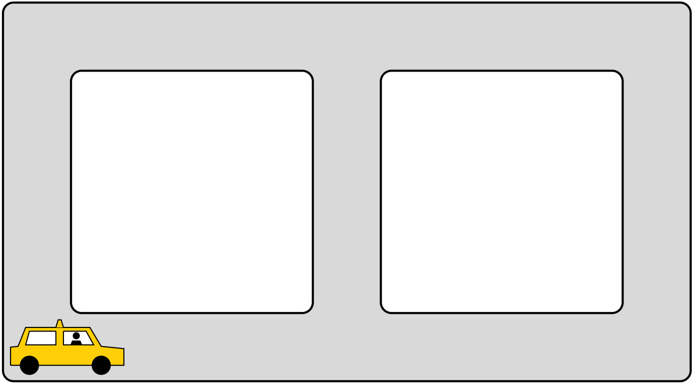
Tight Analysis of the Lazy Algorithm for Open Online Dial-a-Ride

Júlia Baligács, Yann Disser, Farehe Soheil, David Weckbecker

TU Darmstadt, Germany

We acknowledge funding by DFG through grant DI 2041/2.

Example: taxi driver



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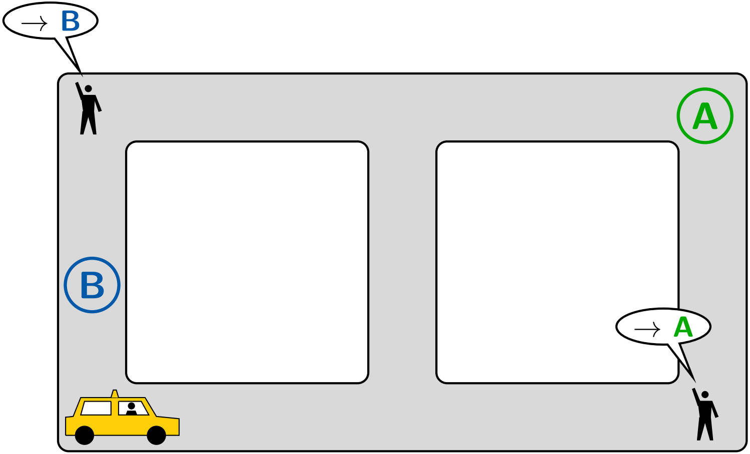

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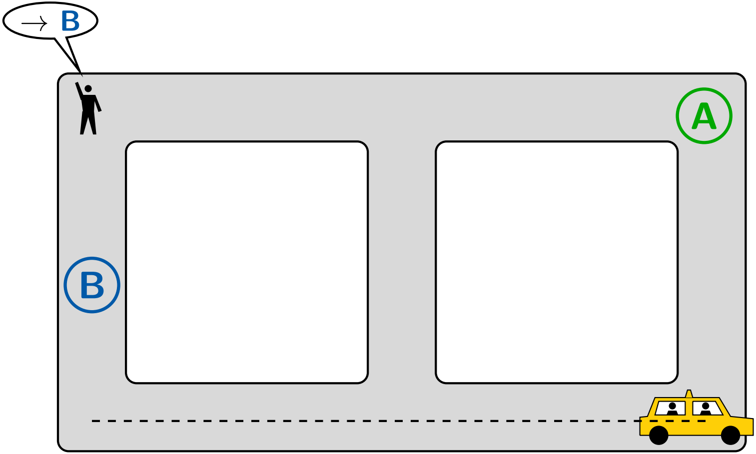

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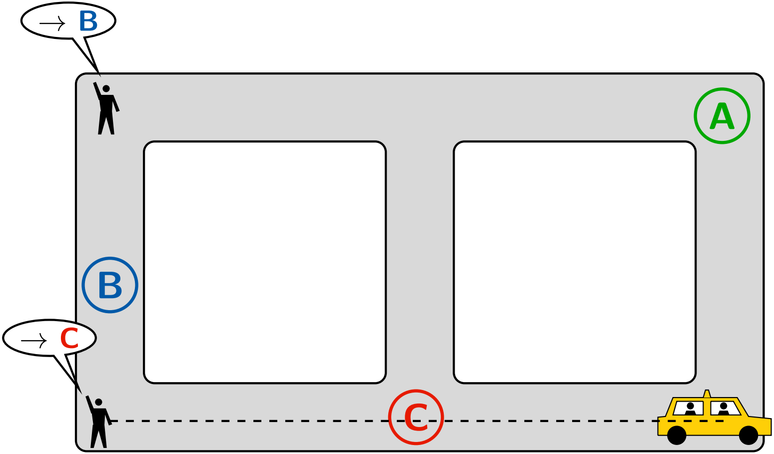

