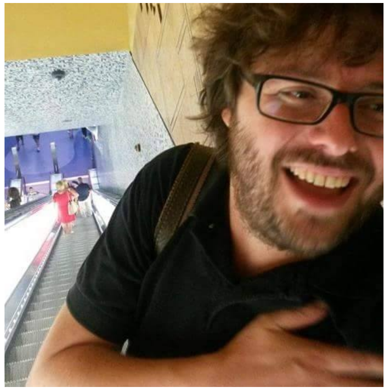
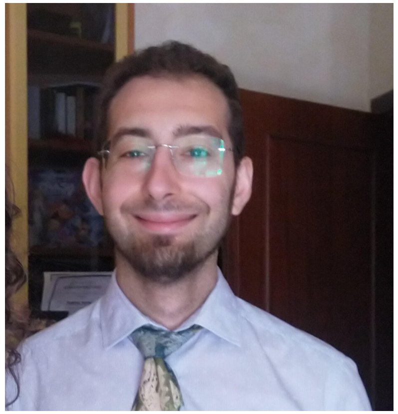
Finding Diameter-Reducing Shortcuts in Trees

Davide Bilò, Luciano Gualà, Stefano Leucci, Luca Pepè Sciarria





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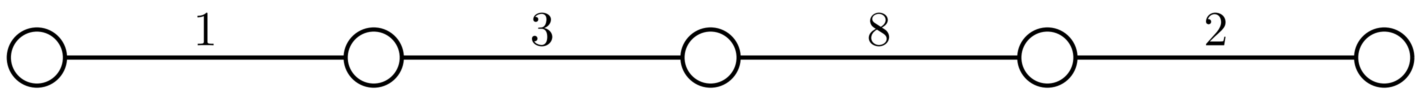
Diameter-Optimally Augmenting a Path

Input:

- A weighted path P with n vertices and non-negative edge costs



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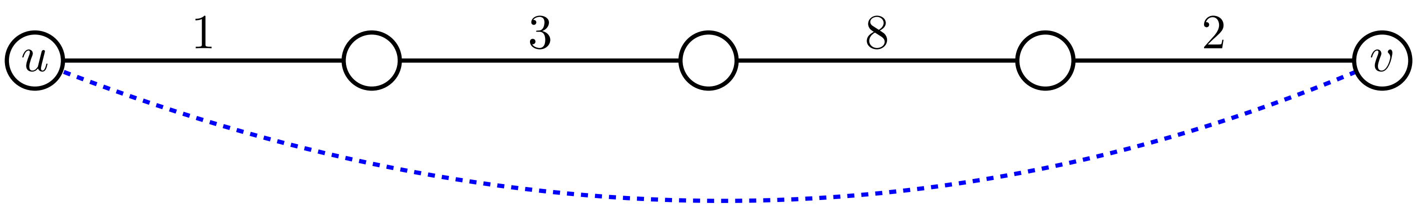

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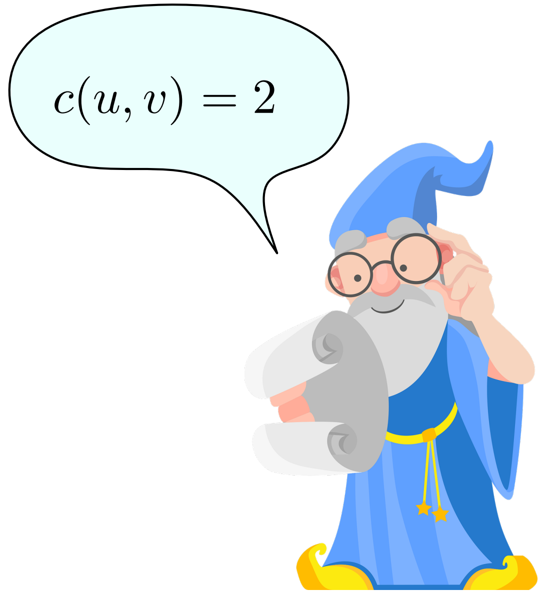

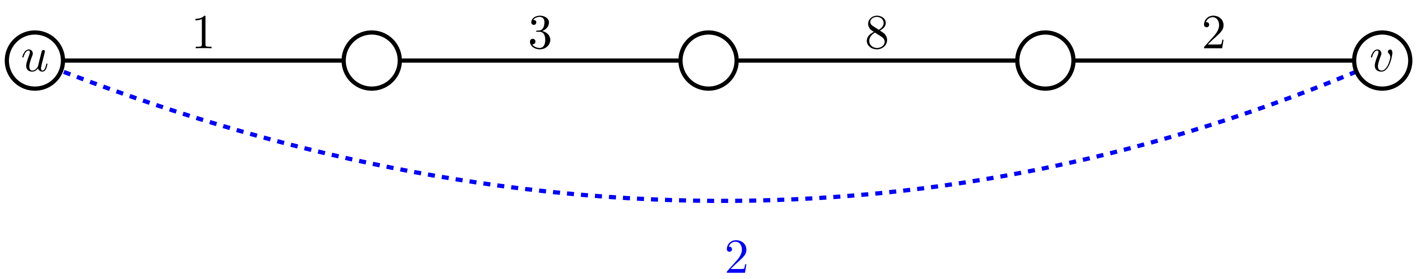

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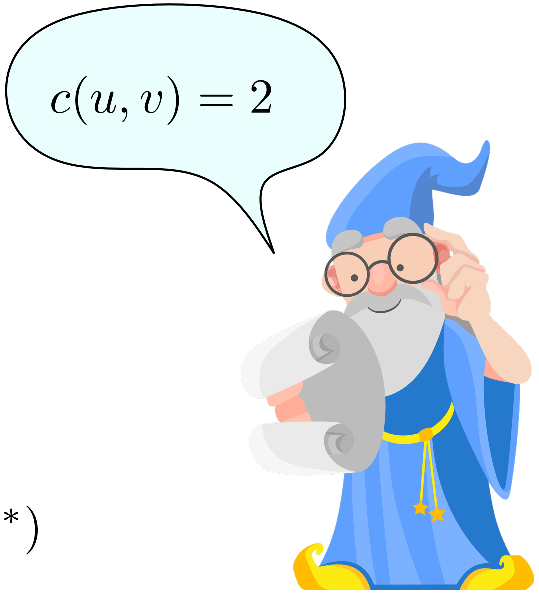

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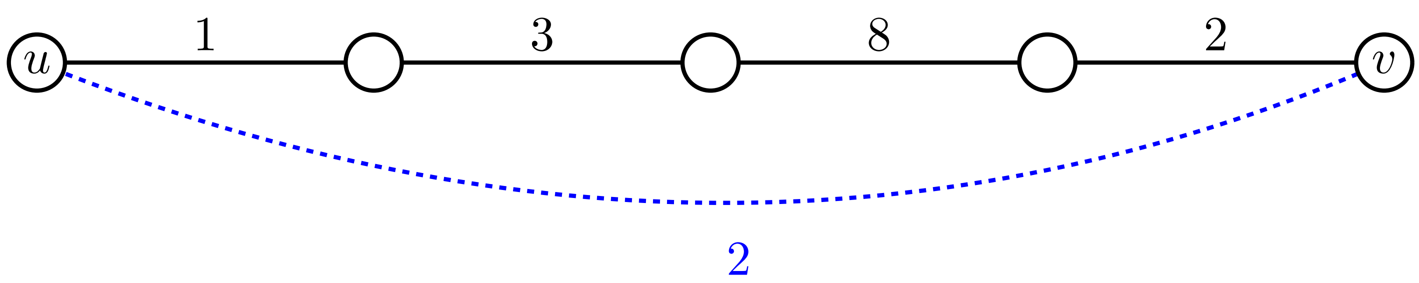

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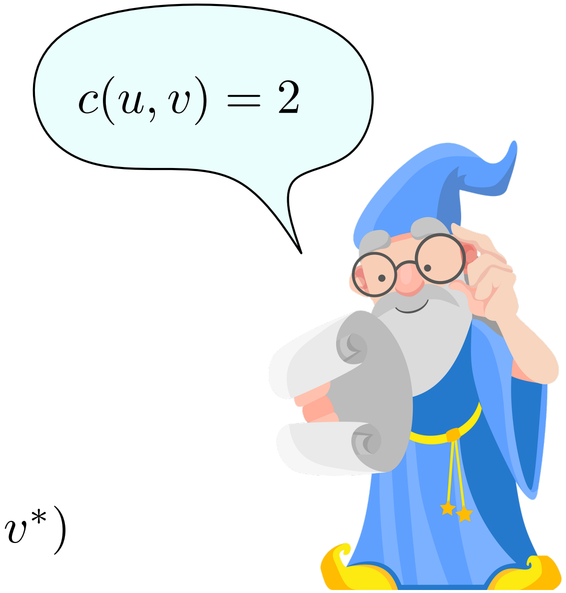

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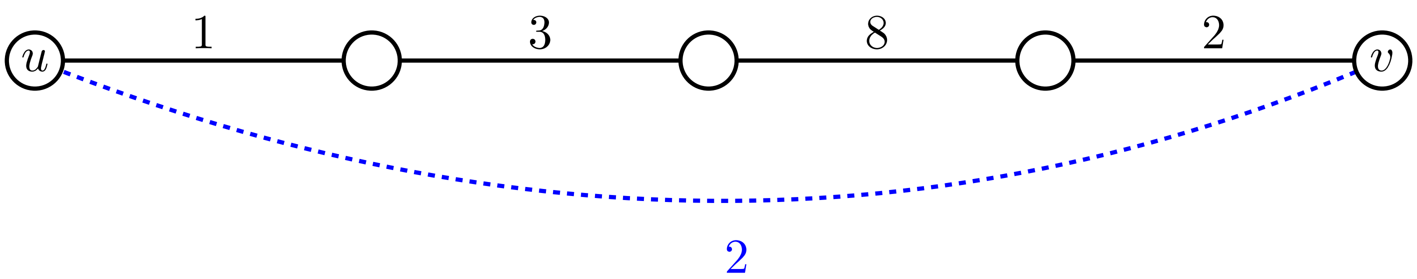
diameter$(P + (u, v))$$= 8$



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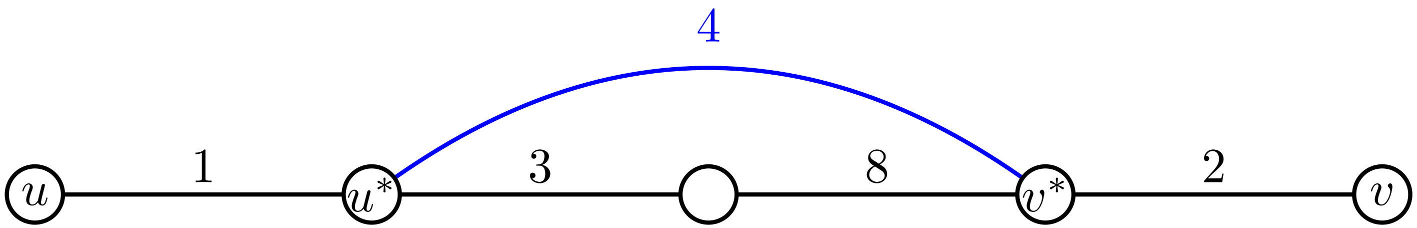
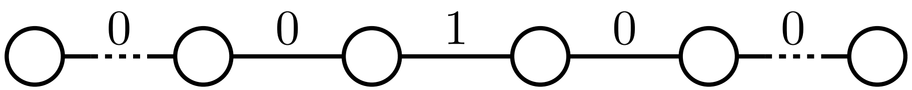


diameter($P + (u, v)$)$= 8$

diameter($P + (u^*, v^*)$)$= 7$

Diameter-Optimally Augmenting a Path

Requires $\Omega(n^2)$ queries/time in general:



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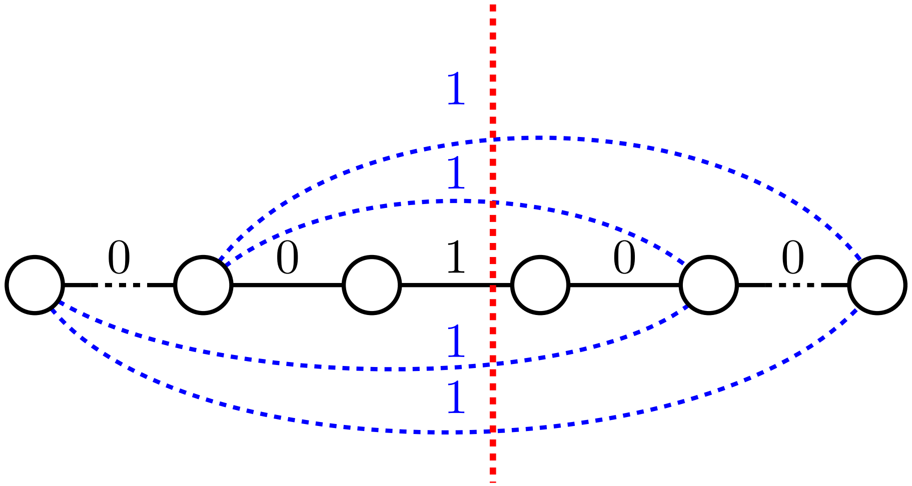

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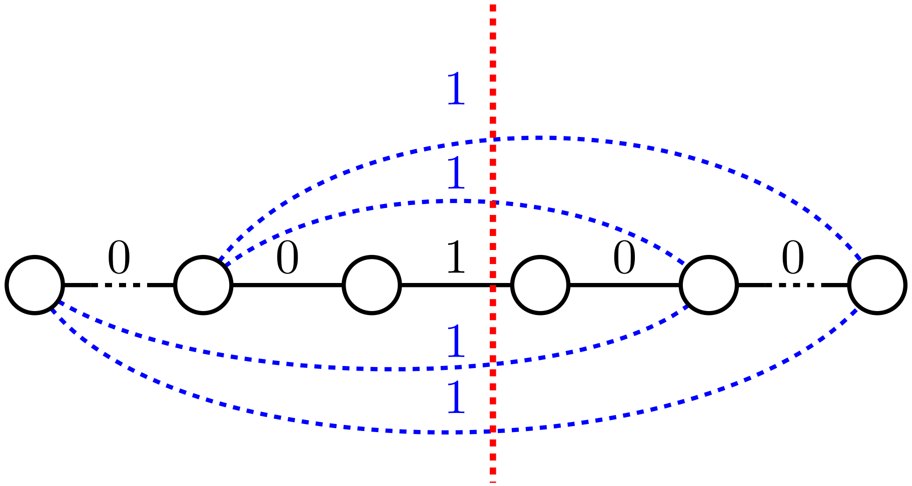

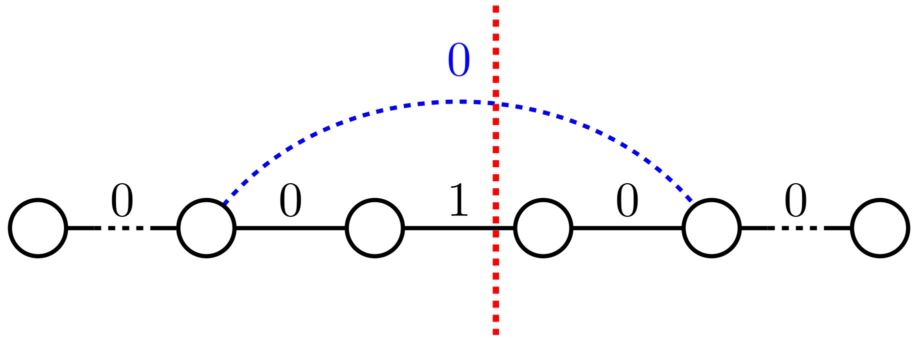

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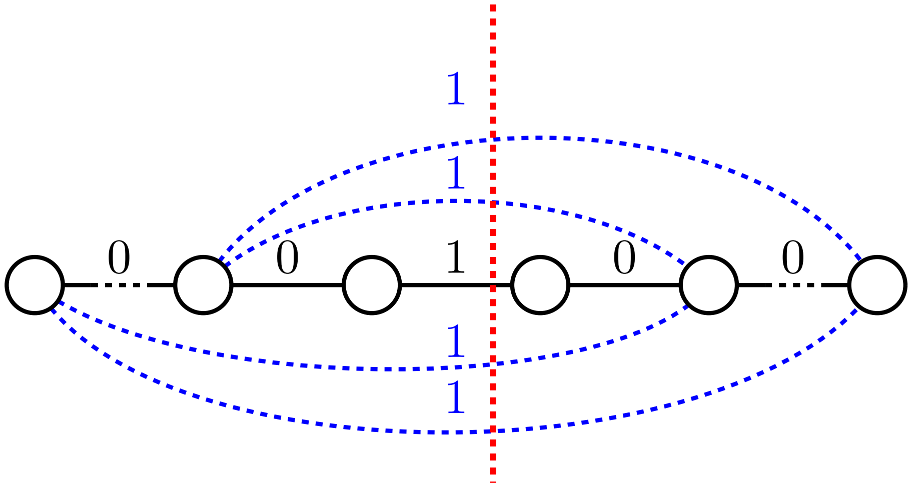

Can be solved in:

- $O(n^2 \log n)$ time, $O(n)$ space [Wang & Zhao TCS 2021]
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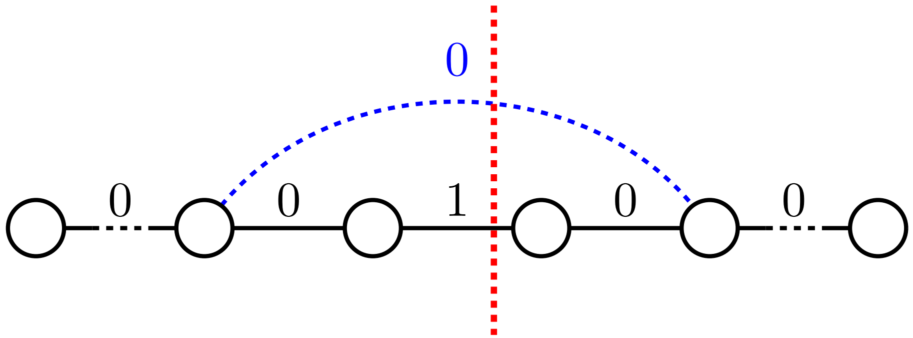

