Generalized Partial Vertex Cover

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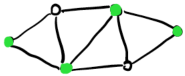
Partial vertex cover

Vertex cover: choose k vertices that cover all the edges.



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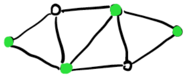
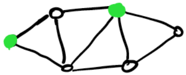


Partial vertex cover: choose k vertices that cover at least c edges.
c is the coverage requirement.

c = 5



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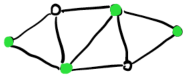

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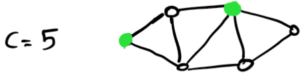
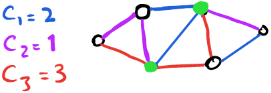


Generalized partial vertex cover: choose k vertices that cover at least c_1 edges from E_1, at least c_2 edges from E_2,..., at least c_m edges from E_m.
E_i is color class i and $E = E_1 \uplus E_2 \ldots \uplus E_m$.



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Known results:

- $O(\log m)$-approx on k where m is the number of color classes. Tight! Bera et al. 2014.
- ≈ 2-approx on k for constant m. Bandyapadhyay et al. 2021.
- find k vertices that cover at least 0.63% of each coverage requirements if m is constant, Chekuri et al. 2009.
- no FPT in k.
- APX-hard even for $m = 1$. So no $(1-\epsilon)$-approx on c_i's in time $f(m, \epsilon) \cdot \mathrm{poly}(n)$.

Theorem (Bandyapadhyay, Friggstad, and M.)

1. *no α-approx for $\alpha \leq 1$ in time $f(k) \cdot n^{o(k)}$ assuming ETH.*
2. *$(1-\epsilon)$-approx on c_i's in time $2^{O(\frac{m \cdot k^2 \cdot \log k}{\epsilon})} \cdot \mathrm{poly}(n)$.*

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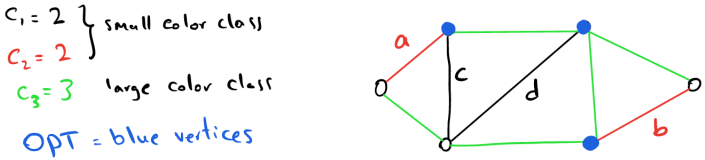
  4. the total number of double counted edges is at most $\binom{k}{2} \leq \epsilon \cdot c$

Our algorithm, small color classes

for small color classes $< \frac{\binom{k}{2}}{\epsilon}$ do label-coding as before and guess the configuration of the vertices in OPT on these color classes.



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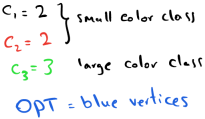

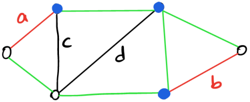

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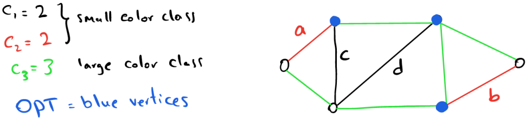

Given the configuration of small color classes in OPT, we use DP to find a solution.

c': the coverage requirements for a subset of large color classes

T: the configuration of some small color classes in OPT.

Order vertices $v_1, v_2, v_3, \ldots, v_n$

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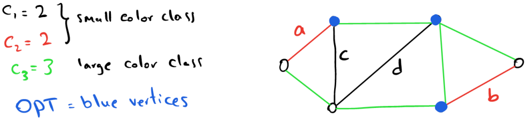

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$g(k', i, T, c') =$ *yes iif there are* k' *vertices among the first* i *vertices compatible with* T *and cover* c' *on large color classes*

Running time

Let's look at our DP table:

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THANK YOU!