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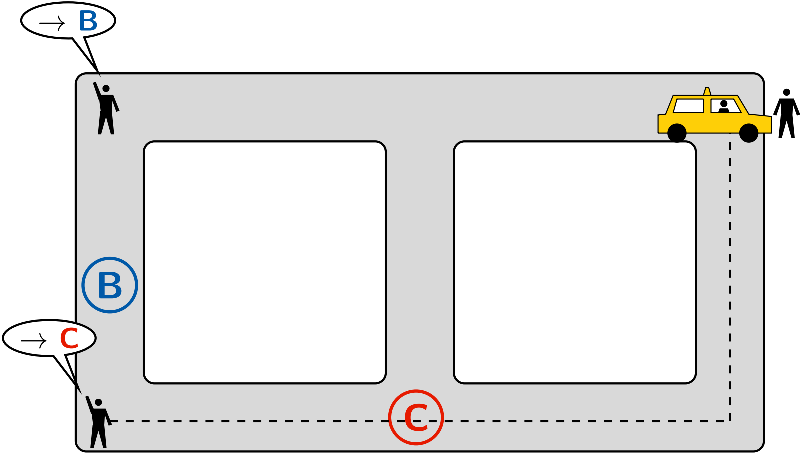

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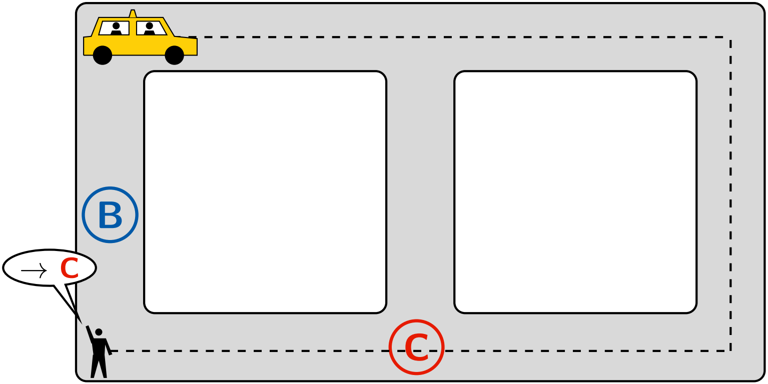

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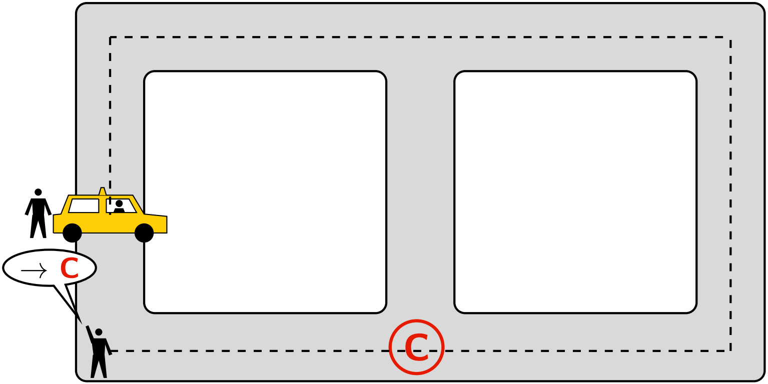

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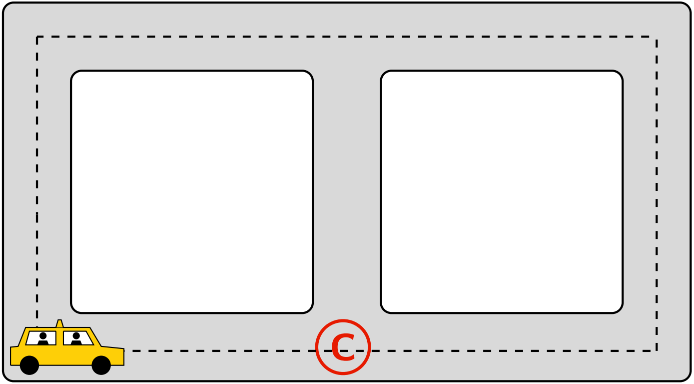

