Tight Approximation Algorithms for Ordered Covering

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Minimum Vertex Cover

Definition

Given a graph $G = (V, E)$, find the smallest cardinality subset $S \subseteq V$ such that for all edges e in E, $e \cap S \neq \emptyset$.

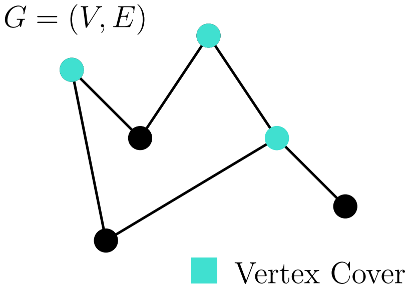

Figure: Vertex Cover in G

Alternate View (Scheduling)

1. One Machine.
2. The vertices are the jobs.

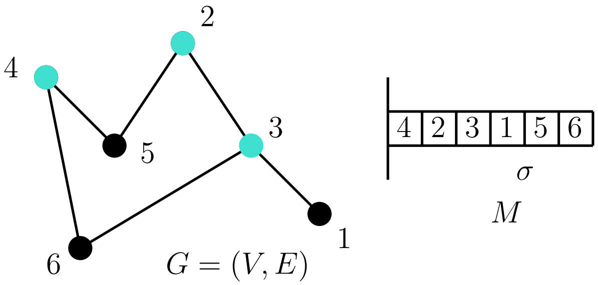

Figure: Scheduling Vertices on a Machine

Objective Function?

Alternate View (Scheduling): Cover Times

Definition

Given a graph $G = (V, E)$ and a schedule $\sigma : V \to [n]$ of its vertices, the cover time of an edge is the first moment a vertex in the edge is scheduled.

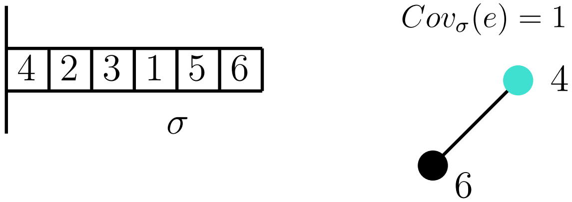

Figure: Cover Time of an Edge

1. Vertex Cover: $\min_{\sigma} \max_{e \in E} Cov_{\sigma}(e)$.
2. Min-Sum Vertex Cover: $\min_{\sigma} \sum_{e \in E} Cov_{\sigma}(e)$ [FLT04, BBFT21, Sta22]

Can we unify?

Ordered Optimization [AS, CSa, BSS18, CSb, CS19]

Given a schedule σ, arrange e in descending order as:
$Cov_\sigma(e_1) \geq Cov_\sigma(e_2) \geq \ldots Cov_\sigma(e_m)$.

$\min_\sigma Cov_\sigma(e_1)$ $\qquad\qquad$ $\min_\sigma \sum_{j=1}^{m} Cov_\sigma(e_j)$

?

Figure: Generalized Objective Functions

Ordered Optimization [AS, CSa, BSS18, CSb, CS19]

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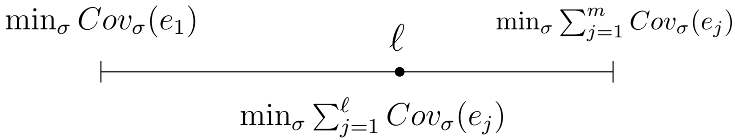

Figure: Top-ℓ Objective Functions

Ordered Objective Functions:

$$\min_\sigma \sum_{j=1}^{m} w_j Cov_\sigma(e_j)$$

for $w_1 \geq w_2 \geq \ldots w_m \geq 0$. This framework has gained a lot of attention for clustering (Ordered k-Median) and Load Balancing problems.

Our Contribution

Figure: One Ring to Rule Them All

Current State

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- Vertex Cover

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  - Hardness of Approximation: $(2 - \varepsilon)$ under UGC [KR08].

