Every Combinatorial Convex Polyhedron Can Unfold with Overlap

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CCCG 2023

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Dürer's Problem: 1525

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11 Cube Unfoldings



Horiyama and Shoji, CCCG 2011:

No edge-unfolding overlap possible for any Platonic solid.

Truncated Icosahedron Overlap



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Overlap: V = 50



Mohammad Ghomi Theorem

- Every convex polyhedron can be stretched via an affine transformation so that it has an edge-unfolding to a net.
- =
- Every combinatorial polyhedron \mathcal{P} has a metric realization P that allows unfolding to a net via some spanning tree \mathcal{T} .

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Malkevitch Q & Theorem

Joseph Malkevitch question:

Is there a combinatorial polyhedron \mathcal{P} such that, for every metric realization P in \mathbb{R}^3 , and for every spanning cut-tree T of the 1-skeleton, P cut by T unfolds to a net?

Answer: NO:

Theorem

Any 3-connected planar graph G can be realized as a convex polyhedron P in \mathbb{R}^3 that has a spanning cut-tree T such that the edge-unfolding of $P \setminus T$ overlaps in the plane.

Combinatorial Cube Overlap



Algorithm: Assume triangulated

Algorithm. Realizing *G* to unfold with overlap.

Input: A 3-connected planar graph G.

Output: Polyhedron P realizing G and a cut-tree T that unfolds P with overlap.

- Select outer face *B* as base.
- **2** Embed B as a convex polygon in the plane.
- Apply Tutte's theorem to calculate an equilibrium stress for G.
- Apply Maxwell-Cremona vertically lifting stressed G to P.
- **(a)** Identify special triangle \triangle .
- O Compress P vertically to reduce curvatures (if necessary).
- **③** Stretch *P* horizontally to sharpen the apex of \triangle (if necessary).
- **I** Form cut-tree T, including 'Z' around \triangle .
- **()** Unfold $P \setminus T \rightarrow \text{Overlap}$.

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Dodecahedron Maxwell-Cremona Lifting



Figure: André Schulz, by permission

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Icosahedron Schlegel diagram



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Icosahedron Lift



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Algorithm

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Icosahedron Lifted



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Icosahedron Overlap



Icosahedron Overlap



Algorithm: Affine Transformations

Algorithm. Realizing *G* to unfold with overlap.

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- Embed B as a convex polygon in the plane.
- Apply Tutte's theorem to calculate an equilibrium stress for G.
- Apply Maxwell-Cremona vertically lifting stressed G to P.
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- **2** Unfold $P \setminus T \rightarrow \text{Overlap}$.

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Compress P vertically to reduce curvatures



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$Curvature < 20^{\circ}$



Stretch angle $\angle a_1 a_2 a_3$



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Algorithm: B and Δ

Algorithm. Realizing *G* to unfold with overlap.

Input: A 3-connected planar graph G.

Output: Polyhedron P realizing G and a cut-tree T that unfolds P with overlap.

- Select outer face *B* as base.
- Embed B as a convex polygon in the plane.
- Apply Tutte's theorem to calculate an equilibrium stress for G.
- Apply Maxwell-Cremona vertically lifting stressed G to P.
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Few $F: B \cap \triangle = v$.



Tetrahedron Overlap

- No two disjoint faces,
- Nor two faces that share just a single vertex:
- Every pair of faces shares two or more vertices
 - ullet \equiv shares two or more edges



Theorem

Any 3-connected planar graph G can be realized as a convex polyhedron P in \mathbb{R}^3 that has a spanning cut-tree T such that the edge-unfolding of $P \setminus T$ overlaps in the plane.

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Open Problems

- Ombinatorial un-zipping:
 - Is there a combinatorial Hamiltonian polyhedron^a whose every metric realization and zipper unfolding avoids overlap?
- Ø Metric conditions: Does any combination of
 - acute angles, and
 - estimate state state

guarantee a cut tree that unfolds to a (non-overlapping) net?

^aE.g., rhombic dodecahedron not Hamiltonian.

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"Edge-Unfolding Nearly Flat Convex Caps," SoCG, 2017