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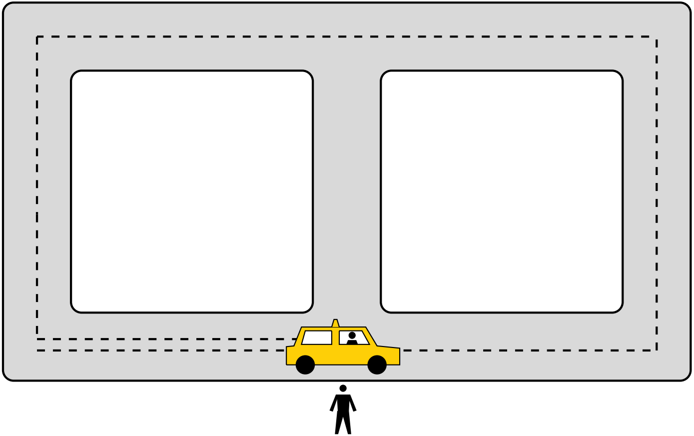

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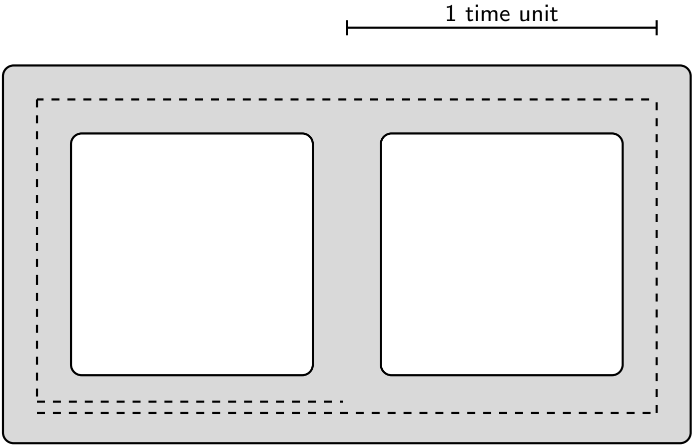

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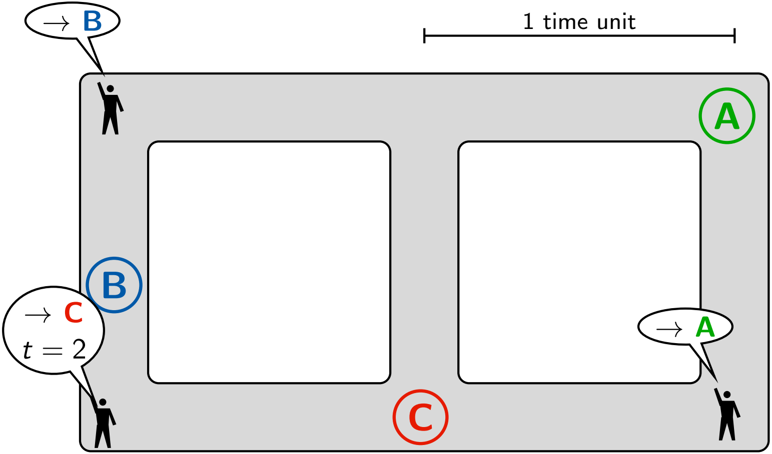