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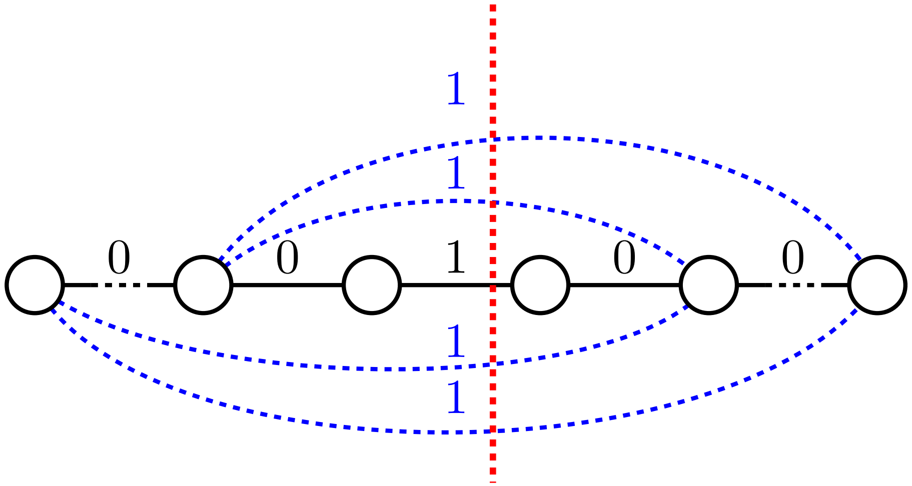

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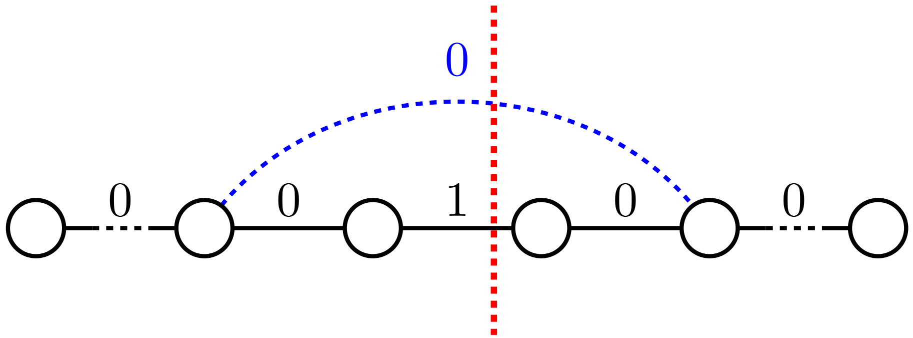

Metric opdimal diameter augmentation: The costs of all edges and non-edges satisfy the triangle inequality

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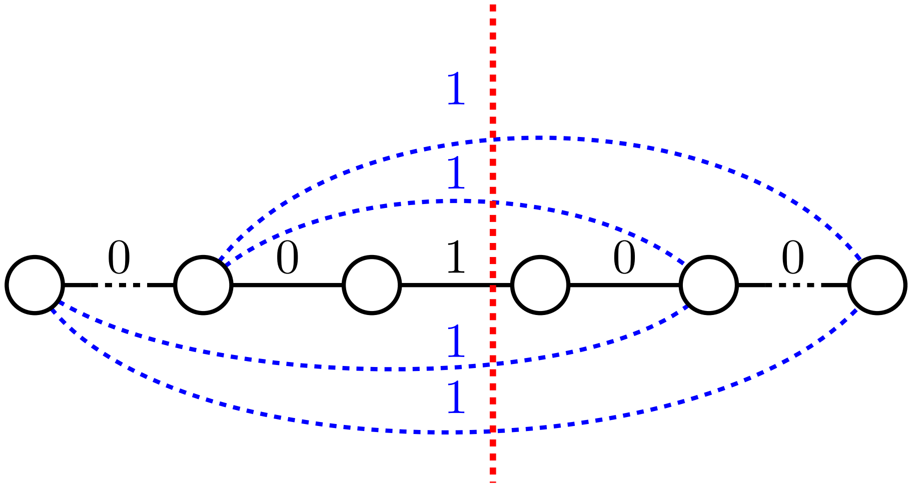

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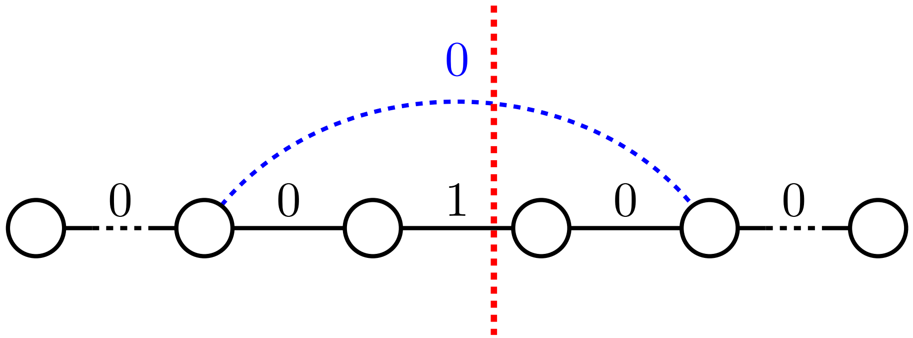

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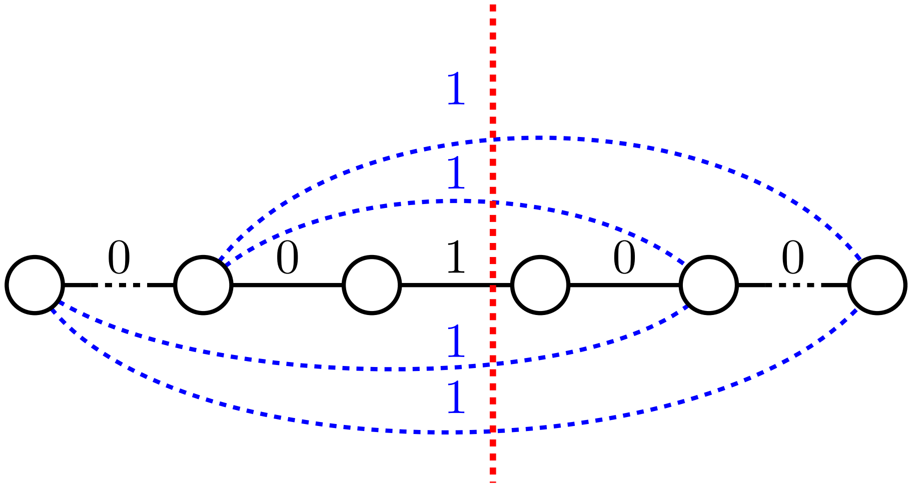

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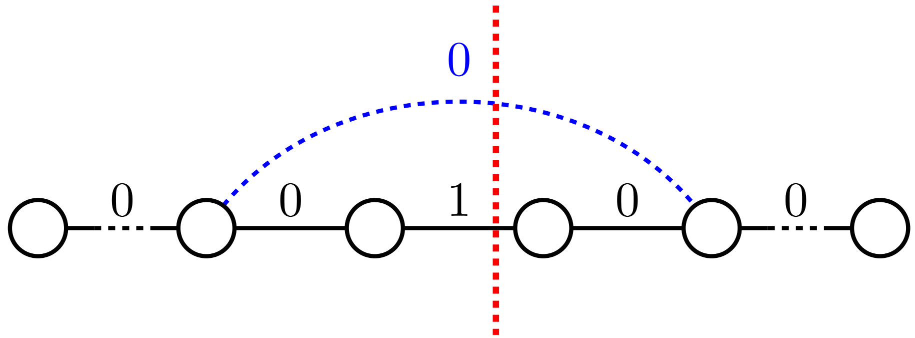

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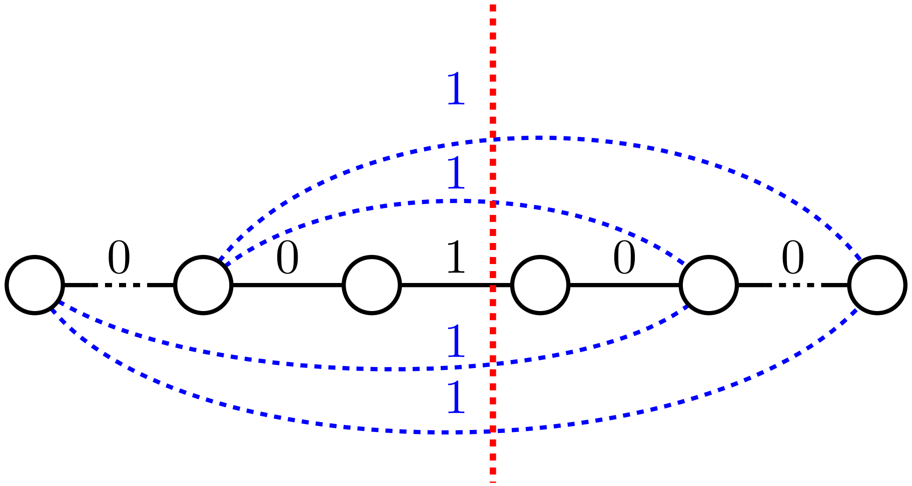

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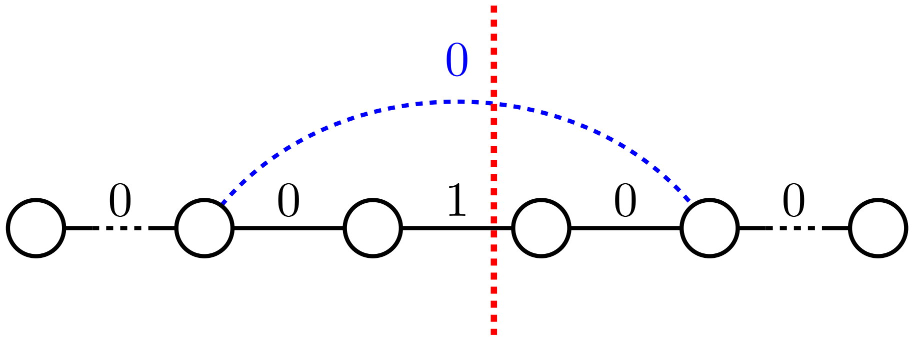

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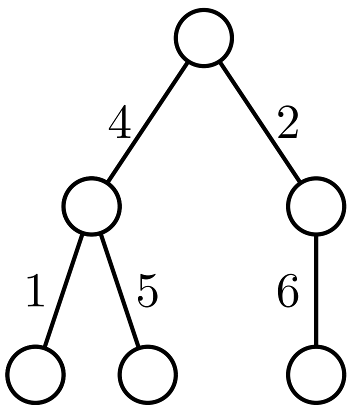
k-Diameter-Optimally Augmenting a Tree

k-DOAT Input:

- A weighted tree T with n vertices and non-negative edge costs
- Acccess to an oracle that can be queried with a *missing* edge (u, v) and reports the cost of (u, v)



$k = 3$

k-Diameter-Optimally Augmenting a Tree

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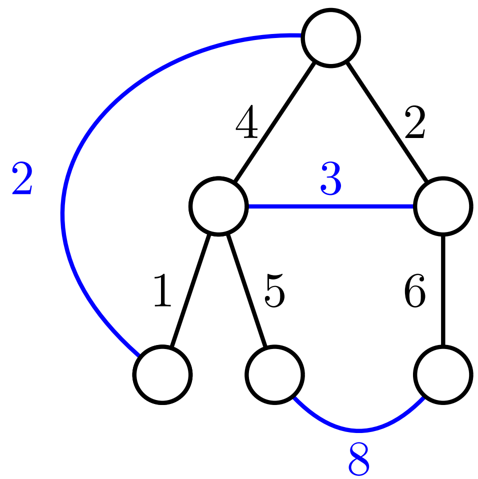
Goal:

- Find a S set of at most k shortcuts that minimizes the diameter of $T + S$



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Our Results

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- For $k \geq 3$: Any $< \frac{10}{9}$-approximation algorithm uses $\Omega(n^2)$ queries.

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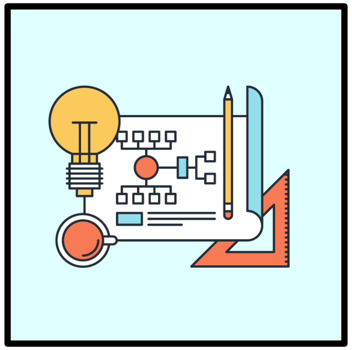

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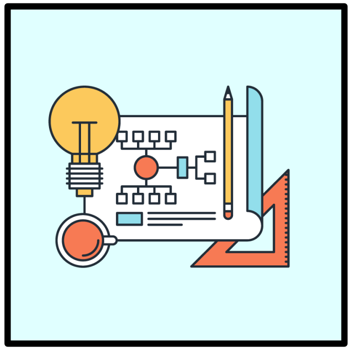

Our Lower Bound for $k \geq 3$

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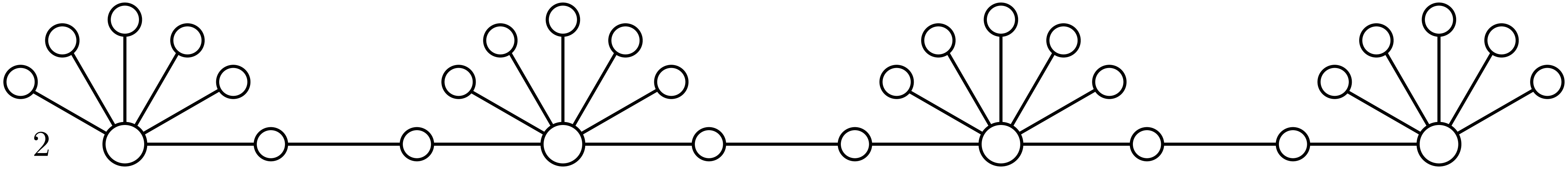
Consider $k = 3$ for simplicity

Our Lower Bound for $k = 3$

Instance $\mathcal{I}$:



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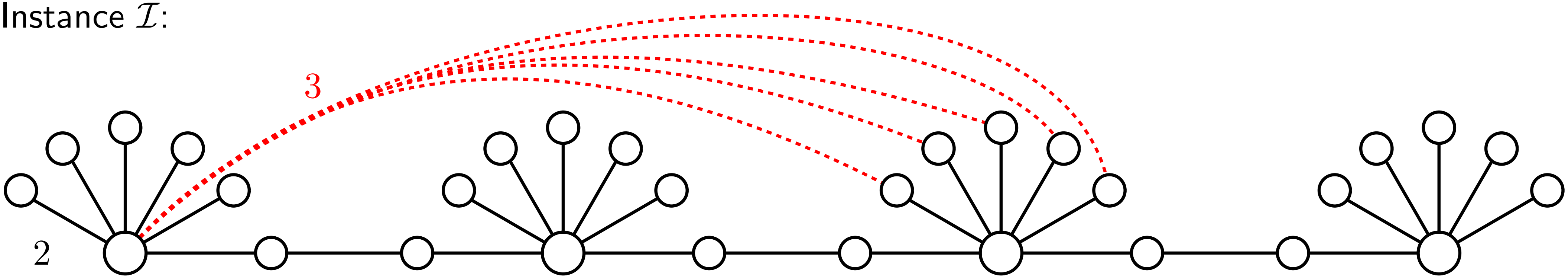

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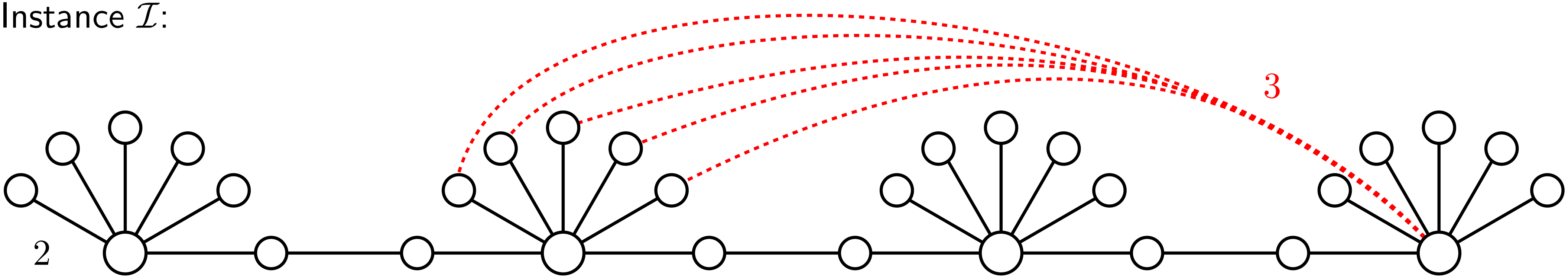

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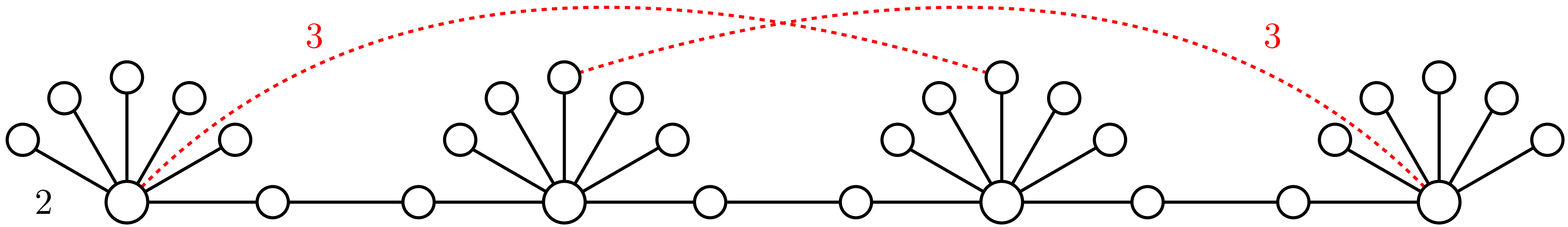

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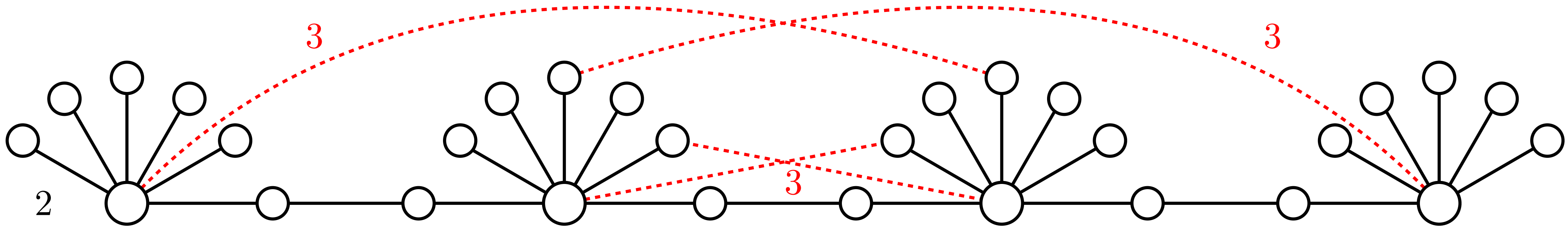

