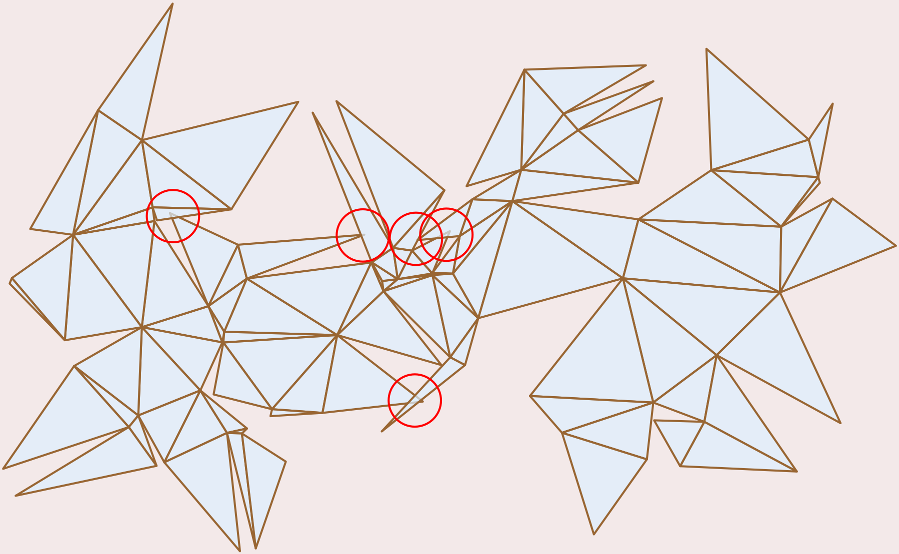
No edge-unfolding overlap possible for any Platonic solid.

Truncated Icosahedron Overlap



Shiota & Saitoh, 2023.

Overlap: $V = 50$



Mohammad Ghomi Theorem

- Every convex polyhedron can be stretched via an affine transformation so that it has an edge-unfolding to a net.
- $\equiv$
- Every combinatorial polyhedron $\mathcal{P}$ has a metric realization P that allows unfolding to a net via some spanning tree T.

Malkevitch Q & Theorem

Joseph Malkevitch question:

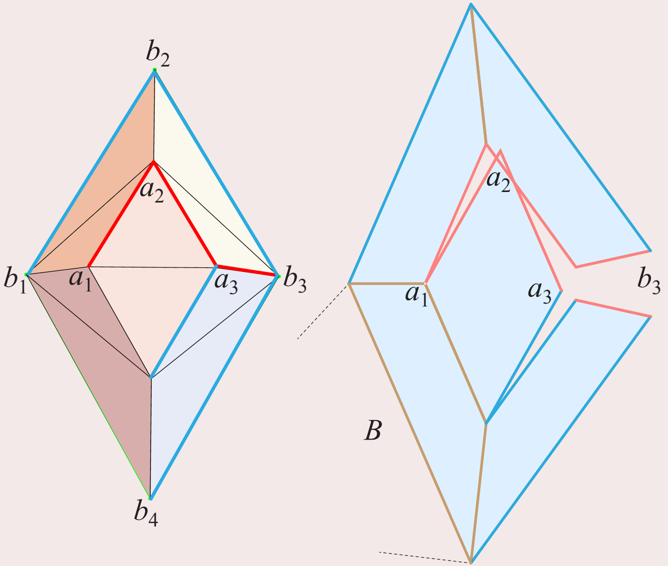
> *Is there a combinatorial polyhedron $\mathcal{P}$ such that, for every metric realization P in $\mathbb{R}^3$, and for every spanning cut-tree T of the 1-skeleton, P cut by T unfolds to a net?*

Answer: NO:

Theorem

Any 3-connected planar graph G can be realized as a convex polyhedron P in $\mathbb{R}^3$ that has a spanning cut-tree T such that the edge-unfolding of $P \setminus T$ overlaps in the plane.

Combinatorial Cube Overlap



Algorithm: Assume triangulated

Algorithm. Realizing G to unfold with overlap.
Input: A 3-connected planar graph G.
Output: Polyhedron P realizing G and a cut-tree T that unfolds P with overlap.

1. Select outer face B as base.
2. Embed B as a convex polygon in the plane.
3. Apply Tutte's theorem to calculate an equilibrium stress for G.
4. Apply Maxwell-Cremona vertically lifting stressed G to P.
5. Identify special triangle $\triangle$.
6. Compress P vertically to reduce curvatures (if necessary).
7. Stretch P horizontally to sharpen the apex of $\triangle$ (if necessary).
8. Form cut-tree T, including 'Z' around $\triangle$.
9. Unfold $P \setminus T \rightarrow$ Overlap.