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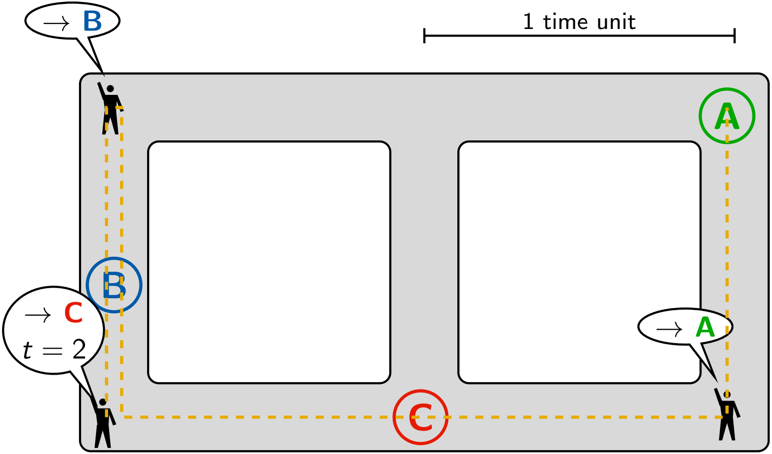
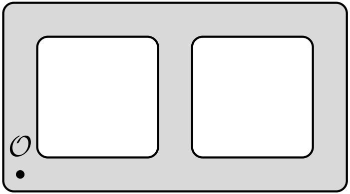


- our solution: completed after 7 time units
- optimal solution: completed after 5 time units

The online dial-a-ride problem

setting:

- metric space (M, d) with origin $\mathcal{O}$



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- metric space (M, d) with origin $\mathcal{O}$
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  - can move with speed ≤ 1
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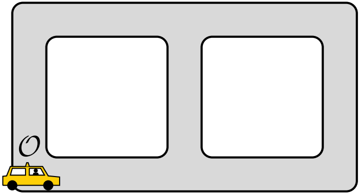

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  - pick up at a_i after time t_i
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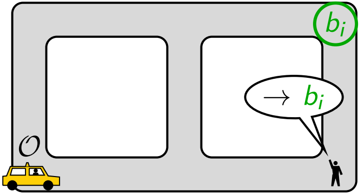

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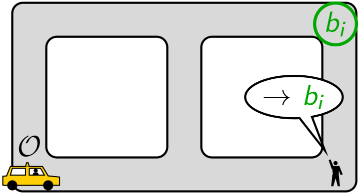

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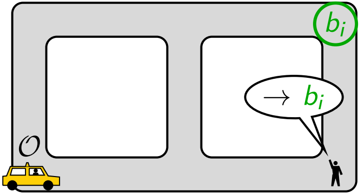

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online algorithm:
learns r_i at time t_i

offline optimum:
knows all requests in advance

Competitive analysis

Definition: for an online algorithm $\textsc{Alg}$

a) $\textsc{Alg}(\sigma)$: completion time on request sequence σ

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Question: What is the best possible competitive ratio for the online dial-a-ride problem?

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- never interrupt a schedule

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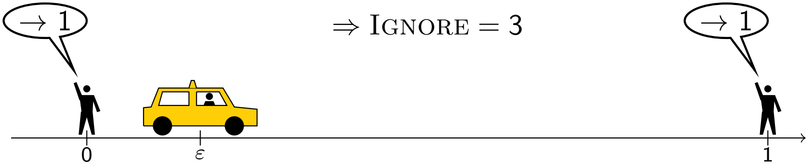

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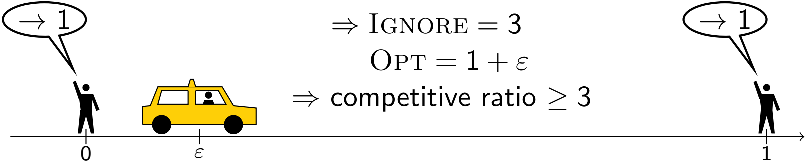

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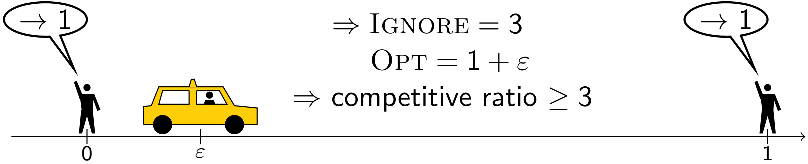

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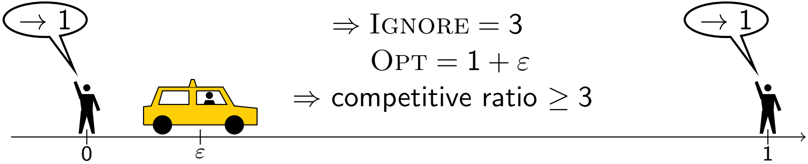