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  - Upper Bound: 8-approx [GGKT08] (All-norm, more general result).
  - Lower Bound: $(2 - \varepsilon)$ under UGC (Captures VC).

However, the techniques for upper bounds are very different across the problems.

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- Set Cover
  - Upper Bound: $O(\log n)$-approx [Joh73, Sla96].
  - Lower Bound: NP-Hard to approximate within $(1 - \varepsilon) \log_e n$ [DS14]

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  - Upper Bound: 4-approx [FLT04].
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  - Lower Bound: NP-Hard to approximate within $(4 - \varepsilon)$ [FLT04].
- Ordered Set Cover
  - $O(\log n)$-approx. [GGKT08] Upper Bound (Much more general result, All-norm) s
  - General Lower Bound: NP-Hard to approximate within $\Omega(\log n)$ (Captures Set Cover). But for specialised Top-ℓ objectives?

Our Results

Definition

Ordered Min-Sum Vertex Cover(OMSVC): Let $G = (V, E)$ be a graph, and a $w_1 \geq w_2 \geq \ldots w_m \geq 0$ be a sequence of non-negative weights. Design a schedule σ of vertices to minimize $\sum_j w_j Cov_\sigma(e_j)$ where e_j is the ordering of edges in descending order of cover times.

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Theorem

There is a $(2+\varepsilon)$-approximation algorithm for OMSVC running in time $n^{O(1/\varepsilon)}$.

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Definition

Top-ℓ Set Cover: Let E be a set and $\mathcal{I}$ be a collection of subsets of E. Design a schedule σ of sets to minimize $\sum_{j=1}^{\ell} Cov_{\sigma}(e_j)$ for a fixed $\ell \in [n]$ where the elements e_j are ordered in descending order of cover times.

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The natural greedy algorithm gives a $(8\log_2(n/\ell) + 16)$*-approx for Top-*ℓ *Set Cover.*

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It is hard to approximate the Top-ℓ *Set Cover problem within a factor better than* $\max(\Omega(1), \Omega(\log(n/\ell)))$.

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It is hard to approximate the Top-ℓ *Set Cover problem within a factor better than* $\max(\Omega(1), \Omega(\log(n/\ell)))$.

First ℓ-based approximation ratio and hardness result for Top-ℓ objectives (to our knowledge).

Overview of the Talk

We will talk about our algorithm which gets a $(2+\varepsilon)$-approx. for Top-ℓ Vertex Cover.

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- We will give a review of the approach of Feige et. al. [FLT04] and why it fails in our problem.
- We will show how a *dependent-rounding*[GKPS06] based approach fixes these problems.

Reduction

Let $\sigma : V \to T$ be a schedule for an instance of Top-ℓ Vertex Cover.

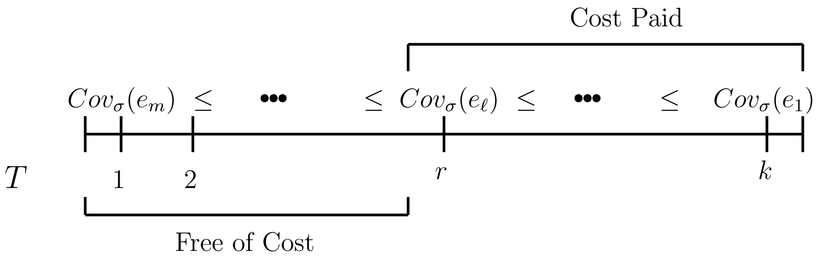

Figure: Top-ℓ Vertex Cover

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Intuition

- Try to write down an ILP for this problem.
- Any schedule should be encoded as a feasible solution to this ILP.
- ILP should know edges which have the Top-ℓ cover times for any schedule σ.
- How to write constraints for this?

Reduction

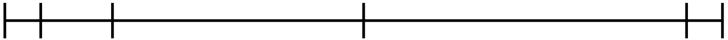
Problem? Do not know the ordering before hand.