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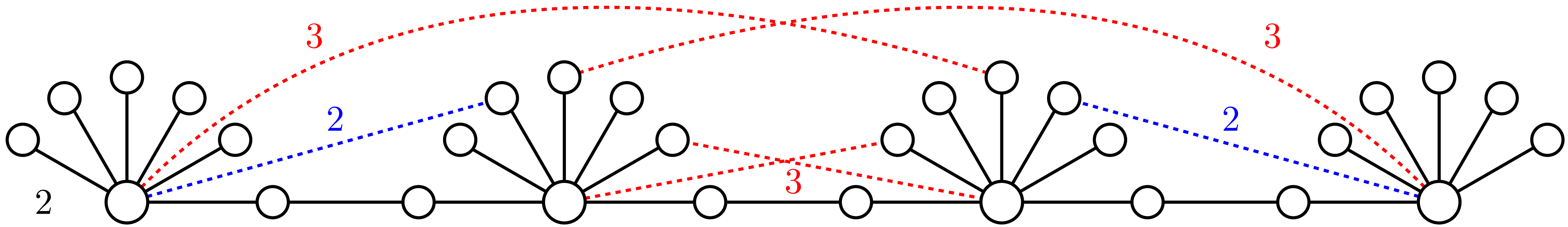

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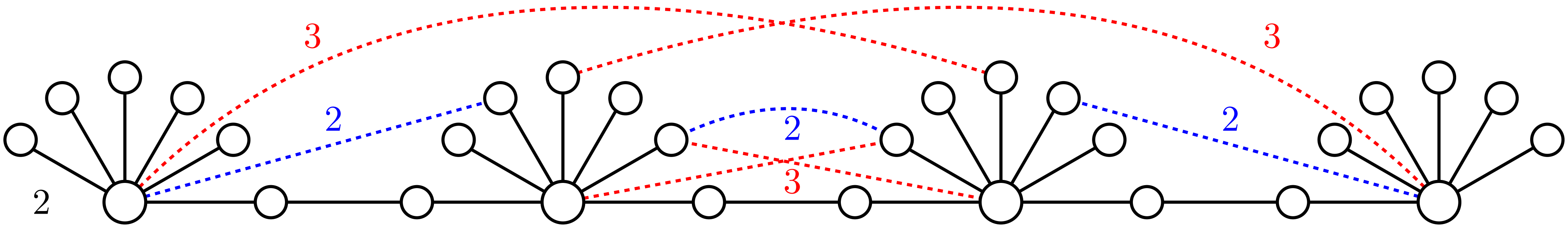

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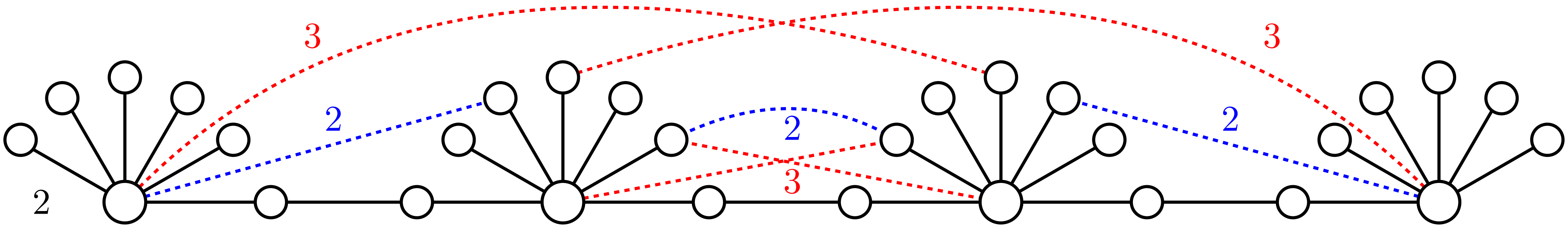

$+$ Metric closure

Our Lower Bound for $k = 3$

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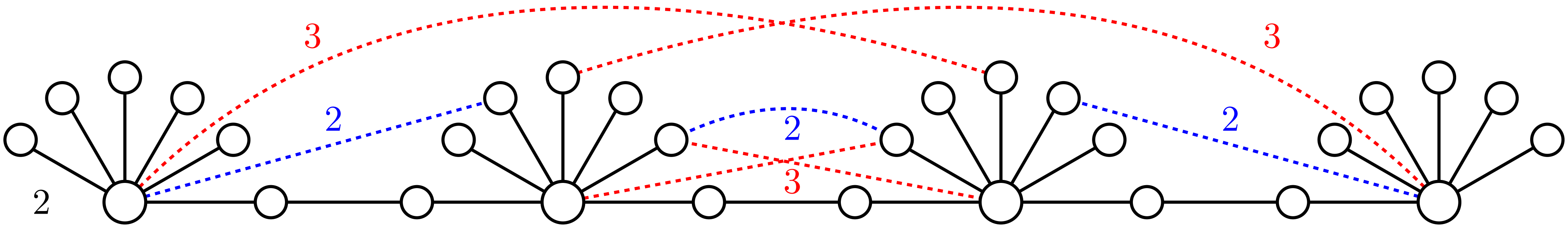
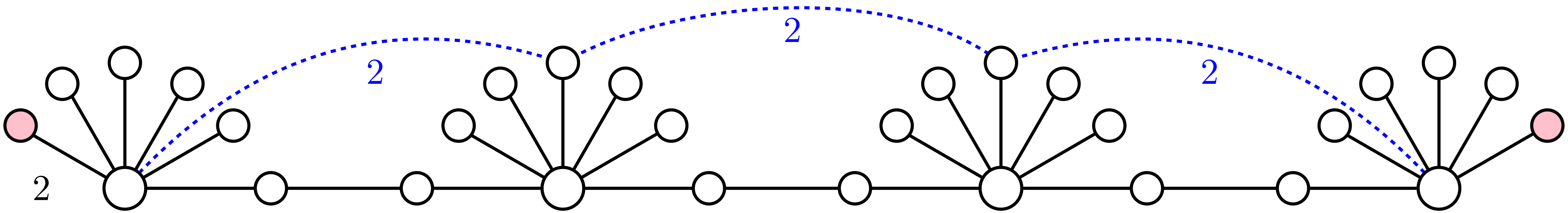


$+$ Metric closure

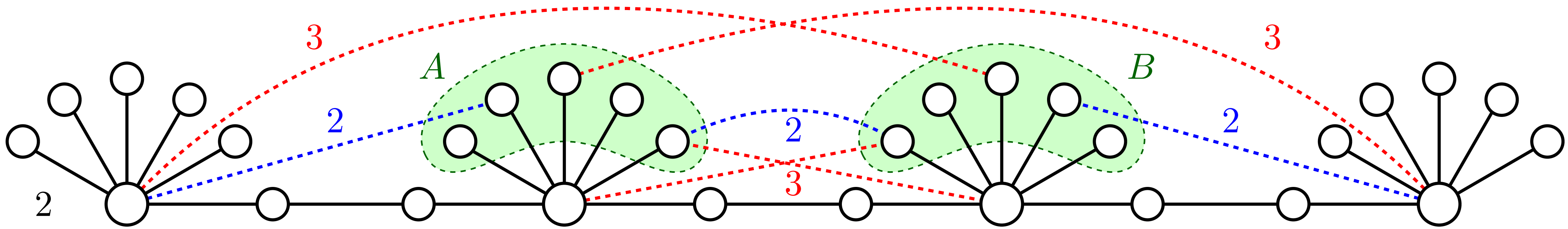
For any set S of ≤ 3 shortcuts, diam$(T + S) \geq 10$



Our Lower Bound for $k = 3$

Pick $a \in A$ and $b \in B$

Instance $\mathcal{I}_{a,b}$:



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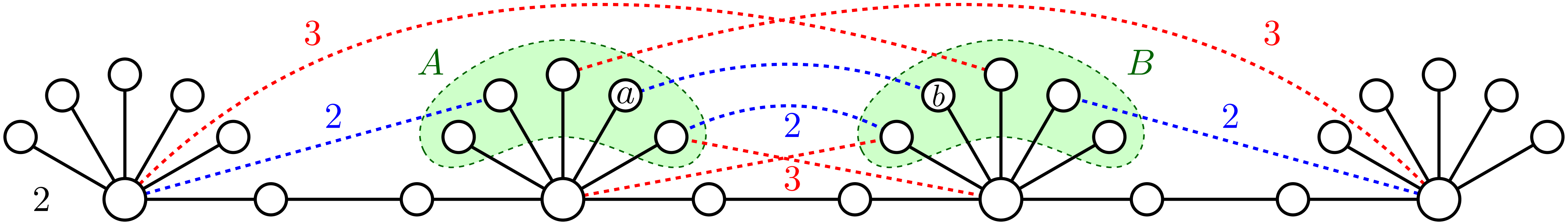
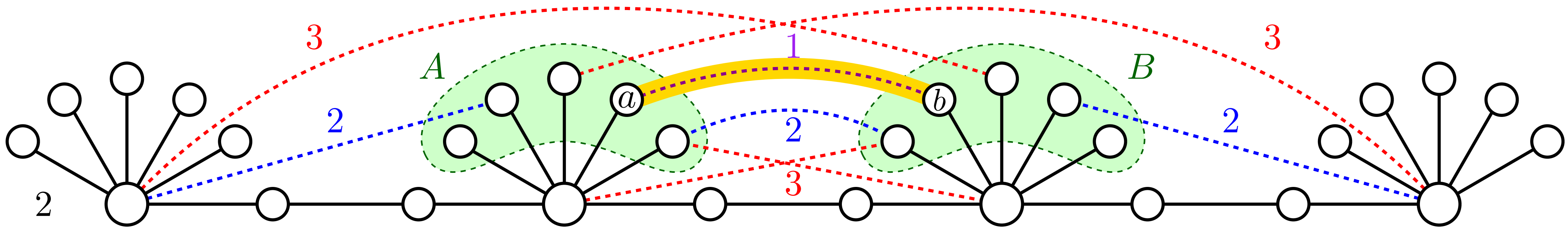


Our Lower Bound for $k = 3$

Pick $a \in A$ and $b \in B$ and lower the cost of (a, b) to 1

Instance $\mathcal{I}_{a,b}$:



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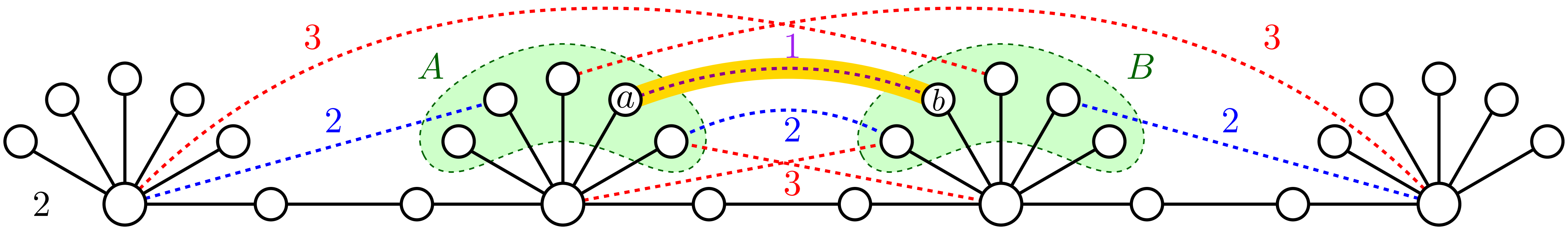

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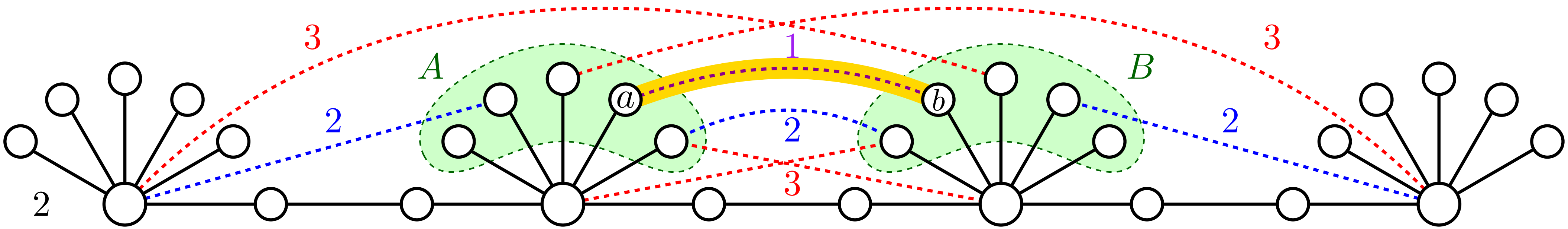
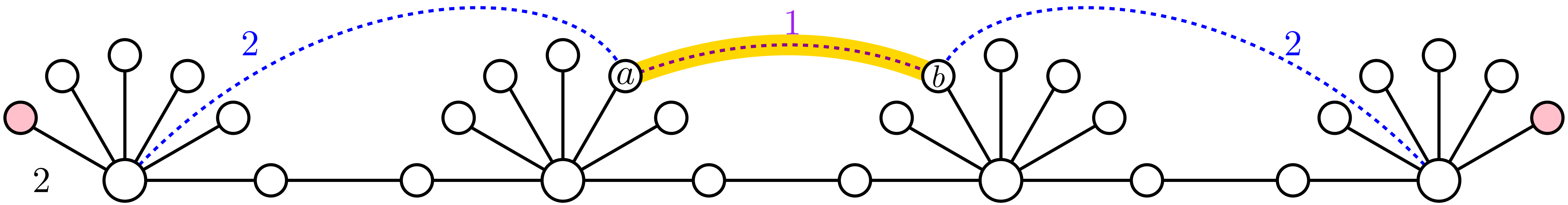


$+$ Metric closure

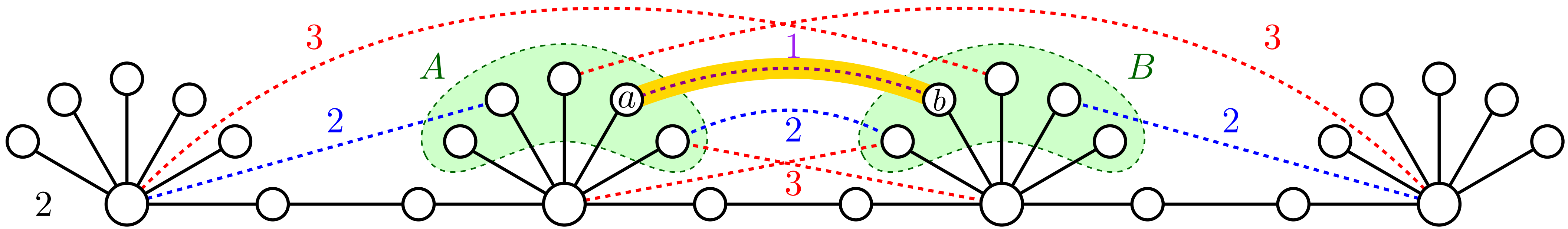
There is a set S of 3 shortcuts such that $\text{diam}(T + S) = 9$



Our Lower Bound for $k = 3$

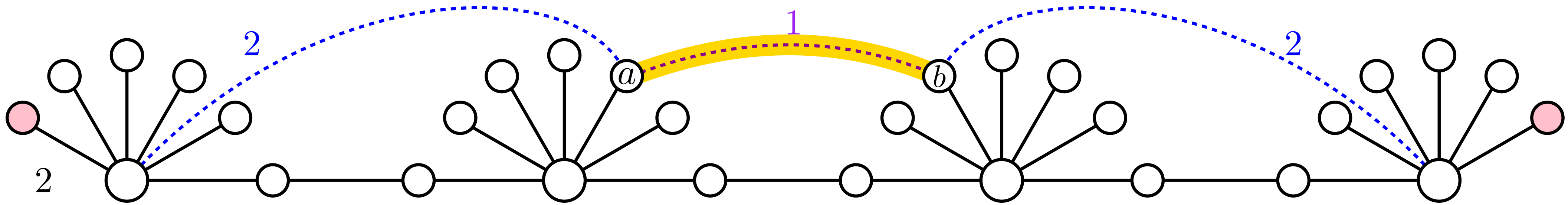
Pick $a \in A$ and $b \in B$ and lower the cost of (a, b) to 1

Instance $\mathcal{I}_{a,b}$:



$+$ Metric closure

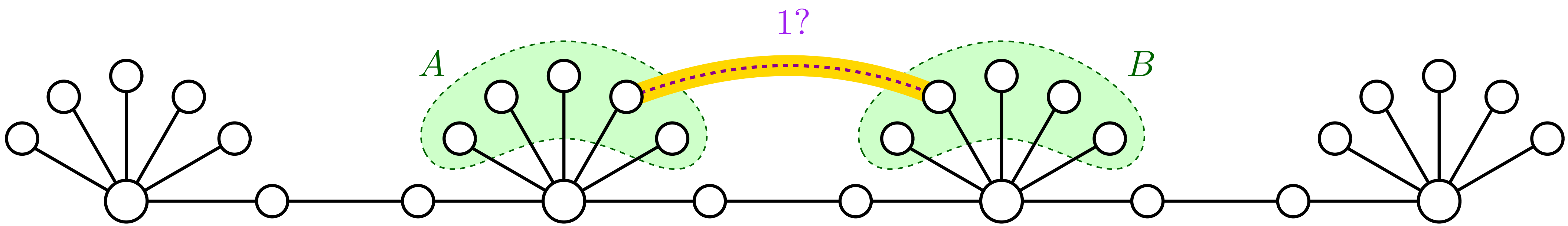
There is a set S of 3 shortcuts such that $\text{diam}(T + S) = 9$



Key Property: the cost all edges, except for (a, b) is the same in $\mathcal{I}$ and $\mathcal{I}_{a,b}$

Our Lower Bound for $k = 3$

Input: Either $\mathcal{I}$ or $\mathcal{I}_{a,b}$ for some $(a, b) \in A \times B$



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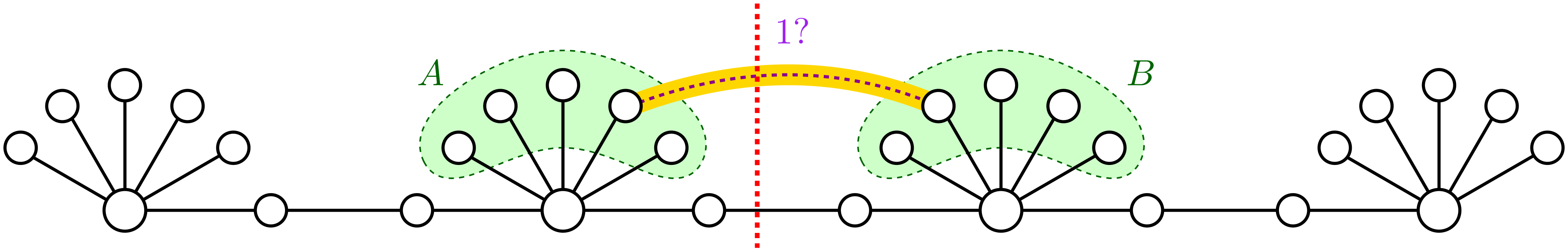

An algorithm that solves 3-DOAT needs to tell $\mathcal{I}$ apart from $\mathcal{I}_{a,b}$