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REPLAN: start optimal schedule whenever a request appears

Some simple algorithms

Ignore:

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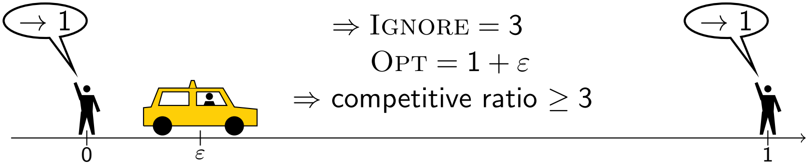

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Replan: start optimal schedule whenever a request appears

Theorem. The competitive ratio of Replan is in [2.5,4]. [Aussiello et al.'01][Birx'20]

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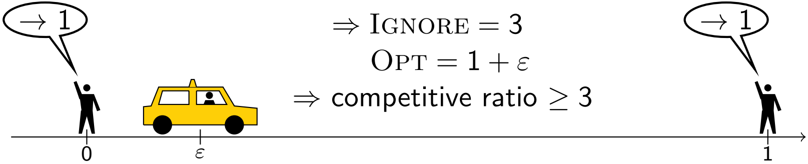

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$\to$ "interpolate" between IGNORE and REPLAN

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$\textsc{Lazy}_\alpha$
new request revealed:
▶ if possible before time $\alpha \cdot \textsc{Opt}(t)$: RESET
when idle:
▶ start schedule at time $t \geq \alpha \cdot \textsc{Opt}(t)$

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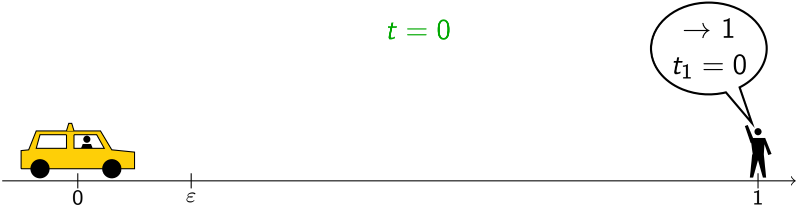

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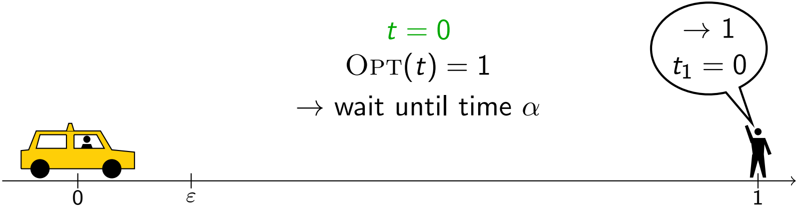

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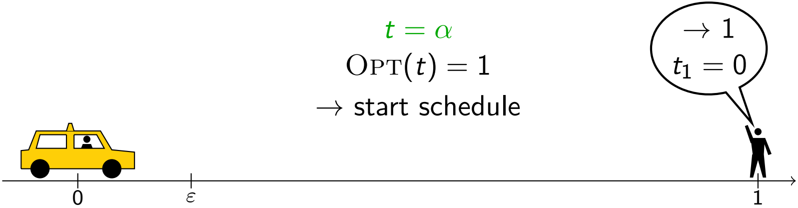

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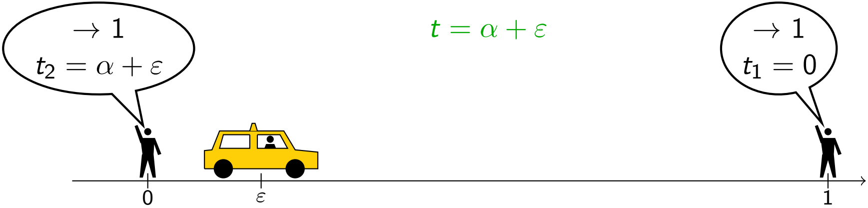