Reduction

Shift by a Discount

Add the Discount back later

$(Cov_\sigma(e_m) - \lambda)_+ \leq \quad \bullet\bullet\bullet \quad \leq (Cov_\sigma(e_\ell) - \lambda)_+ \leq \quad \bullet\bullet\bullet \quad \leq (Cov_\sigma(e_1) - \lambda)_+$



$$\forall \lambda \geq 0 \quad \underbrace{\sum_{e \in E}(Cov_\sigma(e) - \lambda)_+}_{\text{Oblivious to Ordering}} + \underbrace{\lambda \cdot \ell}_{\text{Fixed}} \geq \sum_{i=1}^{\ell} Cov_\sigma(e_\ell)$$

Figure: Top-ℓ Vertex Cover

Reduction

Shift by Good Discount

Add the Discount back later

$$\lambda = Cov_{\sigma^*}(e_\ell)$$

$$(Cov_{\sigma^*}(e_m) - \lambda)_+ \leq \cdots \leq (Cov_{\sigma^*}(e_\ell) - \lambda)_+ \leq \cdots \leq (Cov_{\sigma^*}(e_1) - \lambda)_+$$

$$\sum_{e \in E} (Cov_{\sigma^*}(e) - \lambda)_+ + \lambda \cdot \ell = \sum_{i=1}^{\ell} Cov_{\sigma^*}(e_\ell)$$

Oblivious to Ordering — Fixed

Figure: Top-ℓ Vertex Cover

Reduction

Try to minimize:

$$\sum_{e \in E} (Cov_{\sigma}(e) - \lambda)_+$$

over all possible schedules for a correct guess $\lambda = Cov_{\sigma^*}(e_\ell)$.

Reduction

Theorem

[AS, BSS18, CSa, CS19, CSb] Let σ^* be the optimal schedule to an instance of Top-ℓ Vertex Cover. Then, a schedule σ satisfying:

$$\sum_{e \in E} (Cov_\sigma - 2\lambda)_+ \leq 2 \sum_{e \in E} (Cov_{\sigma^*} - \lambda)_+$$

implies that $\sum_{j=1}^{\ell} Cov_\sigma(e_j) \leq 2 \sum_{j=1}^{\ell} Cov_{\sigma^*}(e_j)$.

Definition

Discounted Min-Sum Vertex Cover (DMSVC) Let $G = (V, E)$ be a graph, and a $\lambda \geq 0$ be a discount. Design a schedule σ of vertices which satisfies:

$$\sum_{e \in E} (Cov_\sigma - 2\lambda)_+ \leq 2 \sum_{e \in E} (Cov_{\sigma^*} - \lambda)_+$$

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1. In the following, we will try to solve the Discounted Min-Sum Vertex Cover problem via an LP-rounding algorithm.

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2. Let us first try to write down the linear program.

Intuition for Linear Program

Variables: $x_{v,t}$, $u_{e,t}$.
Indicator variables.

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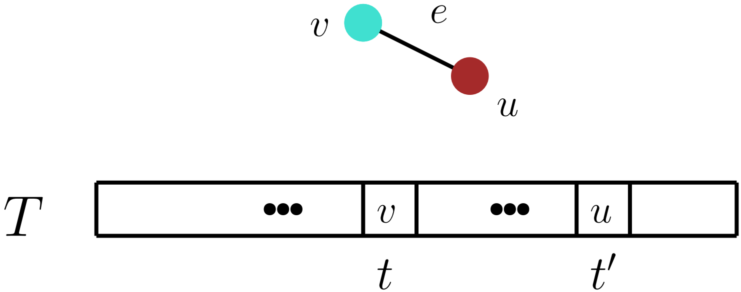
$$x_{v,t} = \begin{cases} 1 & v \text{ is scheduled at } t \\ 0 & \text{otherwise} \end{cases}$$

$$u_{e,t} = \begin{cases} 1 & Cov_{\sigma}(e) \geq t \\ 0 & \text{otherwise} \end{cases}.$$

$u_{e,t}$ tracks at each step if e has not been covered.