... which requires querying all edges in $A \times B$

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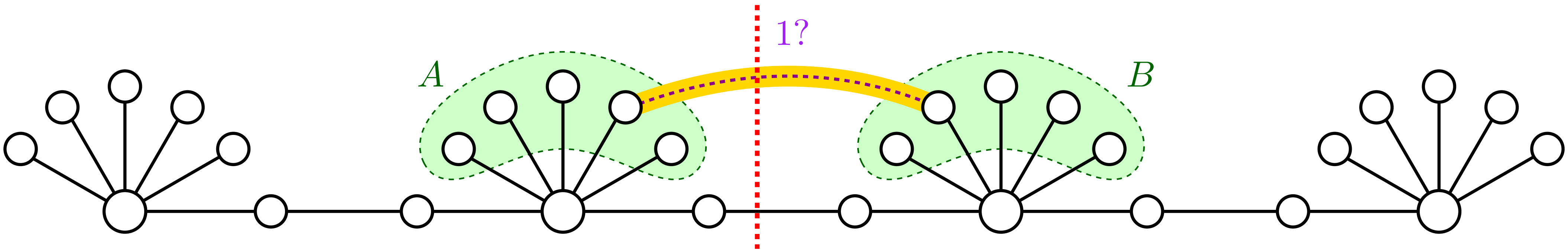

An algorithm that solves 3-DOAT needs to tell $\mathcal{I}$ apart from $\mathcal{I}_{a,b}$

... which requires querying all edges in $A \times B$

Actually shows: there is no $o(n^2)$-queries/time σ-approximation algorithm with $\sigma < \frac{10}{9}$

Our Exact Algorithm

(Speeding up the) Naive Strategy

- For every possible set S of k shortcuts: $O(n^{2k})$ choices
  - Compute the diameter of $T + S$ $\widetilde{O}(n^2)$ time

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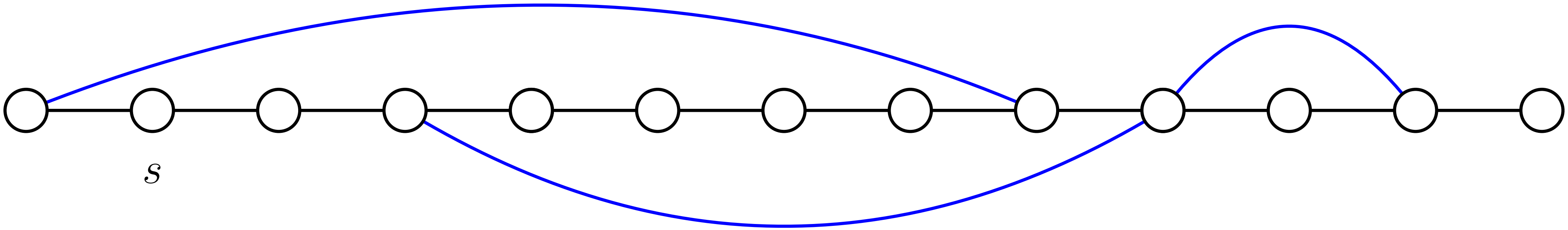
Some remarks:

- Saves a $\Theta(n)$ factor
- Computing the diameter of a graph with $(n-1) + k$ edges is interesting in its own regard

Warm-up: Diameter of an Augmented Path $P + S$



- For every *source* vertex s — $O(n)$ choices
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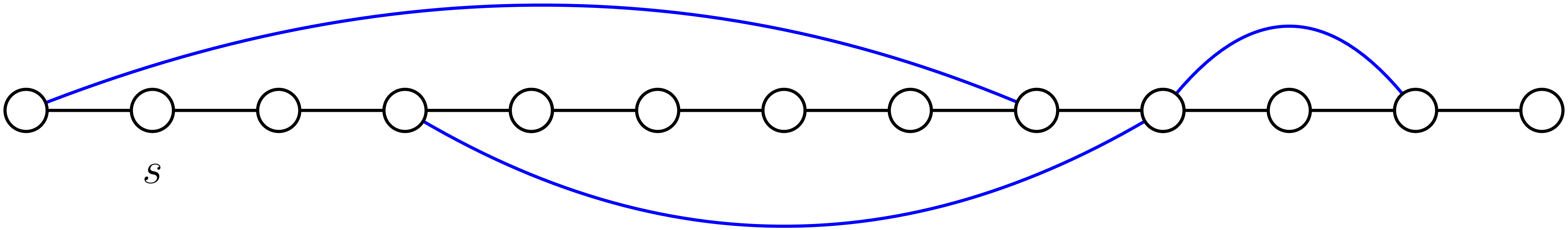

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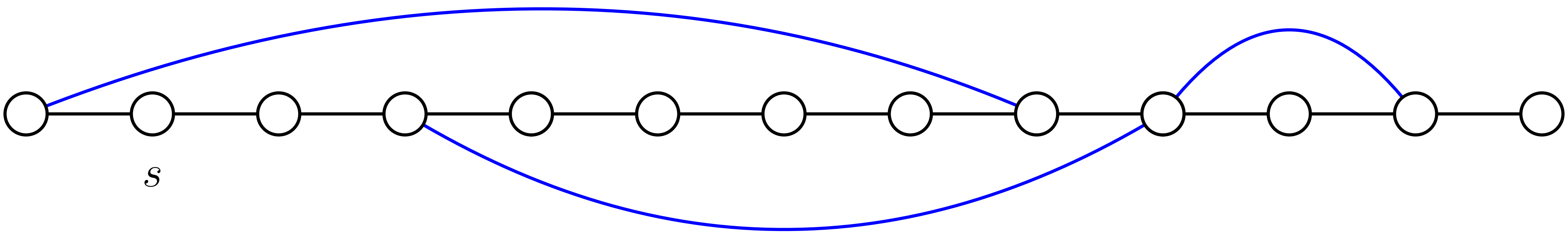

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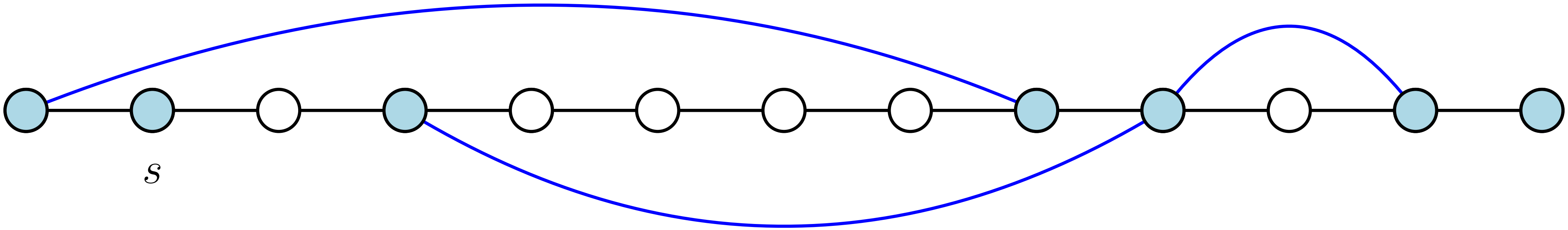

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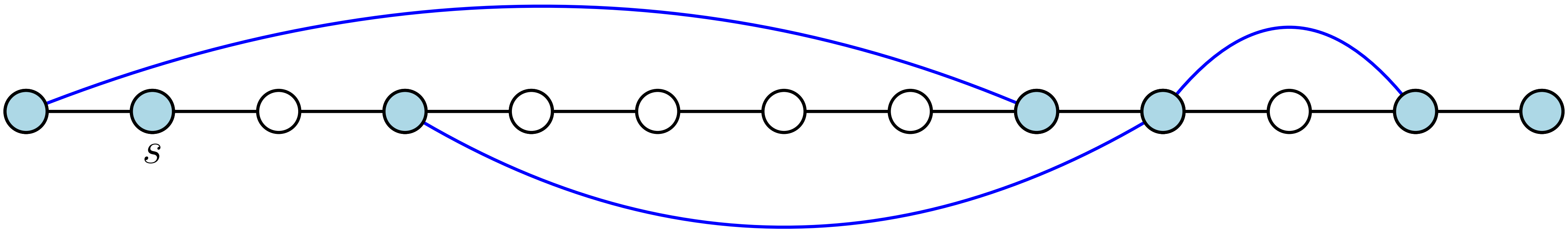
  - Compute the eccentricity $\mathcal{E}(s)$ of s in $P + S$ — Want: $\widetilde{O}(k)$ time — **How?**

Mark s, the endpoints of the path, and all endpoints of some shortcut as **terminals**

Idea: if we know the distances from s to the **terminals**, we can quickly compute all other distances from s

Warm-up: Diameter of an Augmented Path $P + S$

Subgoal: Quickly find all distances $\alpha_v = d_{P+S}(s, v)$ between s and all **terminals** v in $P + S$



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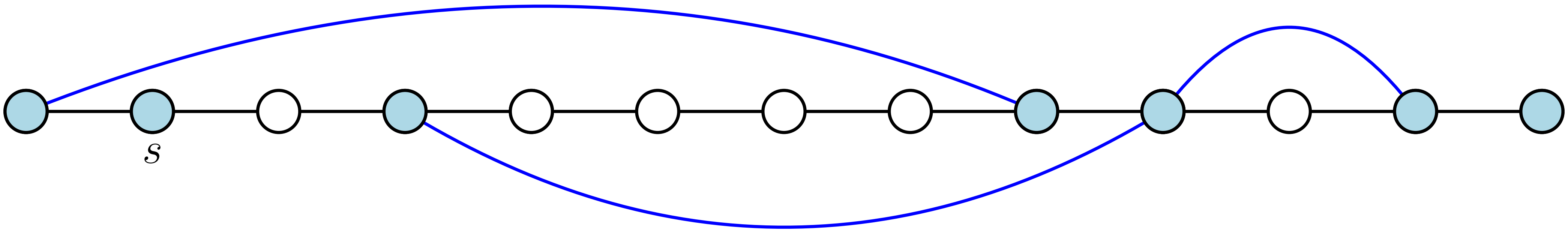

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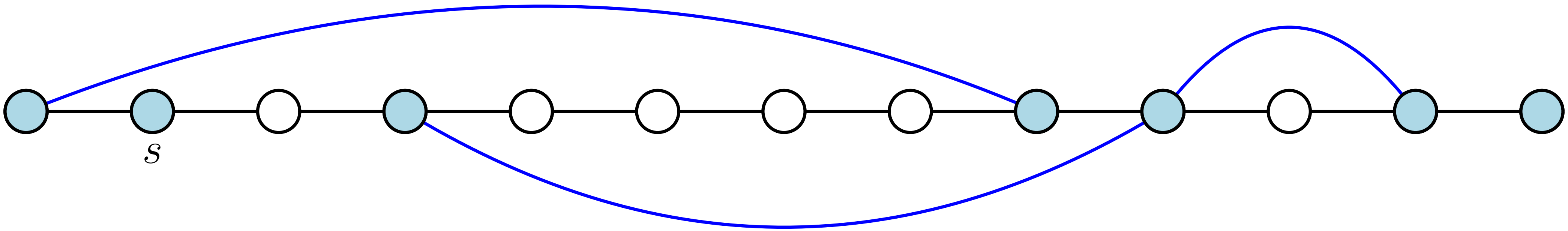
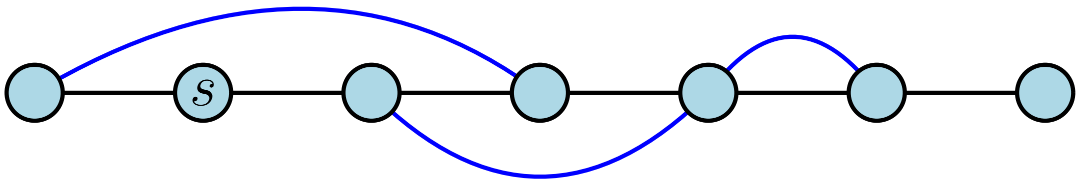


"**Shrink**" $P + S$ into H



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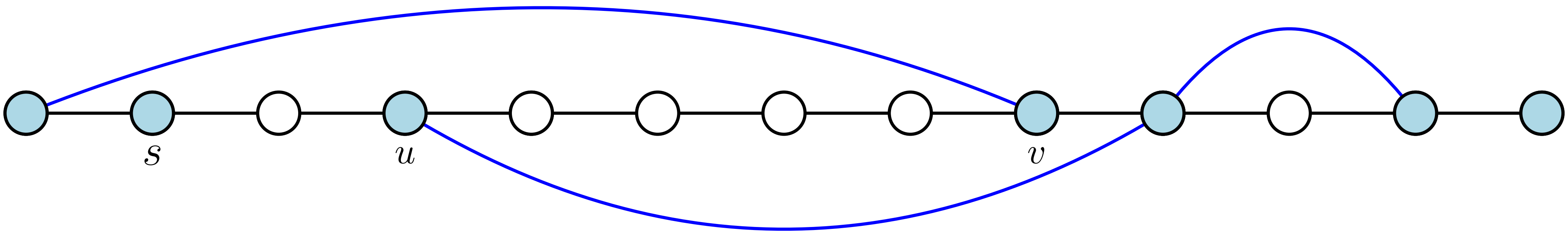

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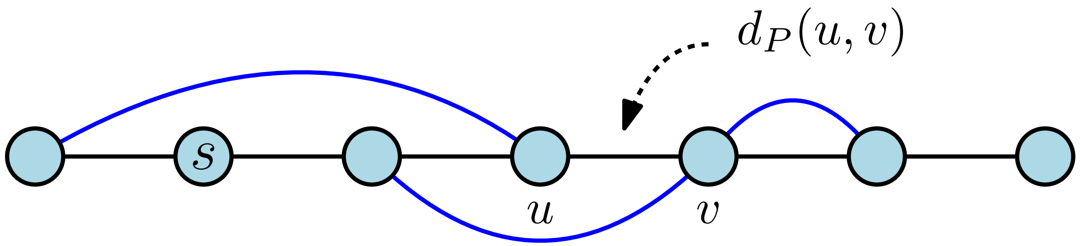

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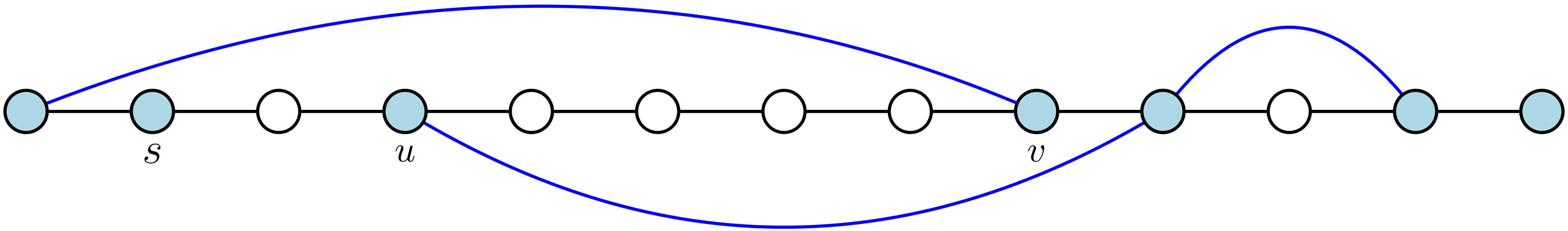

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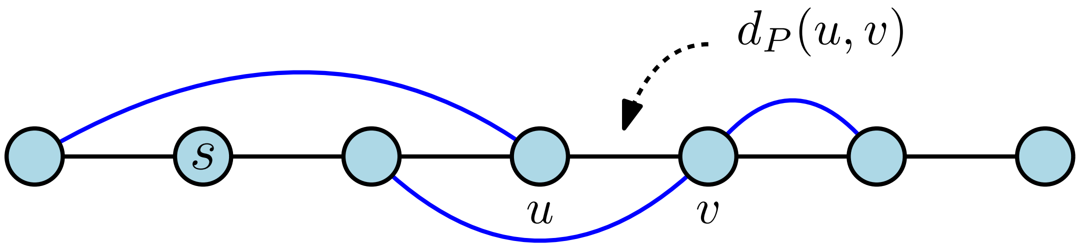

$O(k)$ vertices and $O(k)$ edges.

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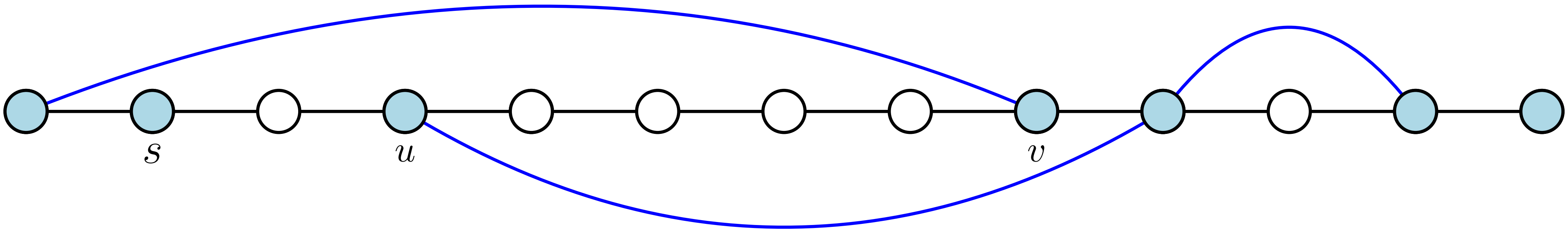

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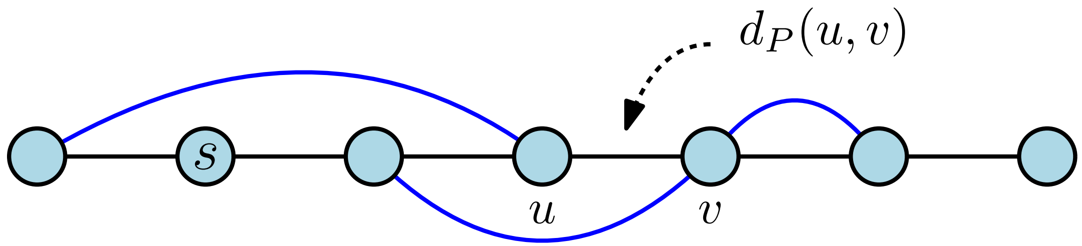
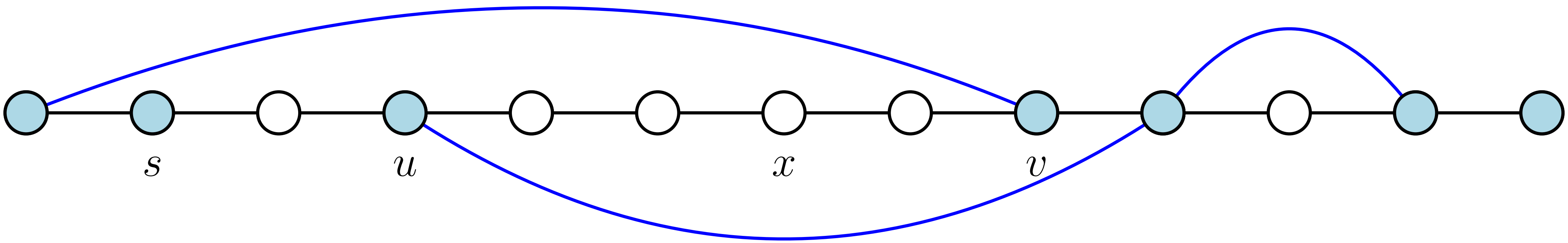


$O(k)$ vertices and $O(k)$ edges.

