Minimum Sum Colorings of Chordal Graphs

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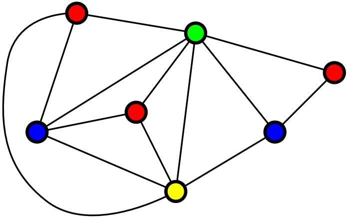
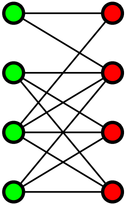
Graph Coloring

Adjacent vertices must receive different colors.





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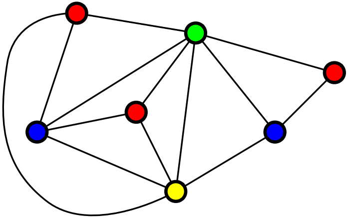

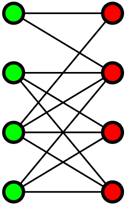

Chromatic Number: $\chi(G)$ = fewest colors required to color G.

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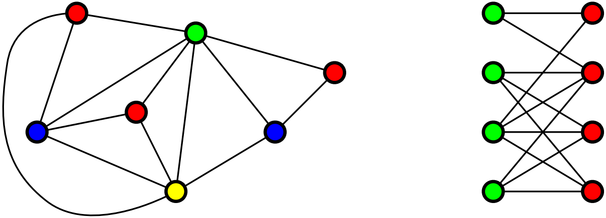

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Scheduling Models

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- Edges: resource conflicts
- Objective: minimize the final completion time

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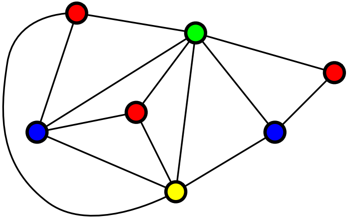

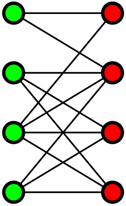

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Scheduling Models

- Vertices: jobs to be processed
- Edges: resource conflicts
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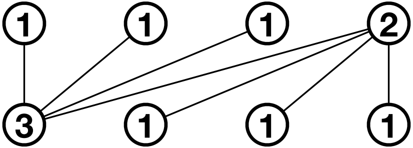
Find a coloring $\phi : V \to \{1, 2, 3, \ldots\}$ which minimizes $\sum_{v \in V} \phi(v)$.



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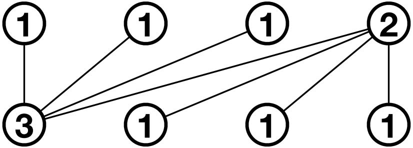

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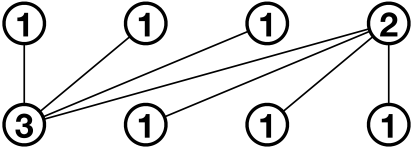

Using 2 colors gives a minimum sum of **12**,

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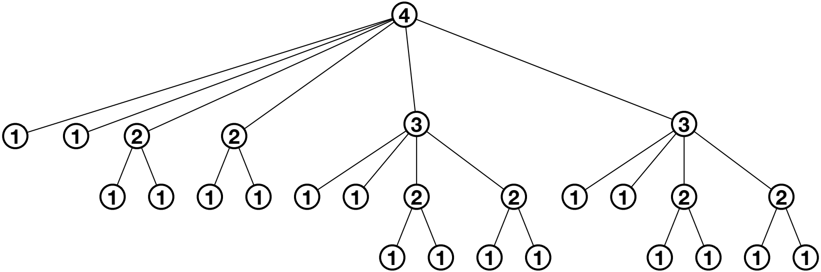
There is no $n^{0.999}$-approximation for arbitrary graphs.
[Bar-Noy et al, 1998]

Min-Sum Coloring on Trees



- The optimum coloring uses $O(\log n)$ colors.
  [Kubicka and Schwenk, 1989]
- An optimal solution can be computed in $O(n \cdot \log^2 n)$ time using dynamic programming.