Intuition For Linear Program

Variables: $u_{e,t}, x_{v,t}$



$$\textcircled{1} \; \Sigma_{v' \in V} x_{v',t} \leq 1$$

Figure: Intuition for variables and constraints

Intuition For Linear Program

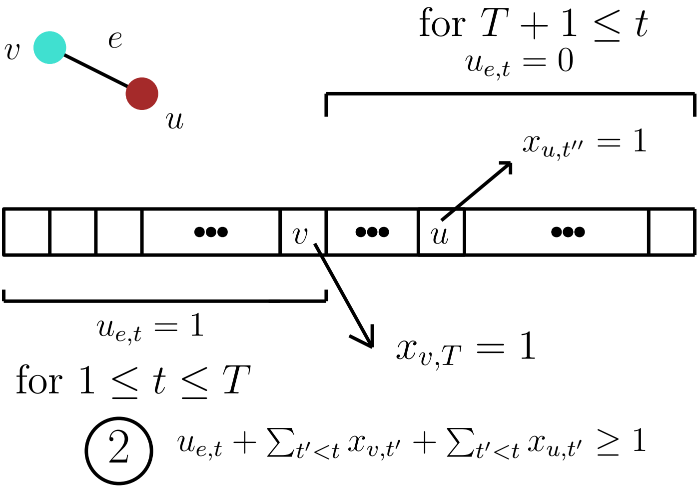


Figure: Intuition for variables and constraints, $Cov_\sigma(e) = \sum_t u_{e,t}$

Intuition For Linear Program

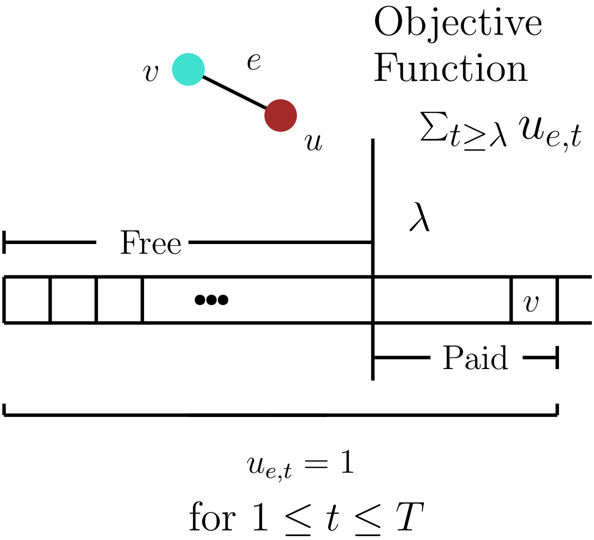


Figure: Intuition for variables and constraints

DMSVC Linear Program

$$\text{Minimize} \quad \sum_{t \geq \lambda} \sum_{e \in E} u_{e,t} \,, \quad s.t.$$

$$\sum_{v \in V} x_{v,t} \leq 1 \,, \quad \forall \; t = 1, 2, \ldots \tag{1}$$

$$u_{e,t} + \sum_{t' < t} x_{u,t'} + \sum_{t' < t} x_{v,t'} \geq 1, \quad \forall \; e = (u, v), t = 1, 2, \ldots \tag{2}$$

$$u, x \geq 0 \,.$$

Bi-criteria algorithm for DMSVC

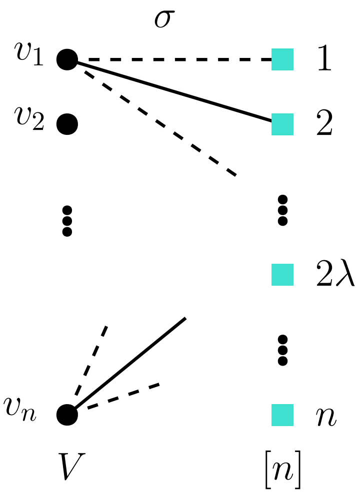


Figure: Assignment Graph

Naive Approach

Solve the above LP.