Compute $\alpha_v = d_H(s, v)$ for all terminals v in time $O(k \log k)$ using Dijkstra's algorithm .

Warm-up: Diameter of an Augmented Path $P + S$

What about the other vertices?



The shortest path to a non-terminal vertex x must pass through one of its two neighboring terminal vertices u, v

Warm-up: Diameter of an Augmented Path $P + S$

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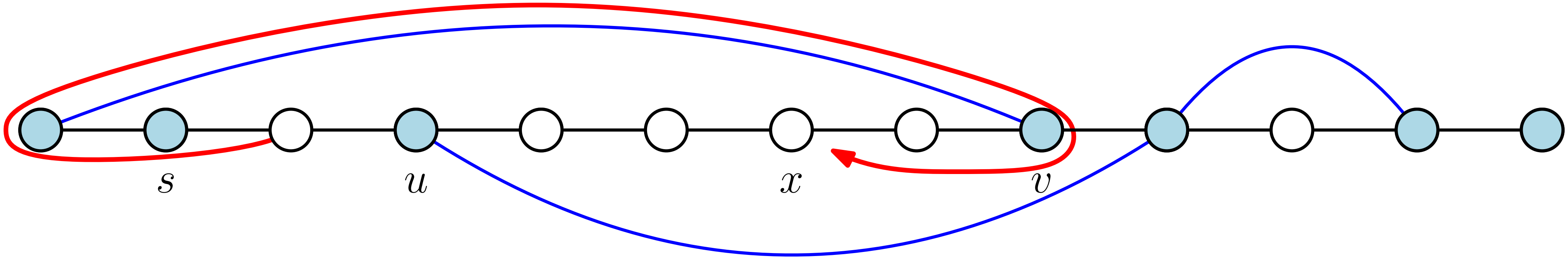

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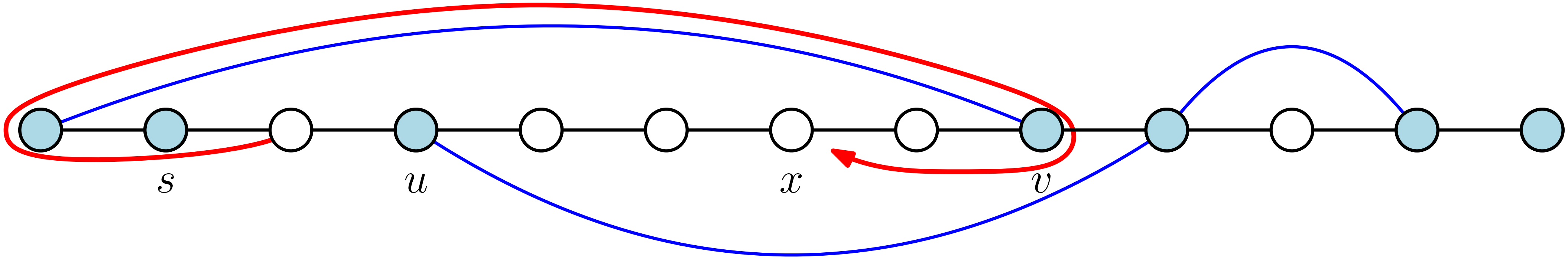

The shortest path to a non-terminal vertex x must pass through one of its two neighboring terminal vertices u, v

$$d_{P+S}(s, x) = \min \begin{cases} \alpha_u + d_P(u, x) \\ \alpha_v + d_P(v, x) \end{cases}$$

Warm-up: Diameter of an Augmented Path $P + S$

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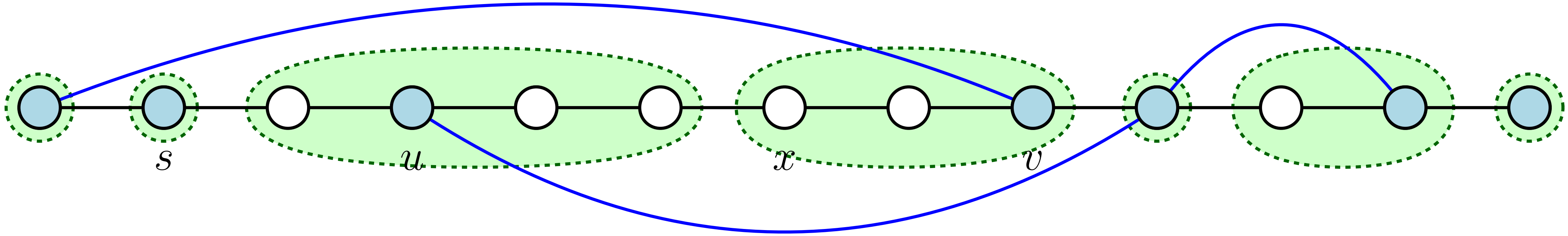

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Assign each node of P to its "closest" terminal

Warm-up: Diameter of an Augmented Path $P + S$

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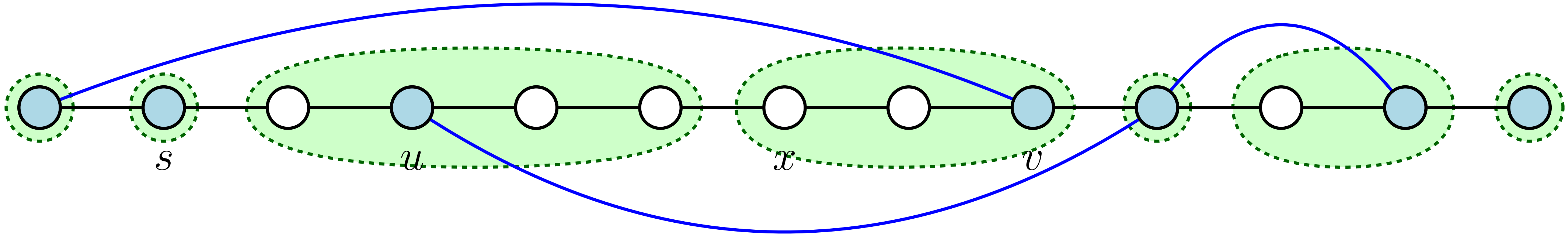

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For each terminal v, we are interested in the distance $\mathcal{E}_v$ from v to the farthest node assigned to v

The eccentricity of s is $\mathcal{E}(s) = \max_{\text{terminal } v} (\alpha_v + \mathcal{E}_v)$

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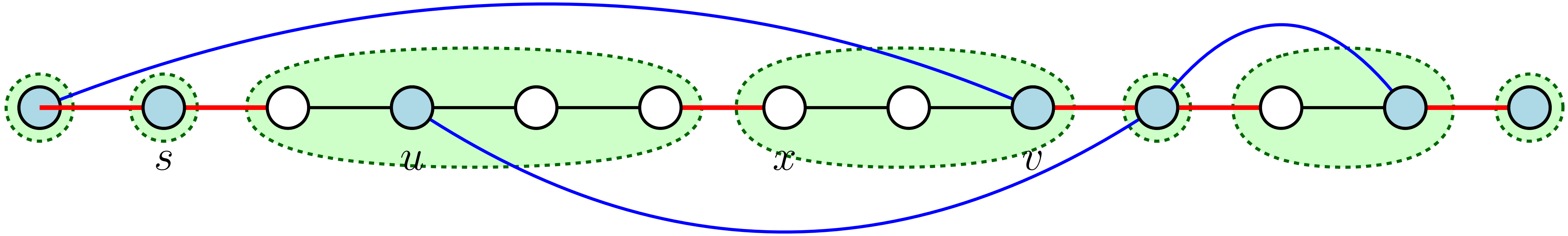

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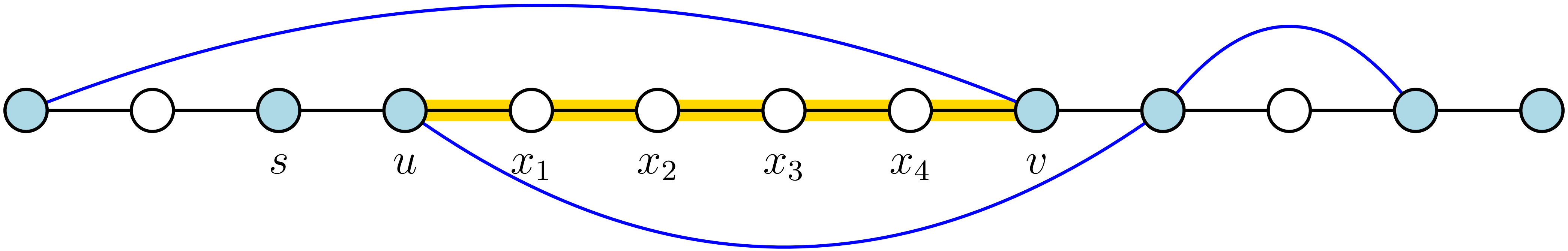
Observation: It suffices to **quickly** find the *boundary* edges

Warm-up: Diameter of an Augmented Path $P + S$

For each subpath between two consecutive terminal vertices u, v:



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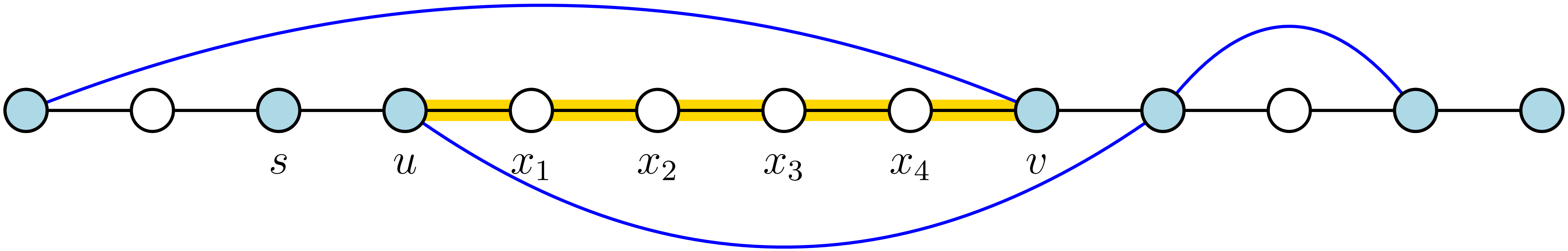

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Monotonically non-decreasing w.r.t. i

Monotonically non-increasing w.r.t. i

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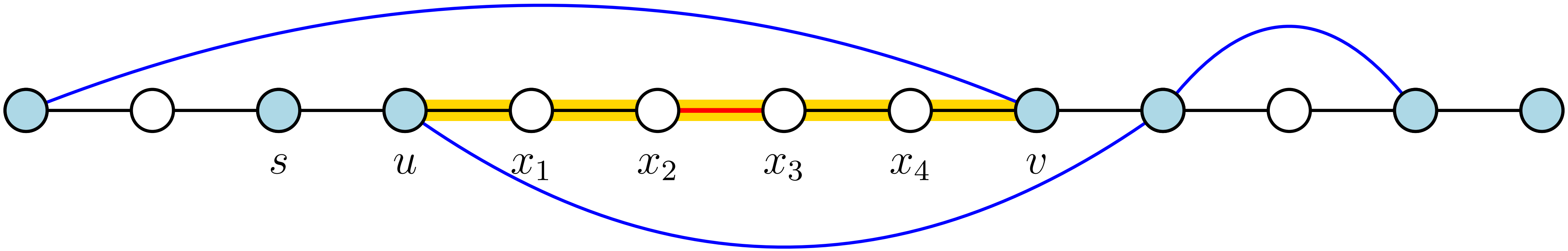

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Monotonically non-decreasing w.r.t. i

Monotonically non-increasing w.r.t. i

Find the cross-over point via binary search in time $O(\log n)$.

From Paths to Trees

- Mark terminals
- $H \leftarrow$ Compress graph
- Compute α_vs on H
- For each terminal v:
  - Find $\mathcal{E}_v$ via binary search
- Return $\max_v (\alpha_v + \mathcal{E}_v)$

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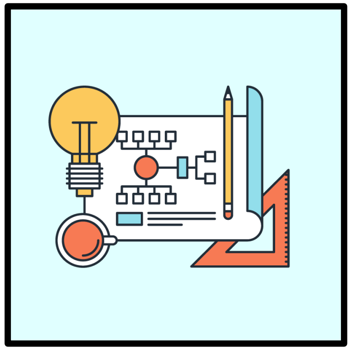

Auxiliar Data
Structure
DS

From Paths to Trees

- DS $\leftarrow$ `Build`(T)
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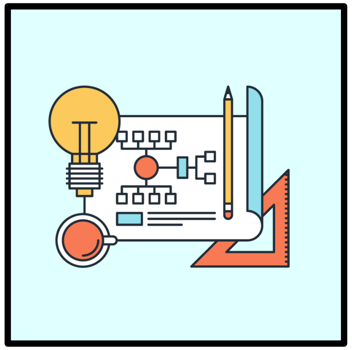

Auxiliar Data
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Build(T): Initializes the data structure on the tree T $O(n)$

From Paths to Trees

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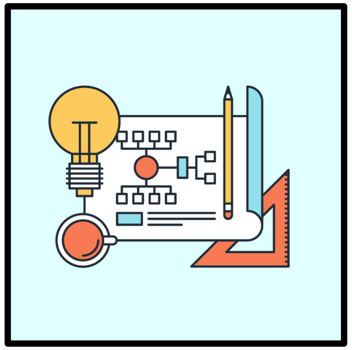

Auxiliar Data
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Build(T): Initializes the data structure on the tree T $O(n)$

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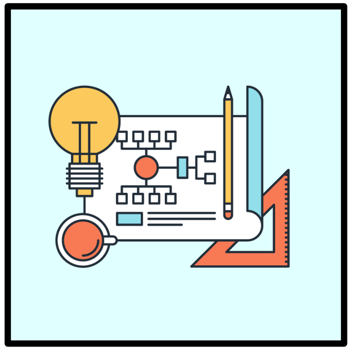

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Build(T): Initializes the data structure on the tree T $O(n)$

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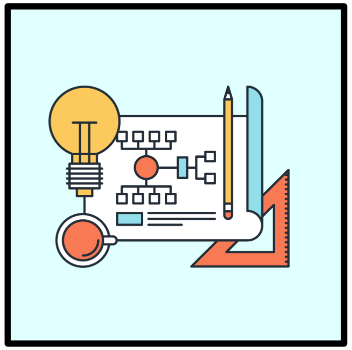

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SetAlpha(v, α_v): Assigns a weight $\alpha_v \geq 0$ to vertex v $O(1)$

From Paths to Trees

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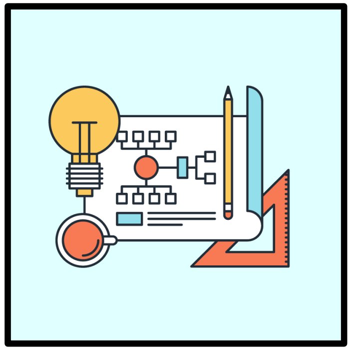
- Return `DS.ReportFarthest`$()$



Auxiliar Data Structure DS

Build(T): Initializes the data structure on the tree T	$O(n)$
MakeTerminal(v): Marks vertex v as a terminal vertex	$O(\log n)$
Shrink$()$: Returns a compact representation of T that contains all terminals	$O(\#\text{ terminals})$
SetAlpha(v, α_v): Assigns a weight $\alpha_v \geq 0$ to vertex v	$O(1)$
ReportFarthest$()$: Return the vertex that is "farthest" from all terminals	$O(\#\text{ terminals} \cdot \log n)$

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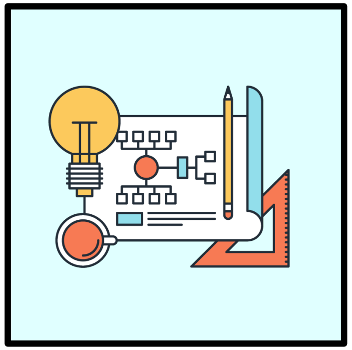

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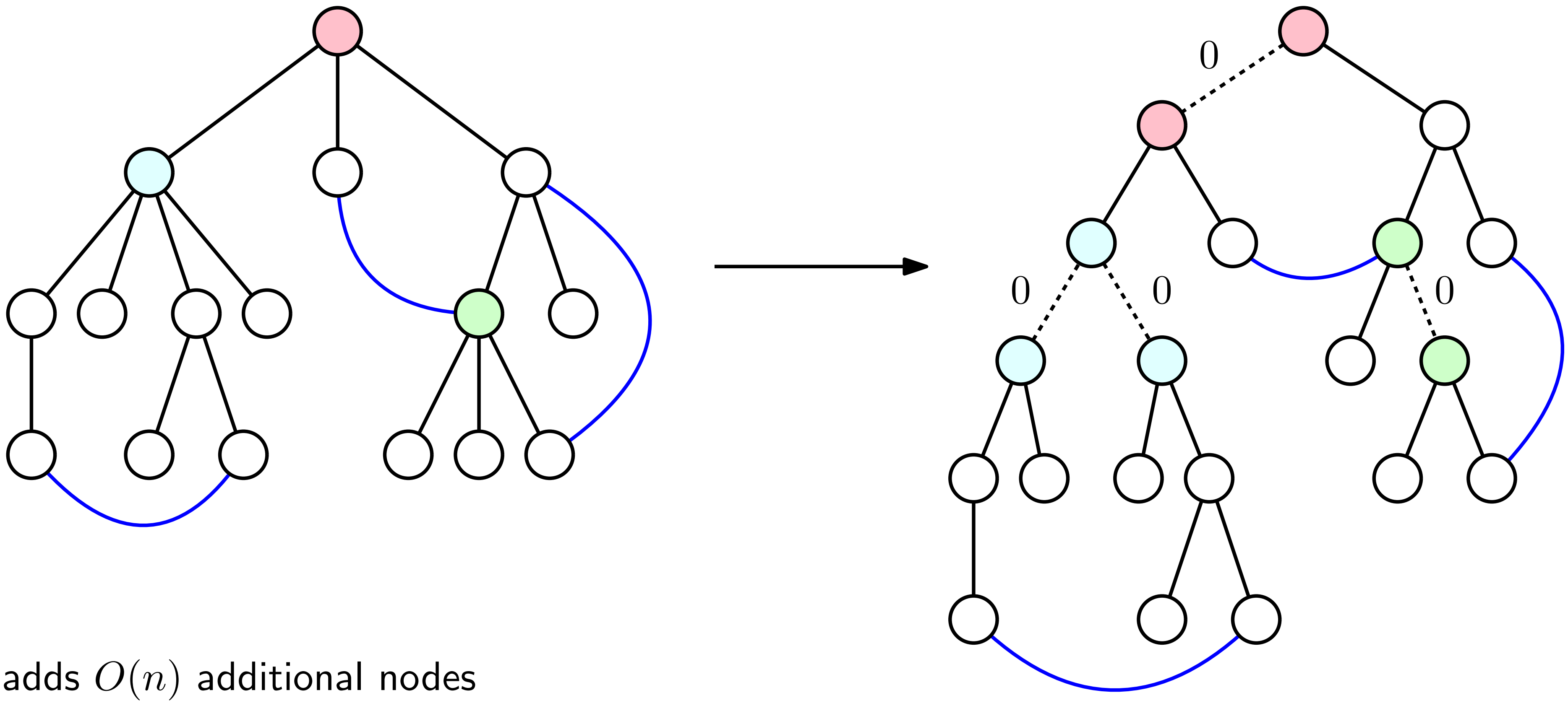
SetAlpha(v, α_v):	Assigns a weight $\alpha_v \geq 0$ to vertex v	$O(1)$
ReportFarthest$()$:	Return the vertex that is "farthest" from all terminals	$O(\#$ terminals $\cdot \log n)$