Dodecahedron Maxwell-Cremona Lifting

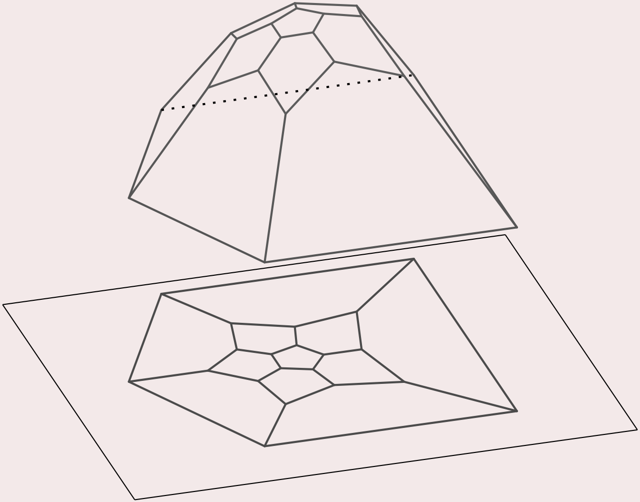
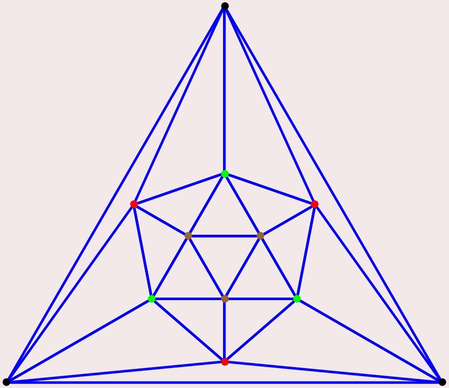
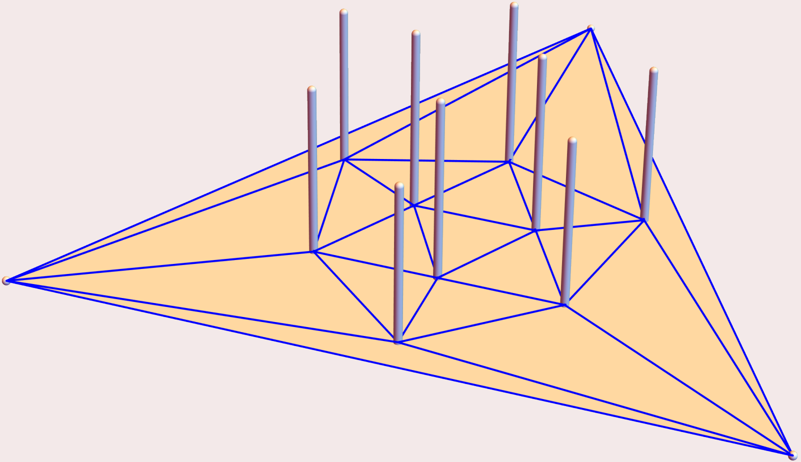


Figure: André Schulz, by permission

Icosahedron Schlegel diagram



Icosahedron Lift



Algorithm

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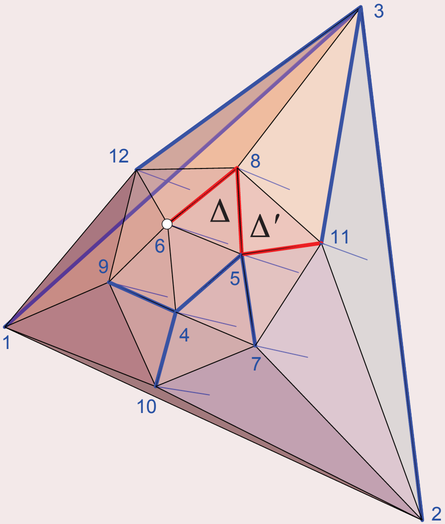
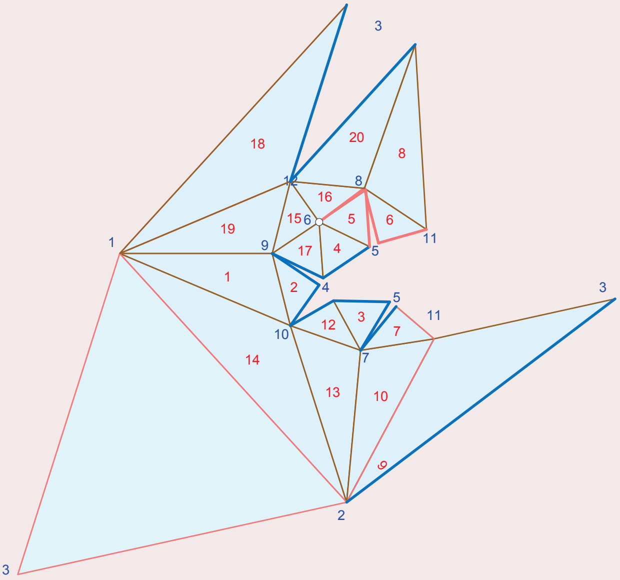
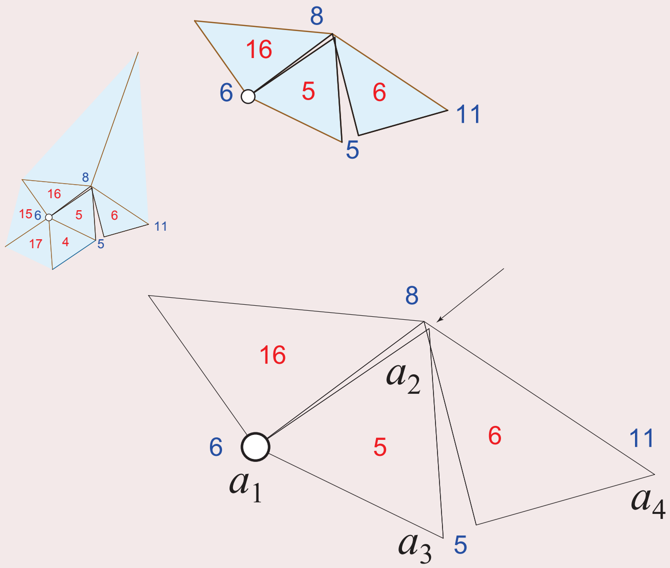
Icosahedron Lifted