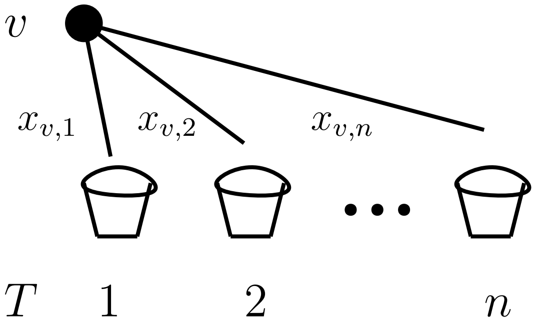


Figure: Try to throw each vertex in bins with probability $x_{v,t}$

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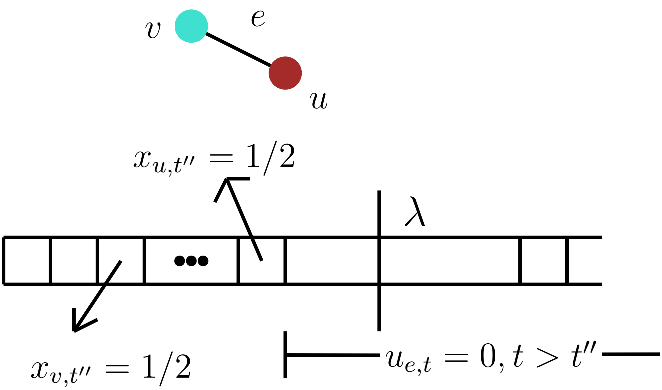

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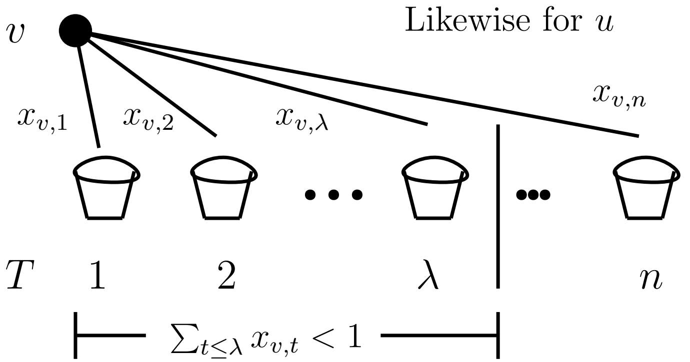

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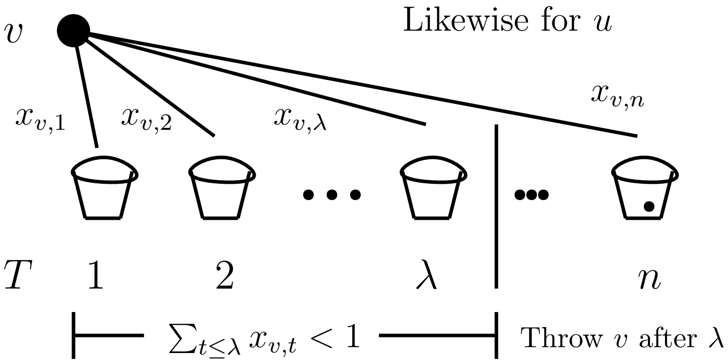

Figure: Ball might be thrown in the last bin

Naive Approach

Independently throw each vertex into the bins using scaled probabilities (x).

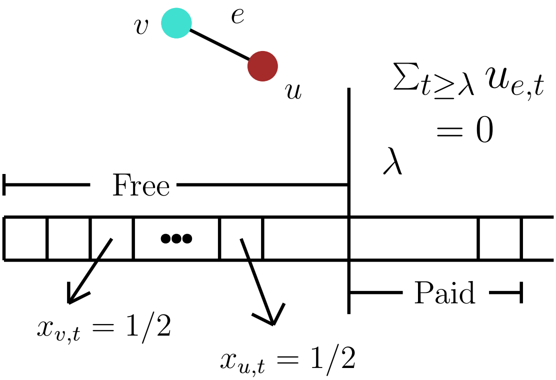

Figure: Linear Program does not pay anything, While we pay a lot.