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It is APX-HARD to compute an optimum min-sum coloring in bipartite graphs [Bar-Noy and Kortsarz, 1998],

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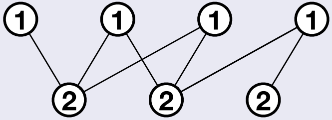
Example:

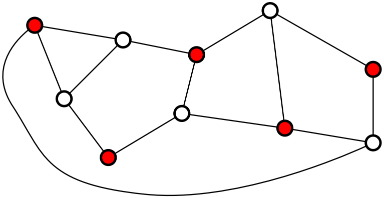
The natural 2-coloring is a 1.5-approximation.

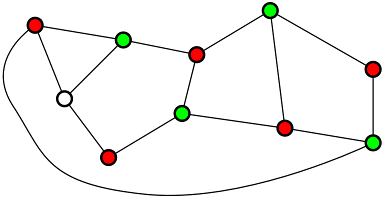


Analysis: The sum of colors is at most $n + \frac{1}{2} \cdot n = 1.5 \cdot n \leq 1.5 \cdot OPT$.

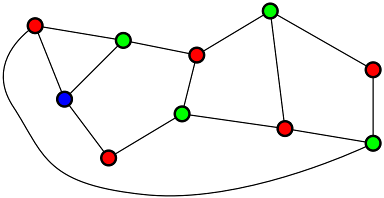
A Greedy Algorithm

Greedily coloring maximum independent sets yields a 4-approximation. [Bar-Noy et al, 1998]





Red = 1
Green = 2
Blue = 3



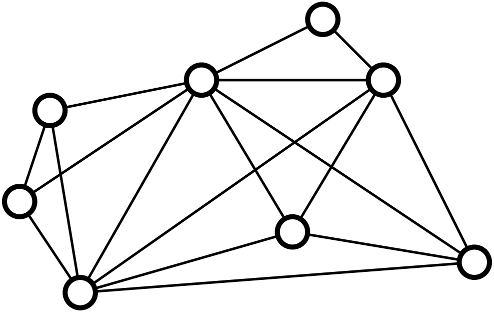
Some Known Approximation Bounds for Min-Sum coloring

Graph Class	Upper Bound	Lower Bound
Perfect	3.591	APX-HARD
Chordal	**1.796** $+ \epsilon$	APX-HARD
Interval	1.796	APX-HARD
Bipartite	27/26	APX-HARD
Planar	PTAS	NP-HARD
Line graphs	1.8298	APX-HARD

Chordal Graphs

Definition (Chordal Graphs)

No induced cycles of length ≥ 4, a.k.a. triangulated graphs.



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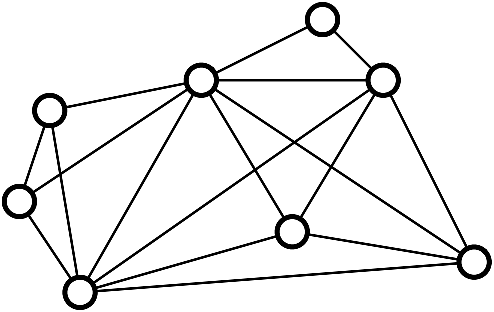

95% of "interference" graphs in the Java 1.5 library are chordal.
[Pereira and Palsberg, 2005]

Interval Graph Algorithm

Suppose G allows us to compute a maximum-size k-colorable subgraph ($\textsc{MkCS}$) in polynomial time for any $k \geq 1$.

Then, a 1.796-approximation is possible. [Halldórsson et al., 2008]

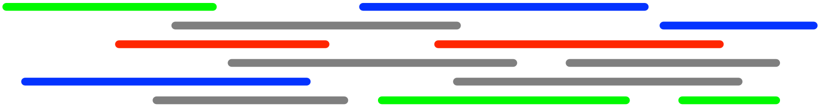

Figure: Interval graphs can be used to represent concurrent tasks.

Interval Graph Algorithm

Set $c := 3.591$ and pick some $\delta \in [0,1)$ uniformly randomly.

for $k = c^{\delta}, c^{\delta+1}, c^{\delta+2}, c^{\delta+3}, \ldots$

- Find a maximum k-colorable subgraph of G.
- Use the next k unused integers to color it.
- Remove these nodes from G.

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This yields an approximation with guarantee

$$\frac{c+1}{2 \cdot \ln c} \approx 1.796$$

MkCS in Chordal Graphs

Unfortunately, MkCS in chordal graphs is **NP**-hard.
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To get an improved approximation for min-sum coloring, we require an approach which can be used with MkCS approximations.