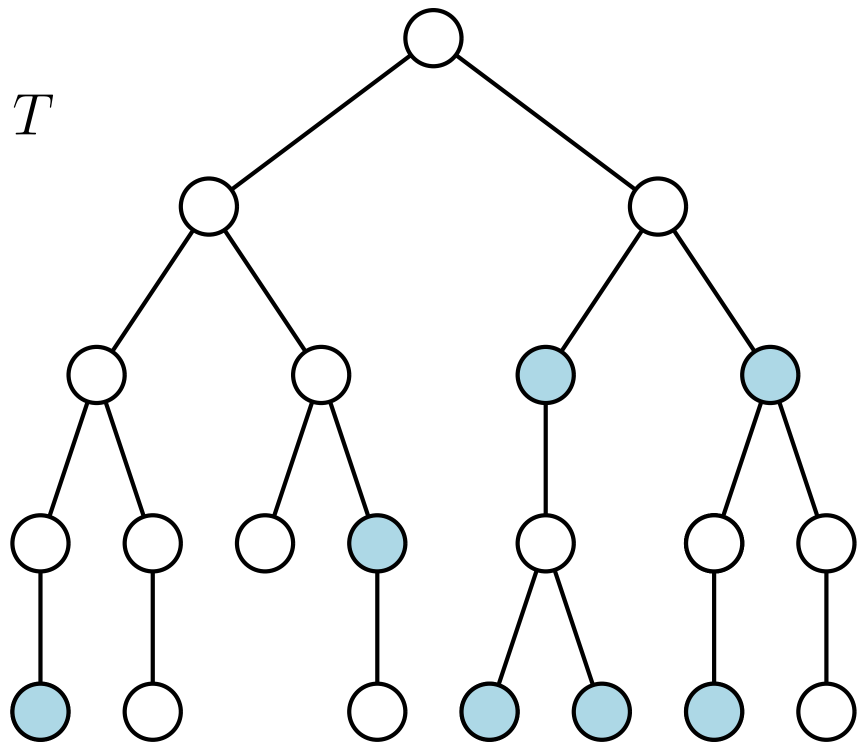
A simplifying assumption

We can assume that T is a binary tree



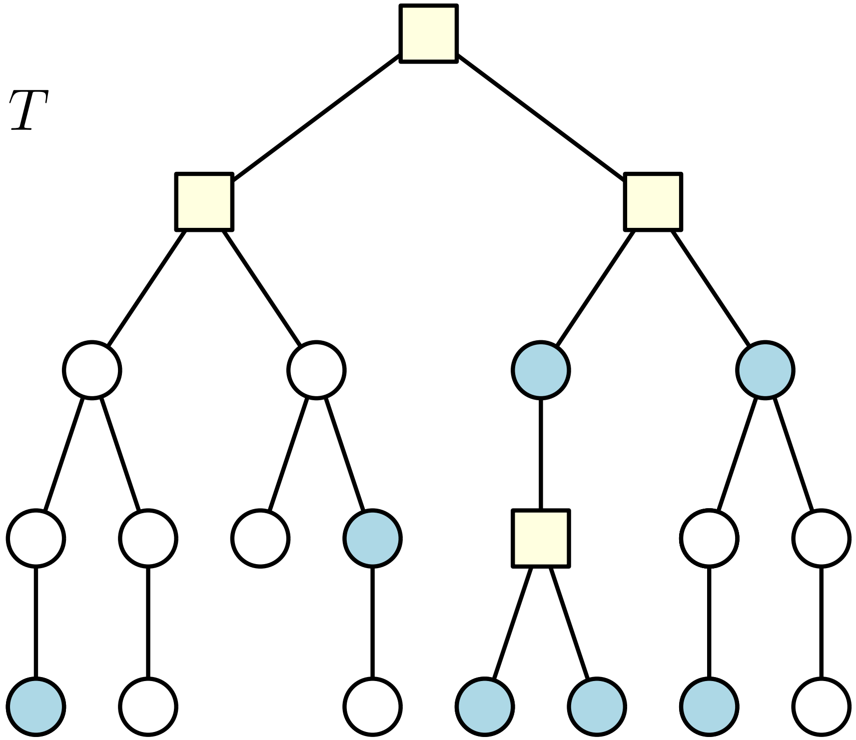
This adds $O(n)$ additional nodes

Shrink()



The vertex set V' of the shrunk tree are all the terminals, plus all the LCAs between pairs of terminals

Shrink()



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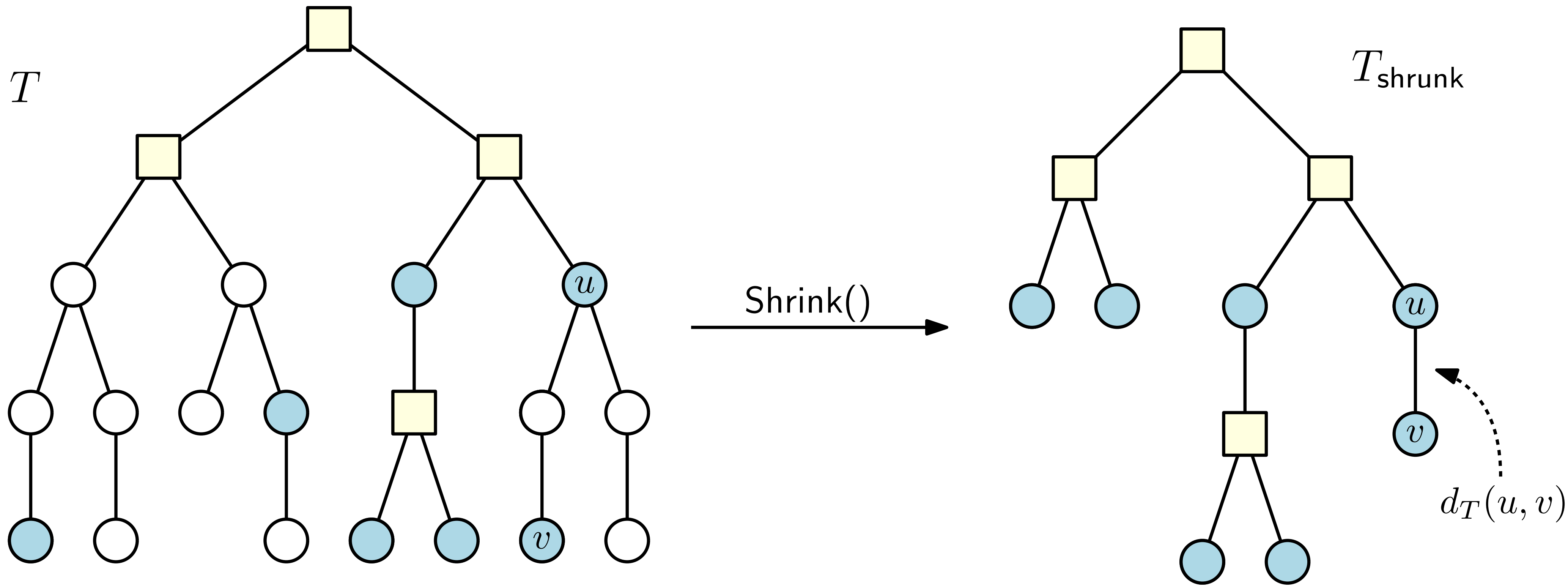

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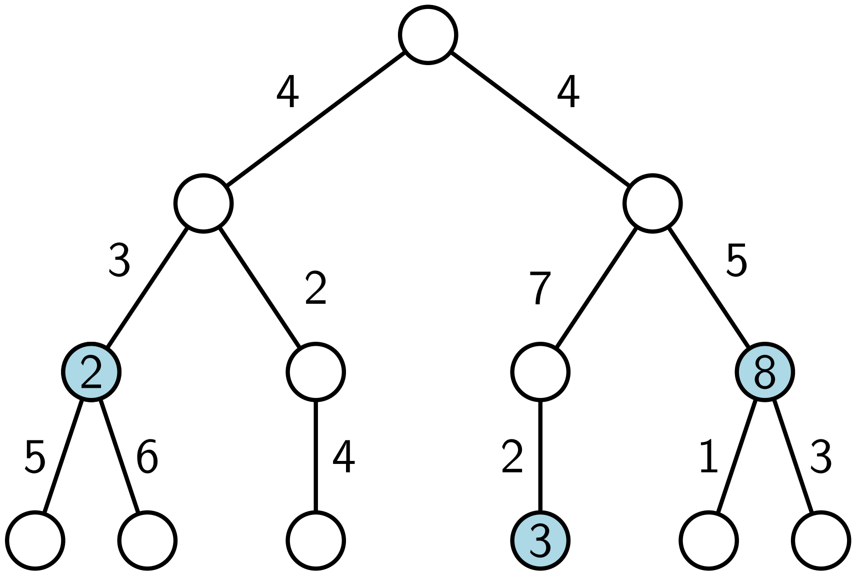
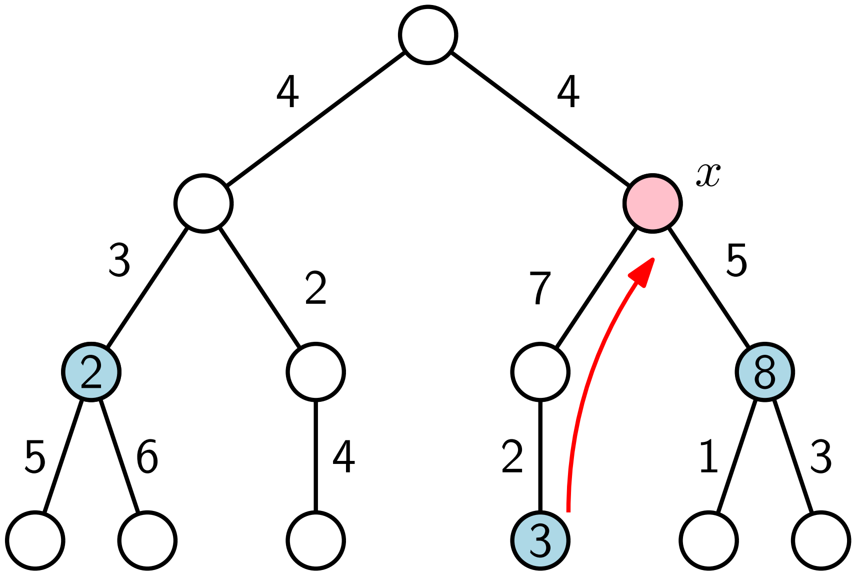
ReportFarthest()



Define the "distance" of a vertex x as $\delta(x) = \min_{\text{terminal } v} \left(\alpha_v + d_T(v,x)\right)$

ReportFarthest()



$\delta(x) = 12$

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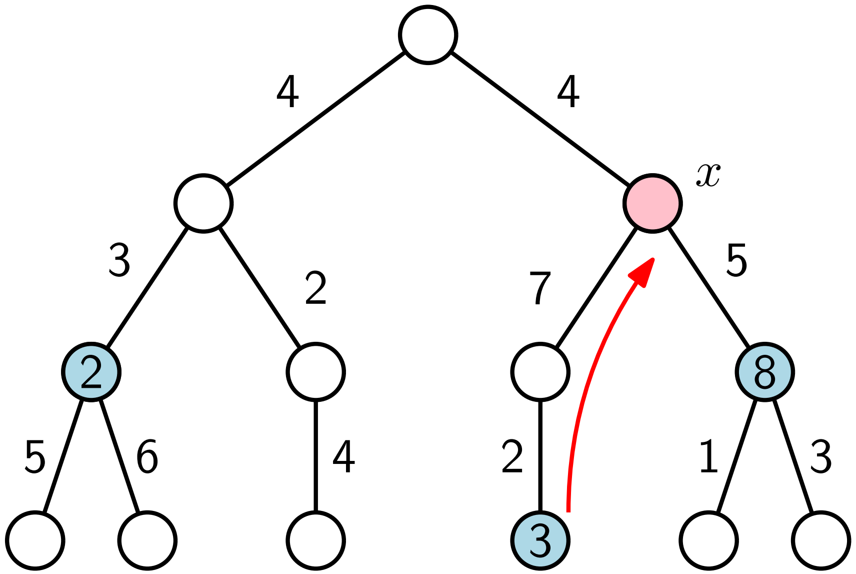

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ReportFarthest() returns a vertex x that maximizes $\delta(x)$

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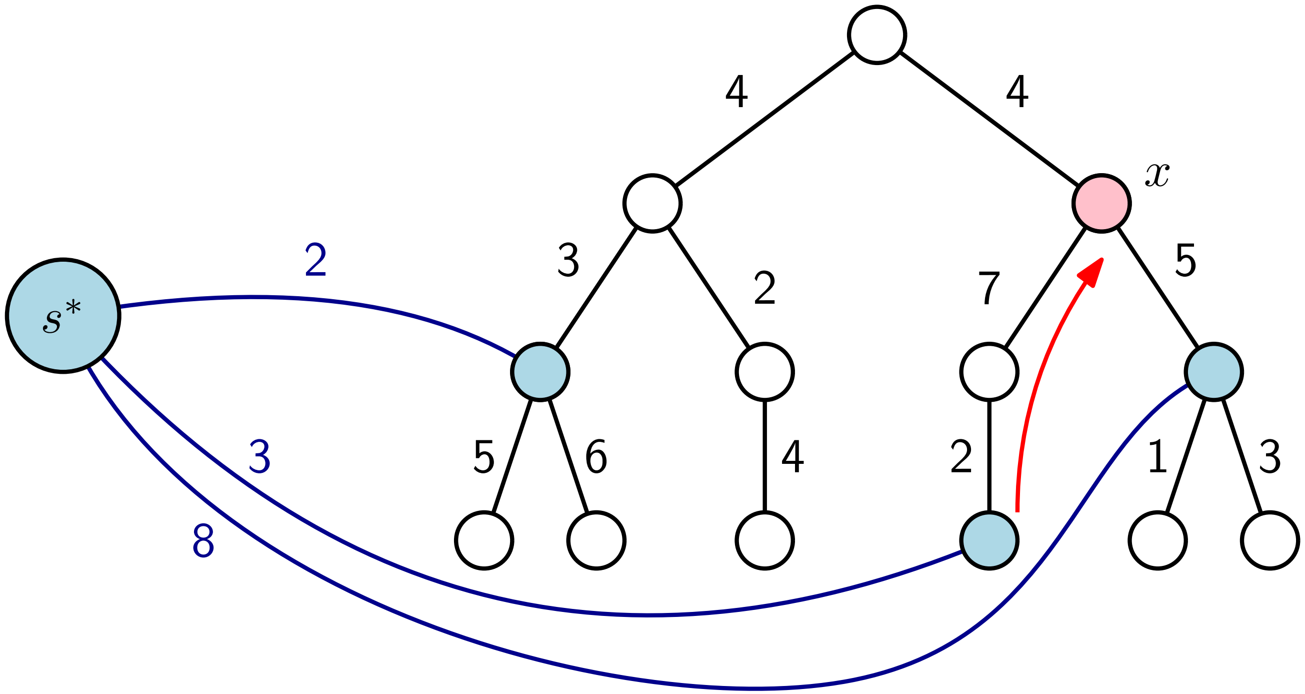

$\delta(x) = 12$ = Eccentricity of s^*

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Our Data Structure: Implementation

- The tree T is stored using a *top-tree* $\mathcal{T}$ time per op: $O(\log \#\text{vertices})$
  - Can add (link) and remove (cut) edges
  - Can mark/unmark vertices as terminals
  - Given a vertex v, it reports the closest ancestor of v that is a terminal
  - Given v, can report the eccentricity of v w.r.t. its tree

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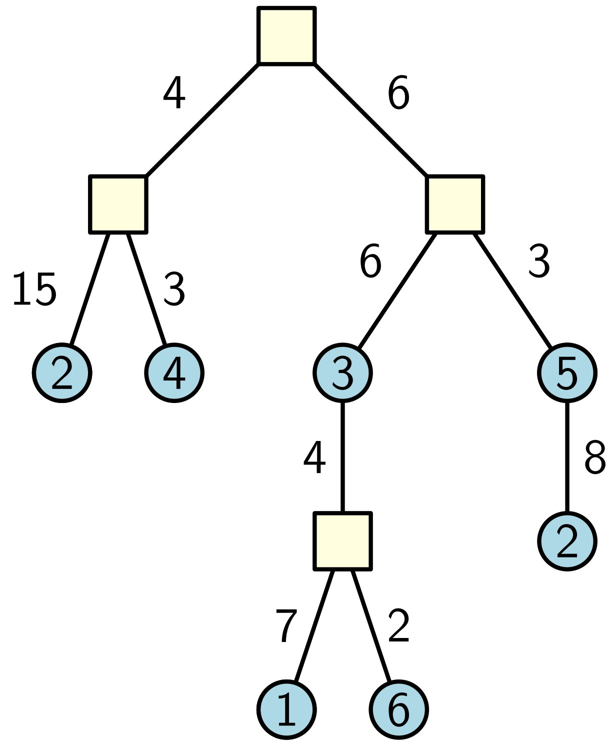
- Oracles with linear size that can report: — time per op: $O(1)$
  - The lowerst common ancestor of a pair of vertices in T
  - The level ancestor of a vertex in T
  - The distance/hop-disance between a pair of vertices in T

Implementing ReportFarthest()

Compute the distance $\beta_v = \delta(v)$ to each vertex v in T_{shrunk}



Can be done in time $O(k)$ using a postorder DFS vits followed by a preoder DFS visit