Approach of [FLT04]

If $e = \{v, u\}$ is fractionally scheduled before λ, then either $\sum_{t<\lambda} x_{v,t} \geq 1/2$ or $\sum_{t<\lambda} x_{u,t} \geq 1/2$.
(Since fractional covering means $\sum_{t<\lambda} x_{v,t} + \sum_{t<\lambda} x_{u,t} \geq 1$.)

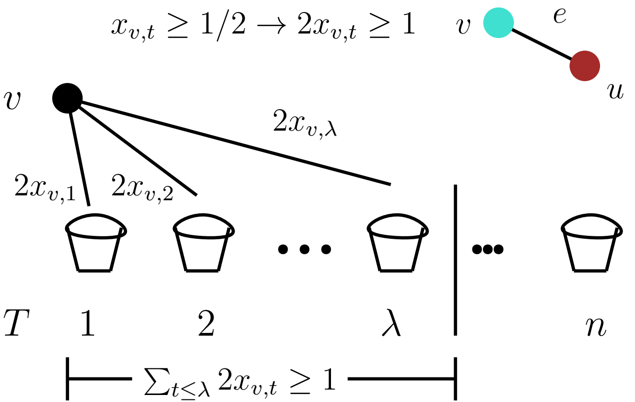

Figure: [FLT04]

Approach of [FLT04]

Independently throw each vertex into the bins using scaled probabilities $(2x)$.

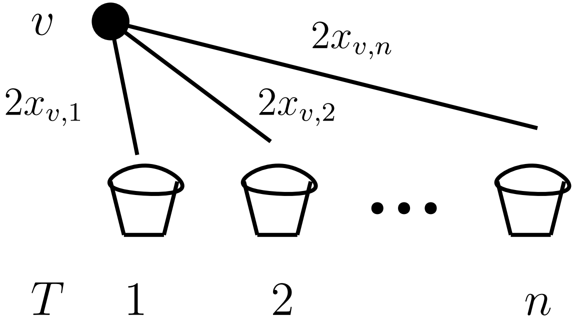

Figure: [FLT04]

Approach of [FLT04]

Break the bins and schedule a uniformly random permutation of vertices inside the bin.

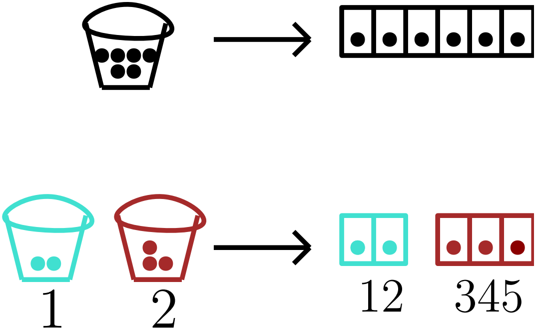

Figure: [FLT04]

Problem?

Recall, $\sum_{v \in V} x_{v,t} \le 1$. Thus
$\mathbb{E}[\#\text{vertices in a bin}] = 2 \sum_{v \in V} x_{v,t} \le 2$.