$\textsc{MkCS}$ in Chordal Graphs

Theorem (D., Friggstad, 2023)

Let $\mathcal{G}$ be a graph class that is closed under taking induced subgraphs. If there is a PTAS for weighted $\textsc{MkCS}$ in $\mathcal{G}$, then for any $\epsilon > 0$, there is a polynomial-time $(1.796 + \epsilon)$-approximation for minimum-sum coloring in $\mathcal{G}$.

MKCS in Chordal Graphs

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General Statement: Given a ρ-approximation for MkCS, our min-sum coloring approximation guarantee is:

$$\inf_{1 < c < \frac{1}{1-\rho}} \quad \frac{\rho \cdot (c+1)}{2 \cdot (1 - (1-\rho) \cdot c) \cdot \ln c}$$

Previous Work

Weighted MkCS in chordal graphs can be solved in $n^{O(k)}$ time.
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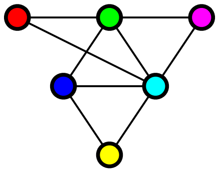
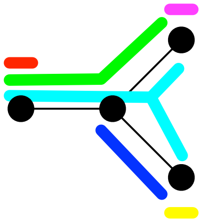
Any chordal graph is the intersection graph of subtrees of a tree $\mathcal{T}$.





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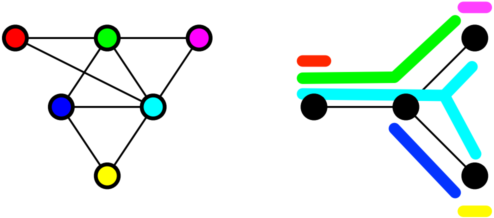

Dynamic Program:

For $v \in \mathcal{T}$ and any $\leq k$ subtrees S spanning v:

$$f(v, S) = \text{best solution in the subtree under } v \text{ that uses } S \text{ at } v$$

Approximations for Large k

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To get the $(1 - \epsilon)$-approximation for any constant ϵ:

- If $k \leq 8/\epsilon^3$, use the exact algorithm (running time $n^{O(1/\epsilon^3)}$).
- Otherwise, use our new $(1 - 2/k^{1/3}) \geq (1 - \epsilon)$ approximation.

Perfect Elimination Orderings

An ordering $v_1, ..., v_n$ of the nodes such that for each v_i, its right neighbors $N^r(v_i)$ form a clique.

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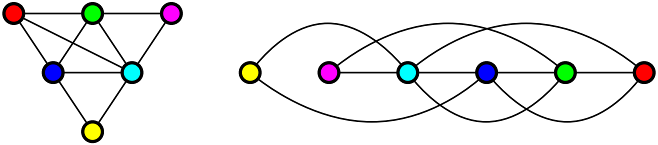
A graph is chordal iff it has a **perfect elimination ordering**.



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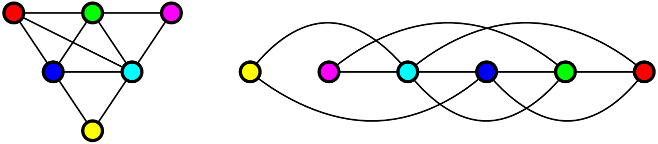

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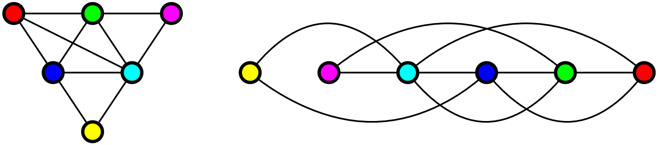

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Remark

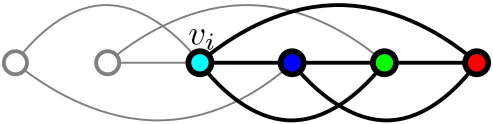
$S \subseteq V$ induces a k-colorable subgraph iff for each $v_i \in S$,

$$|N^r(v_i) \cap S| \leq k - 1.$$

Linear Programming Formulation

Let x_i indicate if we select v_i in our k-colorable subgraph.

$$\begin{array}{rrcll} \textbf{maximize}: & \sum_i w_i \cdot x_i & & & \\ \textbf{subject to}: & x_i + \sum_{v_j \in N^r(v_i)} x_j & \leq & k & \forall\ i \\ & x_i & \in & [0,1] & \forall\ i \end{array}$$