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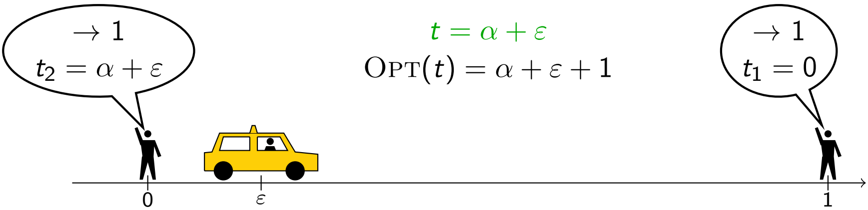

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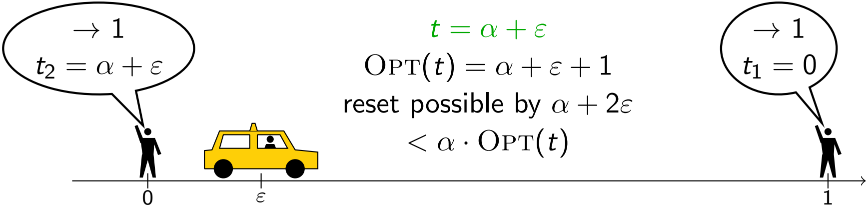

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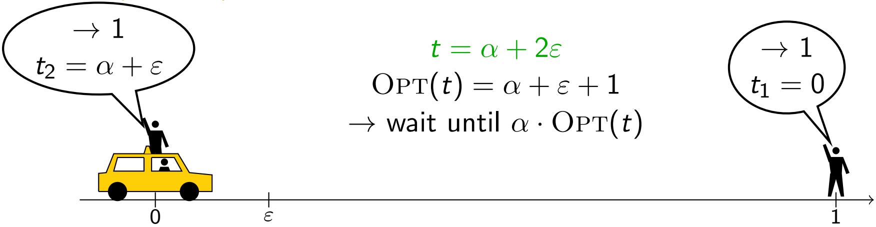

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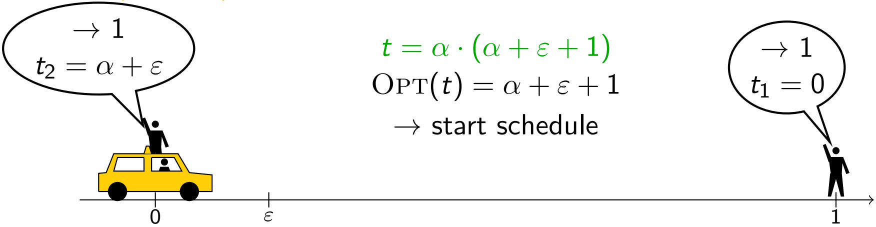

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Example ($\alpha = 1.5$):

$$\text{LAZY}_\alpha = \alpha \cdot (\alpha + \varepsilon + 1) + 1$$
$$\text{OPT} = \alpha + \varepsilon + 1$$

State of the art and our results

Theorem. $\textsc{Lazy}_\alpha$ is

a) 2.457-competitive on every metric space, for $\alpha = 1.457$.
b) 2.366-competitive on the half-line, for $\alpha = 1.366$.
c) There are no better choices for α.

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general	2.05	2.618 [1]	**2.457**
line	2.05 [2]	2.618	2.457
half-line	1.9 [3]	2.618	**2.366**

[1] B., Disser, Mosis, Weckbecker (2022)
[2] Birx, Disser, Schewior (2022)
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$\rightarrow$ key to analysis of $\textsc{Lazy}$: factor-revealing approach

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$\textsc{Lazy}_\alpha(\sigma) \leq (1+\alpha)\textsc{Opt}(\sigma)\ \forall\sigma$

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$\Rightarrow \textsc{Lazy}_\alpha(\sigma) = t^* + s$

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- solve "adversary problem"

$$\max \frac{t^* + s}{\text{Opt}}$$
$$\text{s.t. } t^* \geq \alpha \cdot \text{Opt}$$
$$\vdots$$

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Question: Is Thm implied?