Icosahedron Overlap



Icosahedron Overlap



Algorithm: Affine Transformations

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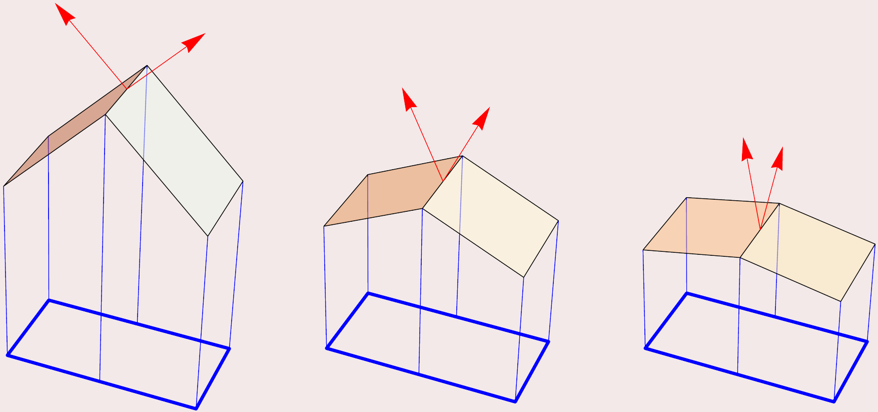
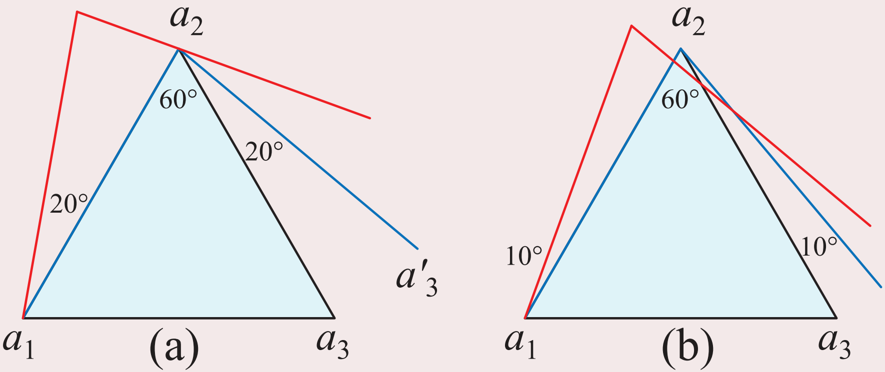
Compress P vertically to reduce curvatures



Curvature $< 20°$



Stretch angle $\angle a_1 a_2 a_3$

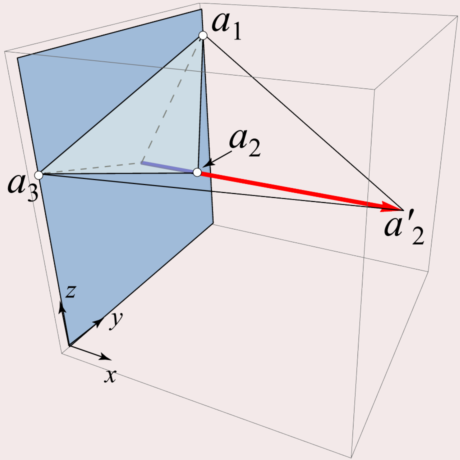


Figure: a_2 angle: $108° \to 53°$.