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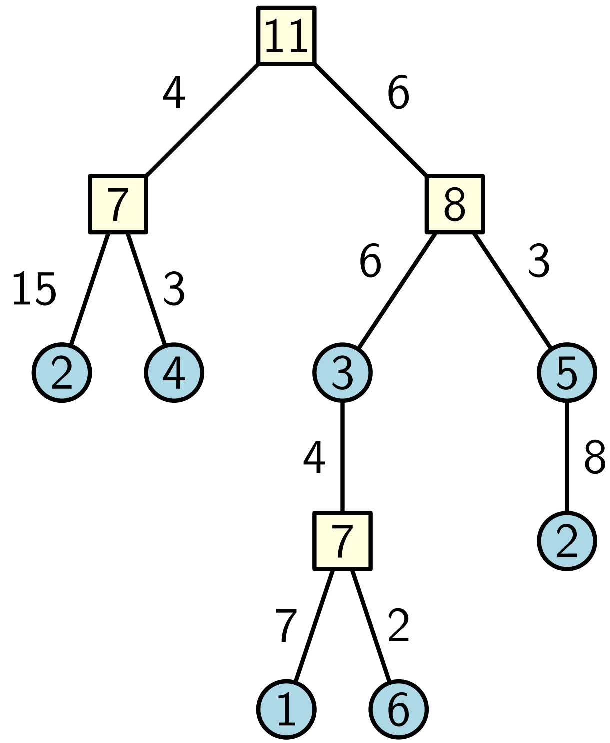

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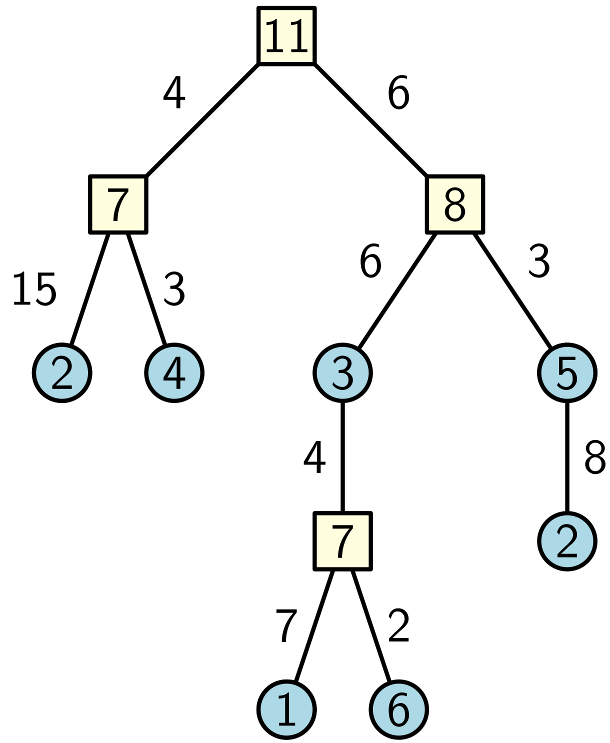

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This allows us to treat all vertices in T_{shrunk} as if they were terminals

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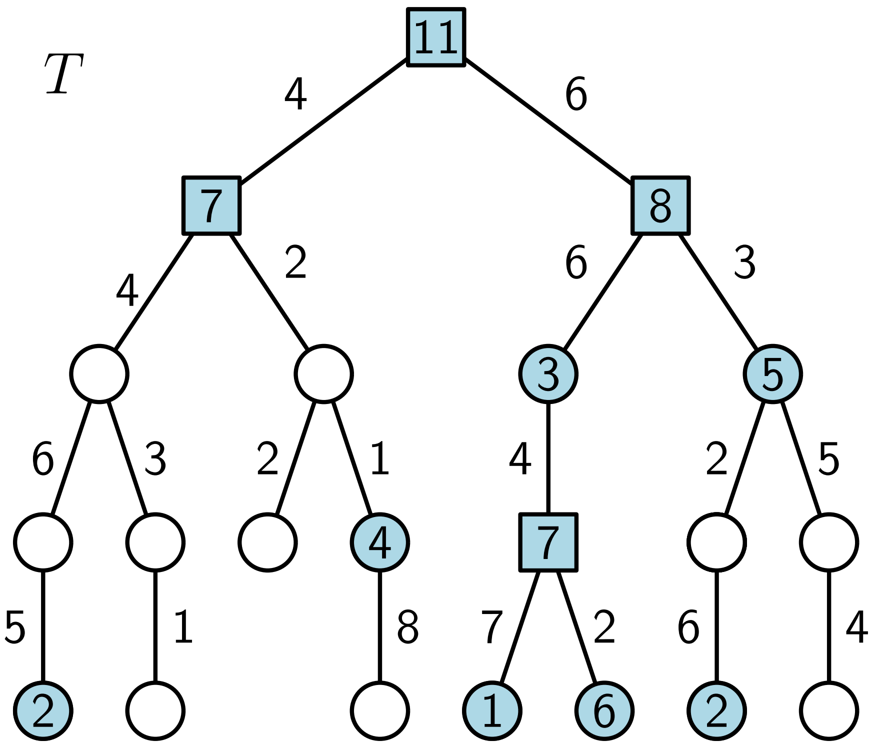

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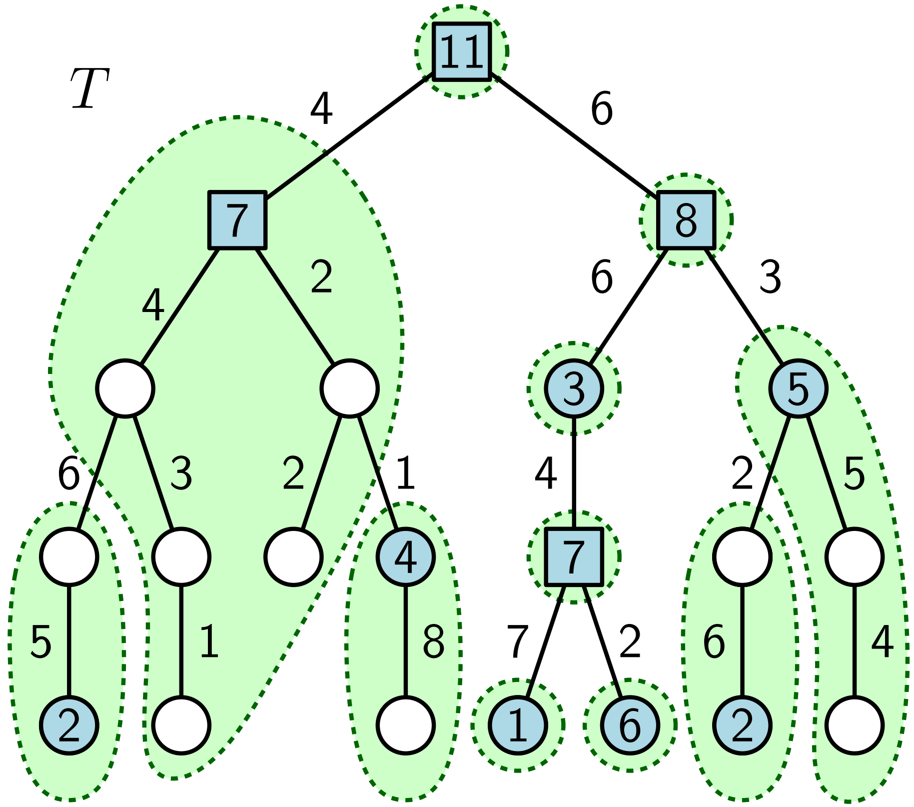
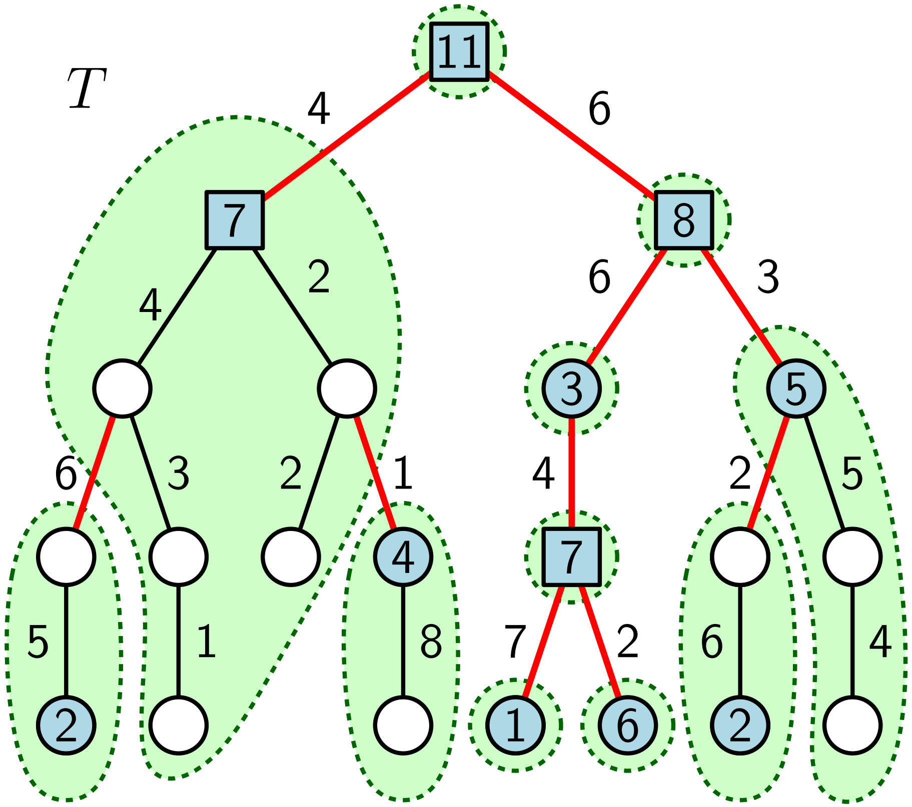


Assign each node of T to the "closest" node of T_{shrunk}

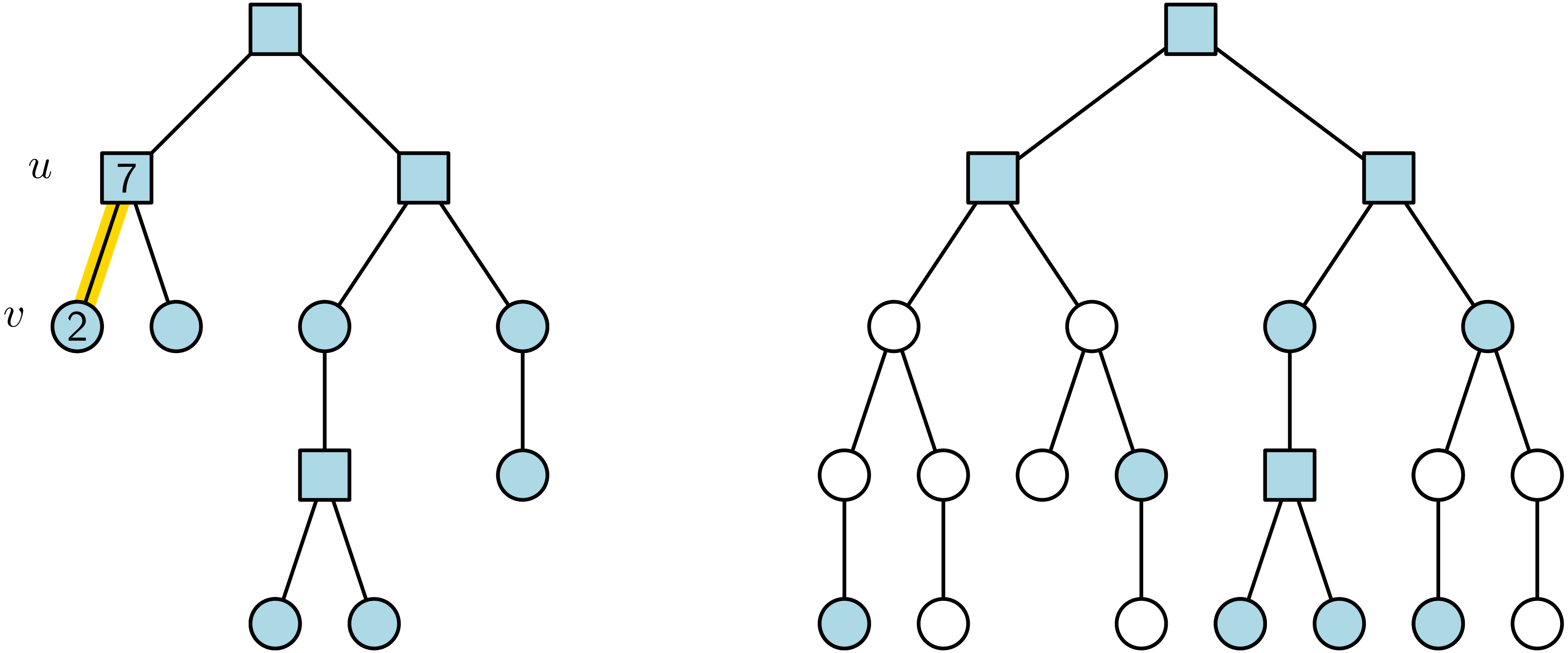
Implementing ReportFarthest()



Assign each node of T to the "closest" node of T_{shrunk}

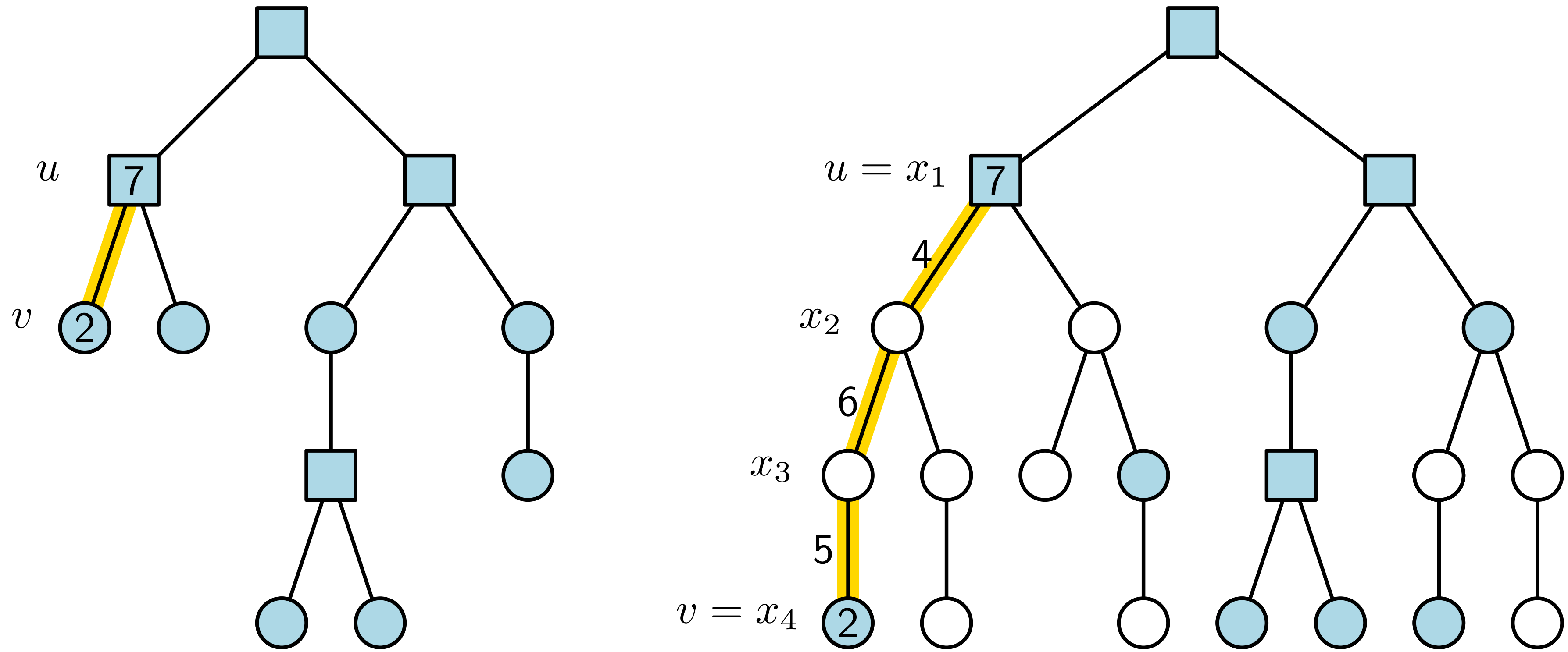
We want to **quickly** find the boundary edges

Implementing ReportFarthest()



Each edge (u, v) in T_{shrunk} corresponds to a vertical path $\langle u = x_1, x_2, \ldots, x_k = v \rangle$ in T

Implementing ReportFarthest()



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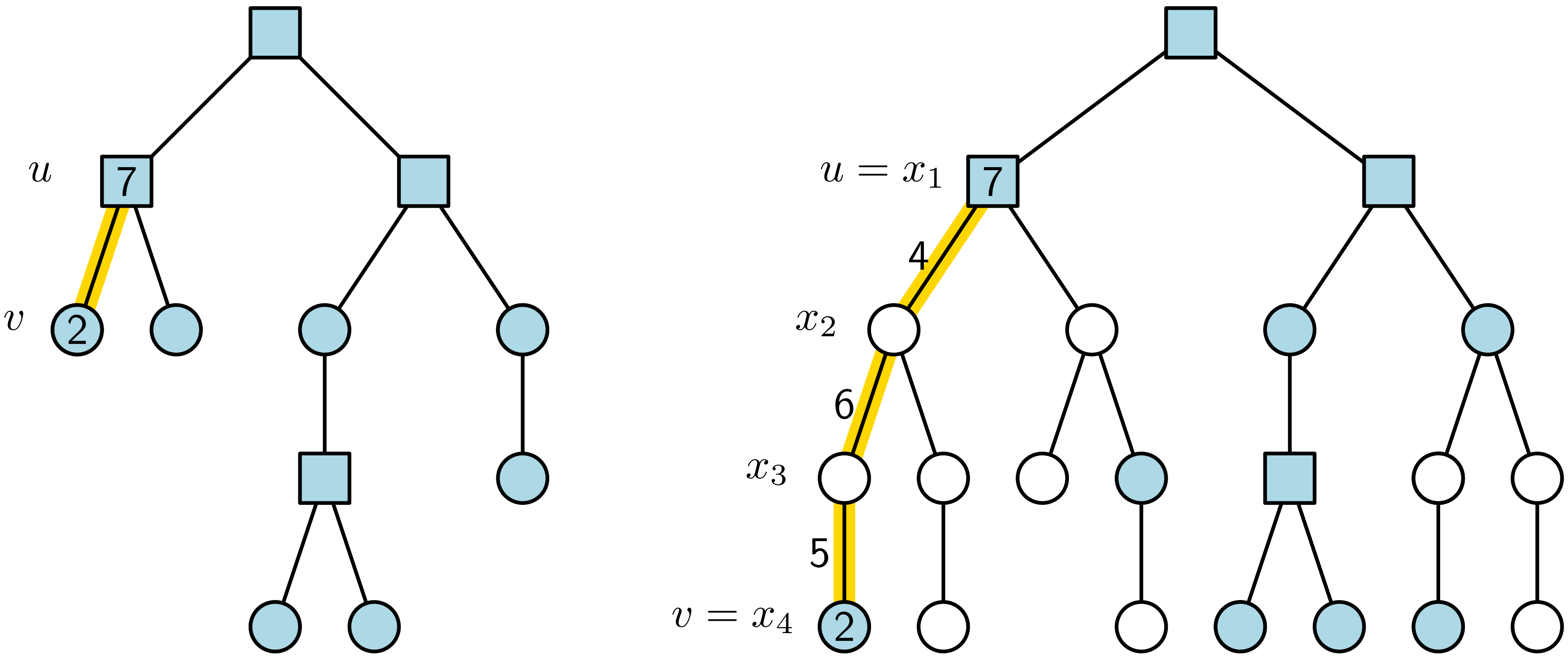