$\textsc{Lazy}_\alpha$
new request revealed:
▶ if possible before $\alpha\textsc{Opt}(t)$: RESET
when idle:
▶ start schedule at $t \geq \alpha \cdot \textsc{Opt}(t)$

- solve "adversary problem"

$$\max \frac{t^* + s}{\textsc{Opt}}$$
$$\text{s.t. } t^* \geq \alpha \cdot \textsc{Opt}$$
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- rescaling σ gives

$$\text{(ALP)} \quad \max\ t^* + s$$
$$\text{s.t. } \textsc{Opt} = 1$$
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Analysis of LAZY

Thm. For $\alpha = 1.457$,
$\text{LAZY}_\alpha(\sigma) \leq (1+\alpha)\text{OPT}(\sigma)\ \forall\sigma$

LAZY_α
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1st step: define suitable variables

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$\to$ add inequality $s \leq 2 \cdot \textsc{Opt}$

Factor-revealing approach [Bienkowski, Kraska, Liu (2021)]

- useful to assemble linear inequalities (for analysis of a fixed algorithm)

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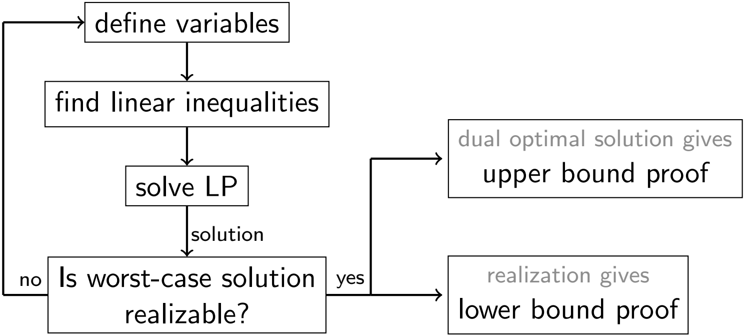

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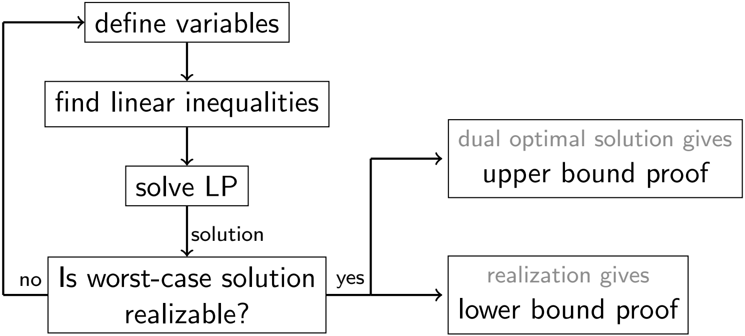

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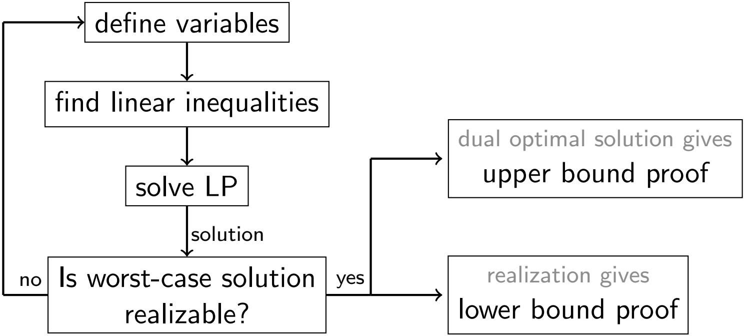

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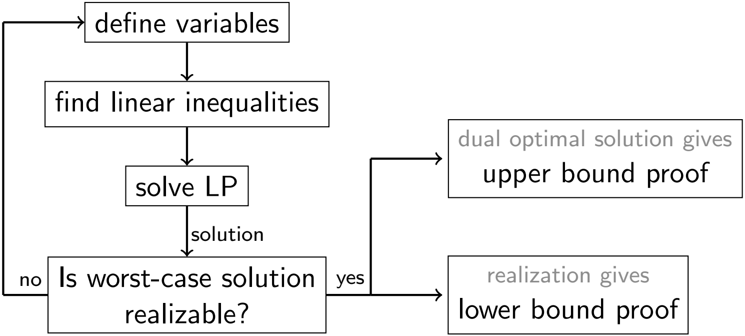

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  → need dual solution for every Branch & Bound node
- our work: purely analytic proof informed by factor-revealing

Takeaways

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  - minimize completion time

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- key to analysis: factor-revealing approach
  - method to assemble linear inequalities