Algorithm: B and Δ

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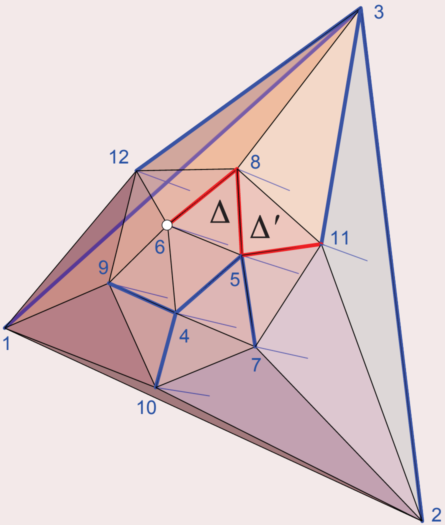
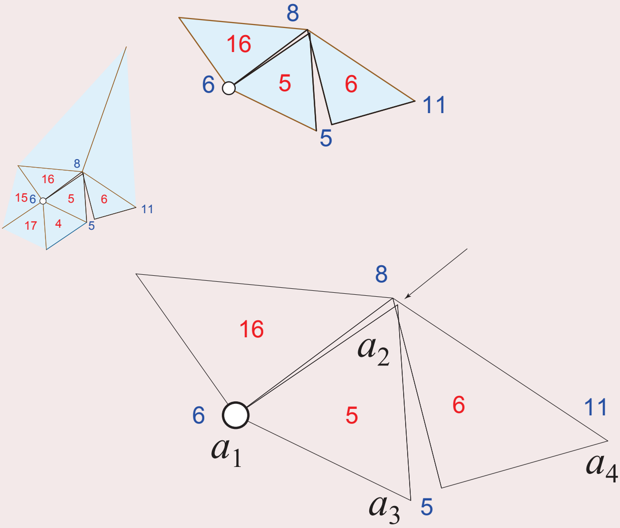
Icosahedron Lifted



Icosahedron Overlap



Few F: $B \cap \triangle = v$.

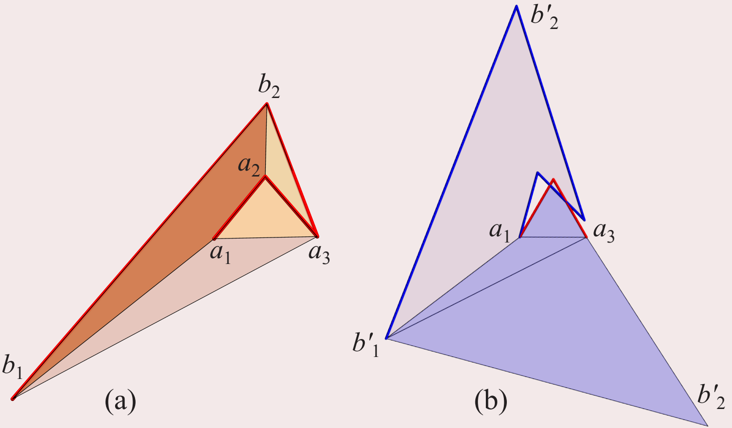
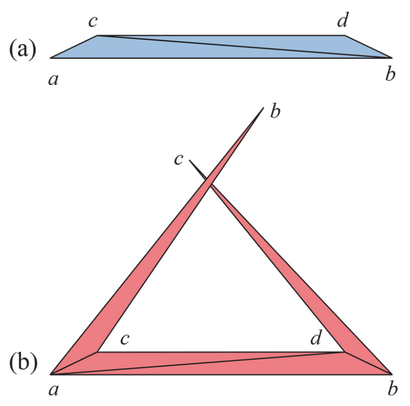


Figure: $\omega(a_3) = 117°$.

Tetrahedron Overlap

- No two disjoint faces,
- Nor two faces that share just a single vertex:
- Every pair of faces shares two or more vertices
  - $\equiv$ shares two or more edges



Theorem

Any 3-connected planar graph G can be realized as a convex polyhedron P in $\mathbb{R}^3$ that has a spanning cut-tree T such that the edge-unfolding of $P \setminus T$ overlaps in the plane.

Open Problems

1. Combinatorial un-zipping:
   1. Is there a combinatorial Hamiltonian polyhedron[a] whose every metric realization and zipper unfolding avoids overlap?
2. Metric conditions: Does any combination of
   1. acute angles, and
   2. small curvatures

   guarantee a cut tree that unfolds to a (non-overlapping) net?

[a] E.g., rhombic dodecahedron not Hamiltonian.

“Edge-Unfolding Nearly Flat Convex Caps,” *SoCG*, 2017