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$$\delta(x_i) = \min \begin{cases} \beta_u + d_T(u, x_i) \\ \beta_v + d_T(v, x_i) \end{cases}$$

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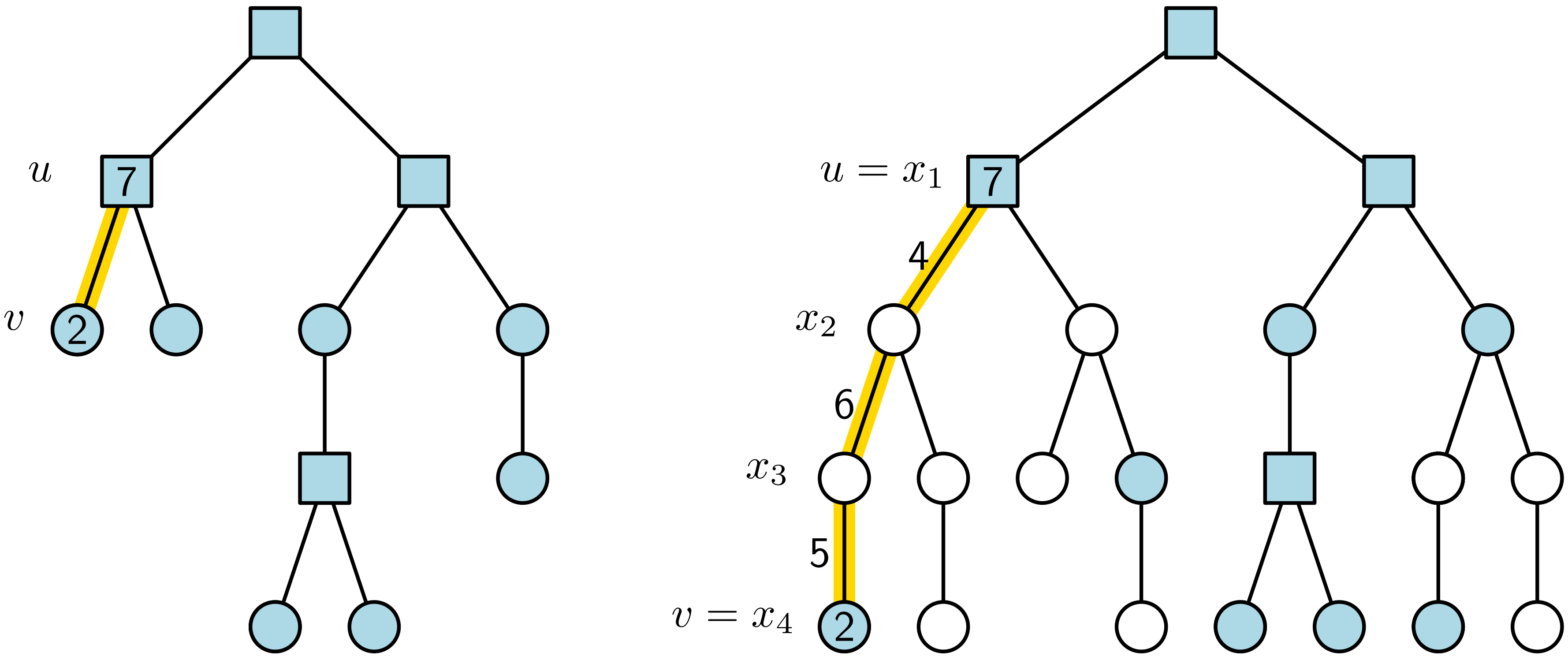

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Monotonically non-decreasing w.r.t. i

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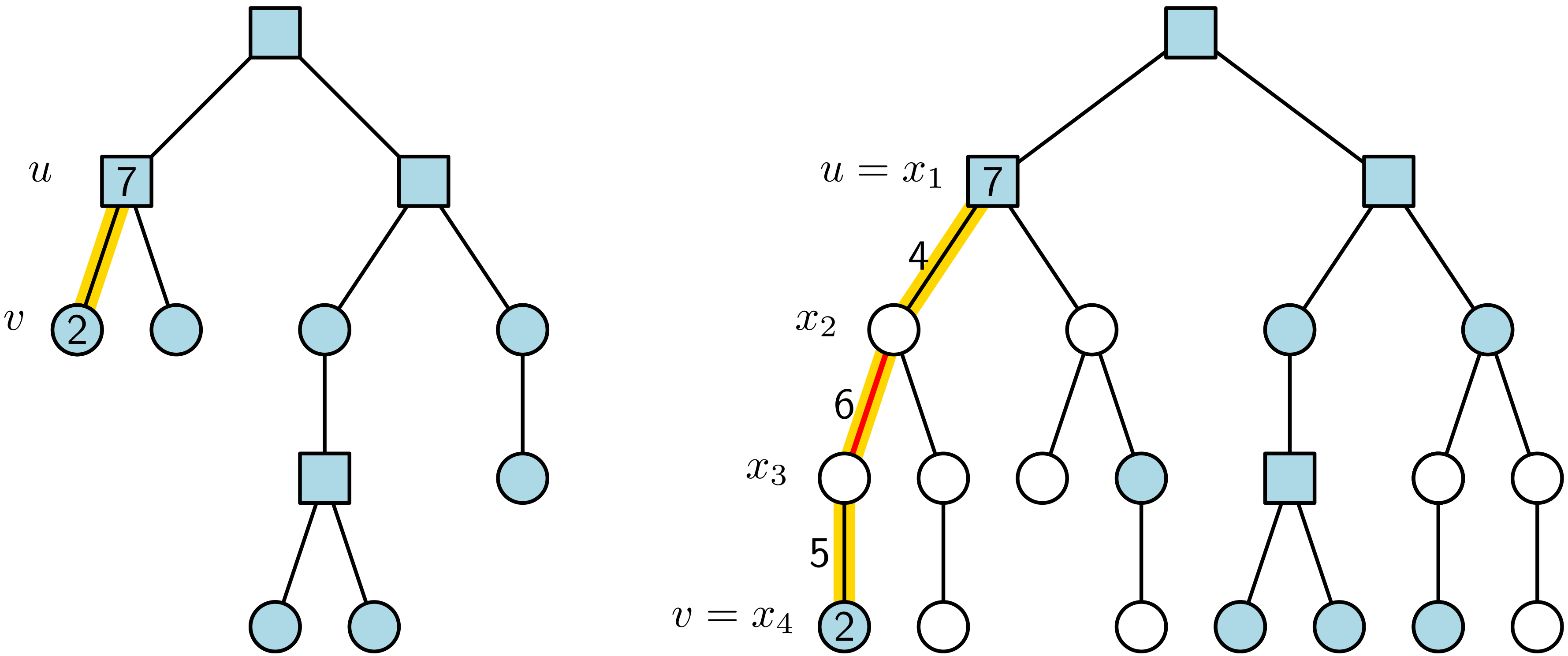

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Monotonically non-decreasing w.r.t. i

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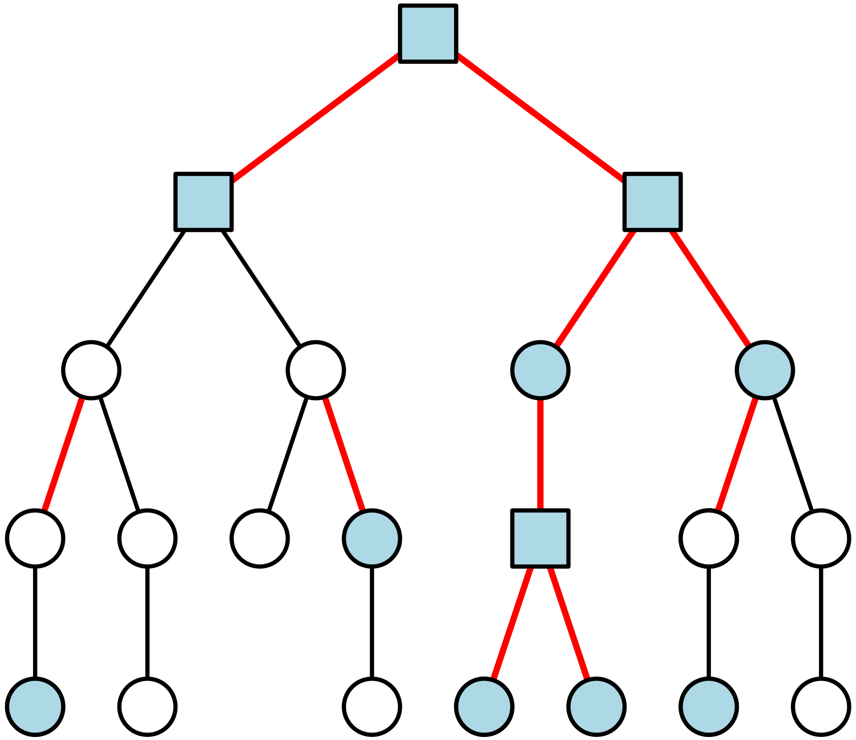
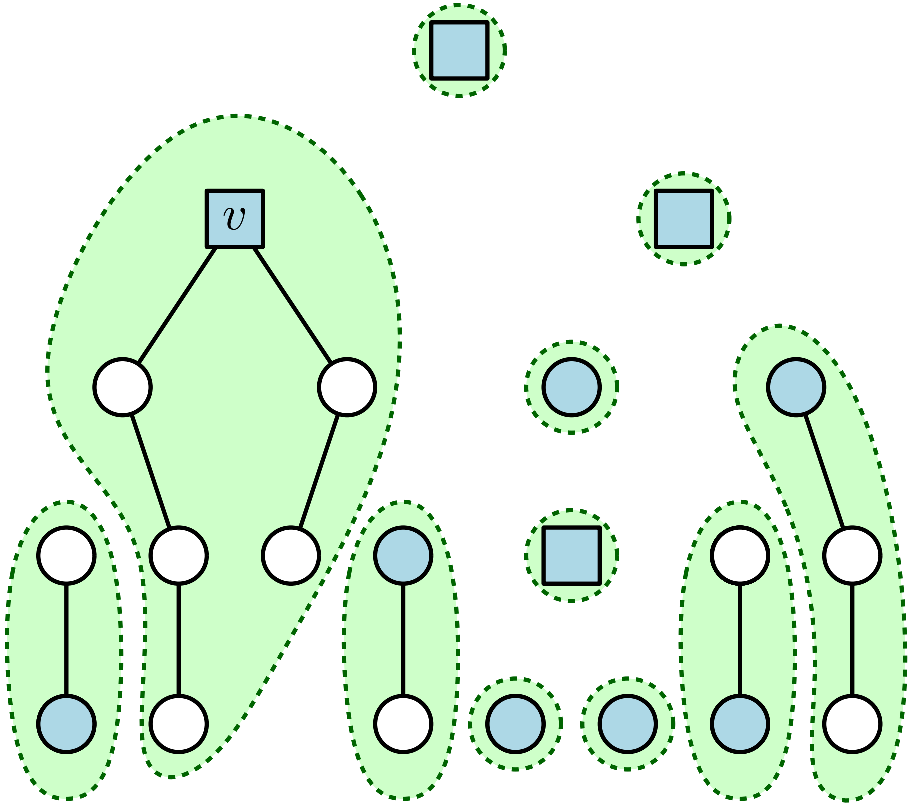
Binary search!

Implementing ReportFarthest()



Cut all bounday edges from $\mathcal{T}$

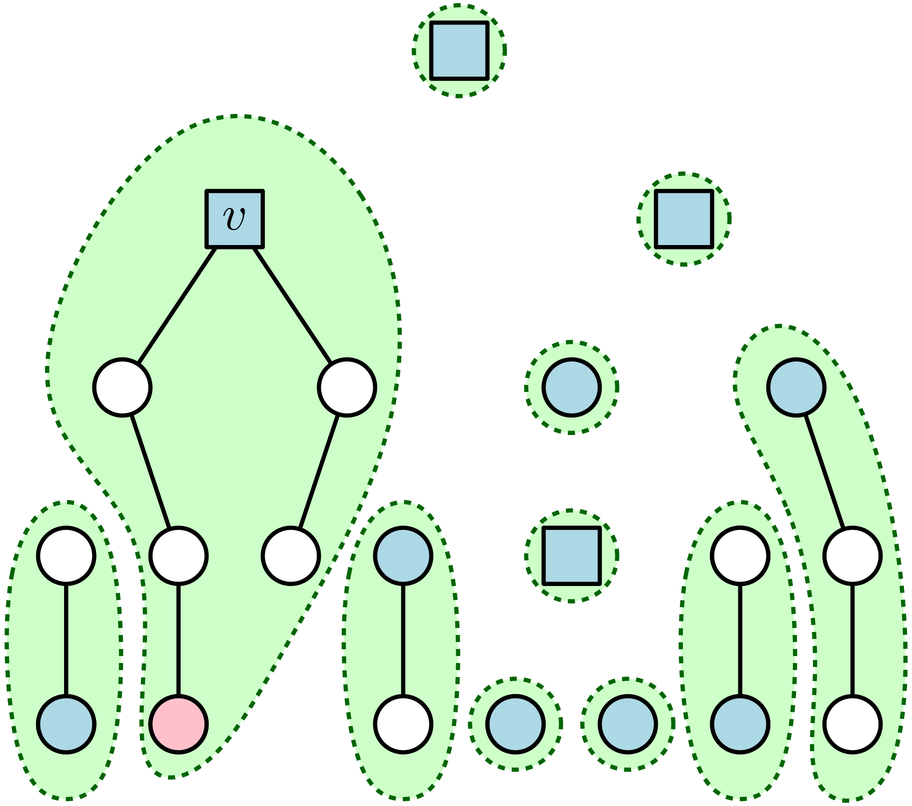
Implementing ReportFarthest()



Cut all bounday edges from $\mathcal{T}$

The resulting forest contains exactly one tree T_v for each vertex v in $T_{\textsf{shrunk}}$

Implementing ReportFarthest()



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For each v: query $\mathcal{T}$ to find the eccentricity $\mathcal{E}_v$ of v in T_v

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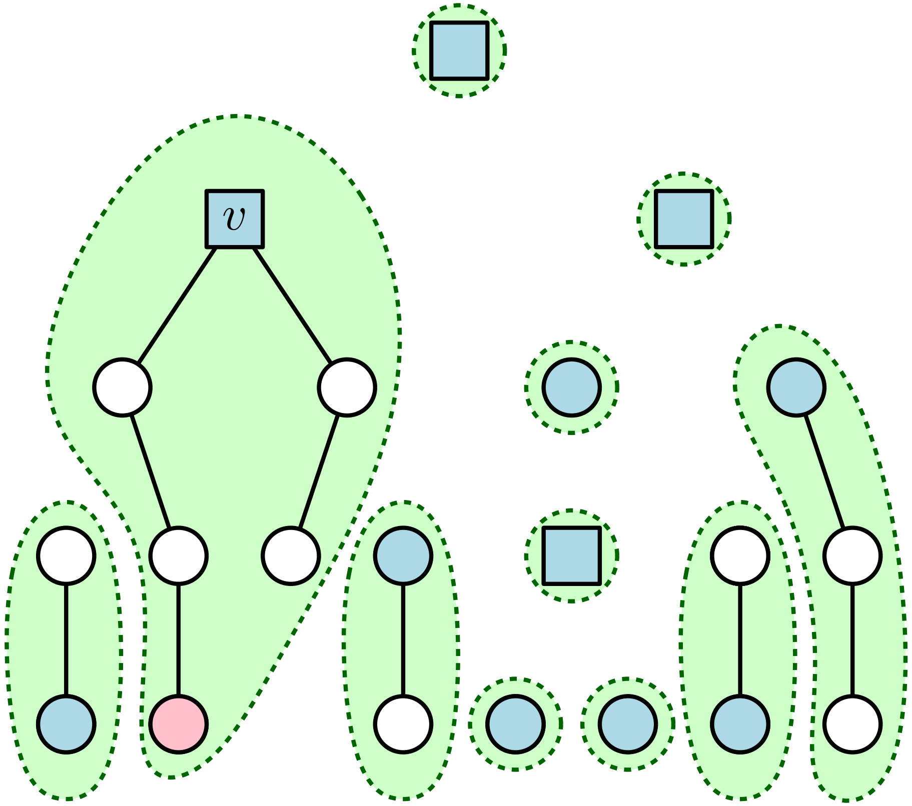

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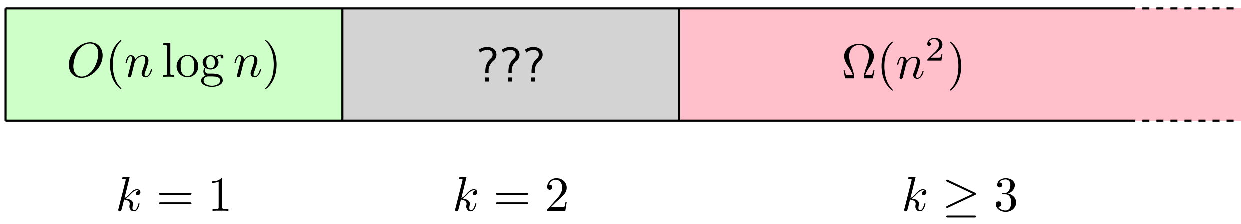
Return $\max_v (\beta_v + \mathcal{E}_v)$ ...and restore the orignal state of $\mathcal{T}$ (**link** the boundary edges)

Open Problems

Faster algorithms for <u>metric</u> k-DOAT?

- Avoid trying all possible shortcuts

Lower bound on the number of queries needed to solve metric 2-DOAT?



Open Problems

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Lower bound on the number of queries needed to solve metric 2-DOAT?

$O(n \log n)$	???	$\Omega(n^2)$
$k = 1$	$k = 2$	$k \geq 3$

Thank you!

Questions?