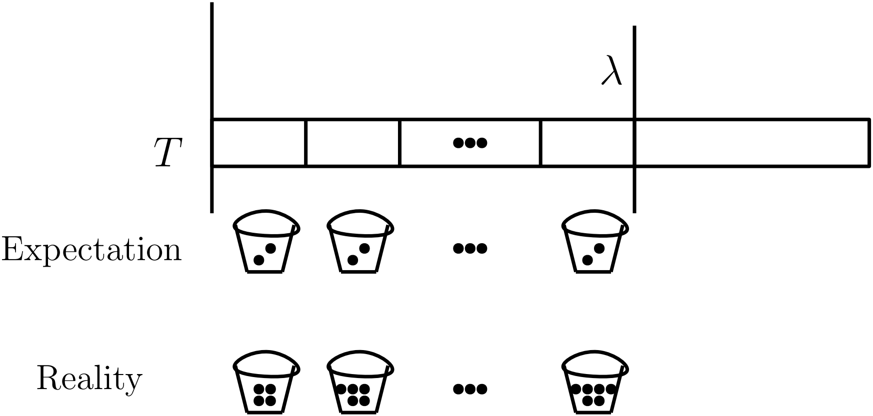

Figure: [FLT04]

Problem?

Want to ensure with high probability that at most 2λ vertices are scheduled within first λ slots.

Ingredients We Need

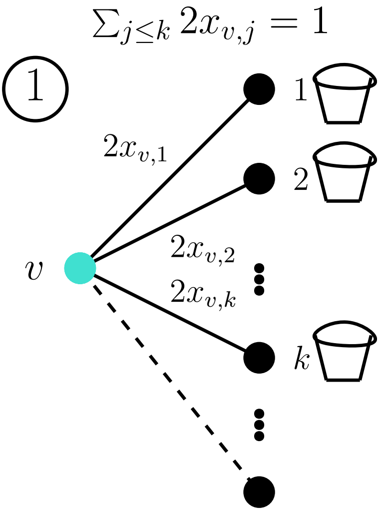

Figure: Marginal Preservation: Throw v into a bin with probability $2x_{v,t}$

Ingredients We Hope for

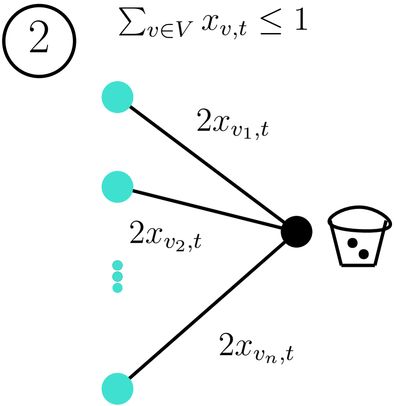

Figure: Dependent Rounding on Side of Slots

- ≤ 2 vertices in each bin with probability 1.
- Thus, $\leq 2\lambda$ vertices in first λ bins with probability 1.

Fix?

Dependent Rounding.

Dependent Rounding [GKPS06]

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Theorem

Given a weighted bipartite graph $(V \cup T, F, z : F \to [0,1])$ *(*$z = 2x$*), it is possible to sample a subset of edges* $S \subseteq F$ *satisfying the following criterion:*

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2. $deg_S(t) = \lceil \sum_{e \in \delta(t)} 2x_{v,t} \rceil \leq 2$ *(Ensures that the load is at most 2).*

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3. $deg_S(v) = \sum_{e \in \delta(v)} 2x_{v,t} = 1$ *(Ensures each vertex is thrown into exactly 1 bin)*

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3. $deg_S(v) = \sum_{e \in \delta(v)} 2x_{v,t} = 1$ *(Ensures each vertex is thrown into exactly 1 bin)*
4. *(Negative Correlation): For any time t,* $\Pr((v,t) \in S, (u,t) \in S) \leq \Pr((v,t) \in S) \Pr((u,t) \in S)$. *[However, we do not use it]*

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3. Run Dependent rounding on the bipartite graph $(V \cup T, F, 2x : F \rightarrow [0,1])$. (The sampled edges give an assignment of vertices to time slot).
4. Break bins of time slots t (Each bin has at most 2 vertices: cost goes up by at most 2).

The End

Thank you.

Open Directions

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- Getting a 2-approx. for All-Norm Vertex Cover (Improving from 8 [GGKT08])

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- Getting a 2-approx. for All-Norm Vertex Cover (Improving from 8 [GGKT08])
- Getting ℓ-dependent or w-dependent approximation algorithms for other problems (such as Load Balancing or k-Median).

References

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