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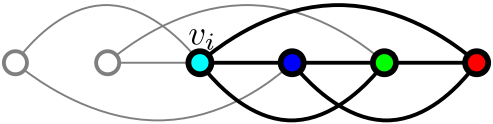

Rounding Algorithm

- $S \leftarrow \varnothing$
- **for** i from n down to 1:
  - Flip a coin c_i with bias $(1 - 1/k^{1/3}) \cdot x_i$ towards heads.
  - If $S \cup \{v_i\}$ is feasible and c_i = heads, add v_i to S.

Analysis of the Rounding Algorithm

For each v_i, the **expected** number of `heads` flipped from $N^r(v_i)$ is at most $k \cdot (1 - k^{-1/3}) = k - k^{2/3}$.

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Thus, v_i will be added with probability at least:

$$\mathbf{Pr}[|\#\texttt{heads} \text{ from } N^r(v_i)| < k \wedge c_i = \texttt{heads}] \geq (1-k^{-1/3}) \cdot (1-k^{-1/3}) \cdot x_i.$$

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$\therefore$ The expected size of S is at least $(1 - 2/k^{1/3}) \cdot OPT_{LP}$.

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This analysis approach was presented because there are generic ways to efficiently de-randomize such algorithms if the analysis only uses second moments.

With more hands-on work, we can de-randomize tighter analysis (using Chernoff bounds) to get a deterministic algorithm with guarantee

$$1 - \frac{O(1)}{k \log k}.$$

k-Colorable Subgraphs to Min-Sum Coloring

Theorem (D., Friggstad 2023)

Let $\mathcal{G}$ be a graph class that is closed under taking induced subgraphs.If there is a PTAS for weighted MkCS *in $\mathcal{G}$, then for any $\epsilon > 0$, there is a polynomial-time $(1.796 + \epsilon)$-approximation for minimum-sum coloring in $\mathcal{G}$.*

Linear Program

- $x_{v,k}$ - indicates that v has color k.
- $z_{C,k}$ - indicates that C is the set of nodes that are $\leq k$-colored.

$$
\begin{aligned}
\textbf{minimize:} \quad & \sum_{v \in V} \sum_{k=1}^{n} w_v \cdot k \cdot x_{v,k} && && (1) \\
\textbf{subject to:} \quad & \sum_{k=1}^{n} x_{v,k} = 1 && \forall\ v \in V && (2) \\
& \sum_{C \in \mathcal{C}_k} z_{C,k} \leq 1 && \forall\ 1 \leq k \leq n && (3) \\
& \sum_{C \in \mathcal{C}_k : v \in C} z_{C,k} \geq \sum_{k' \leq k} x_{v,k'} && \forall\ v \in V, 1 \leq k \leq n && (4) \\
& x, z \geq 0
\end{aligned}
$$

Last constraint: (partial) agreement between z and x on the statement v is colored by color at most k.

Taking the Dual

maximize: $$\sum_{v \in V} \alpha_v - \sum_{k=1}^{n} \beta_k$$

subject to:

$$\alpha_v \leq w_v \cdot k + \sum_{\hat{k}=k}^{n} \theta_{v,\hat{k}} \quad \forall\ v \in V, 1 \leq k \leq n \tag{5}$$

$$\sum_{v \in C} \theta_{v,k} \leq \beta_k \quad \forall\ 1 \leq k \leq n, C \in \mathcal{C}_k \tag{6}$$

$$\beta, \theta \geq 0 \tag{7}$$

Solving and Rounding

Via the ellipsoid method for solving LPs, this yields a solution (x, z) with value $\leq OPT$ for the following slightly modified LP.

$$\begin{array}{rcll} \sum_t x_{v,t} & = & 1 & \forall\ v \\ \sum_S z_{S,t} & = & 1/\rho & \forall\ t \\ \sum_{S \ni v} z_{S,t} & \geq & \sum_{t' \leq t} x_{t',v} & \forall\ v, t \\ x, z & \geq & 0 & \end{array}$$

Rounding Algorithm

$c \leftarrow 3.591$

$\delta \sim [0, 1)$ uniformly at random

For $k := c^{\delta}, c^{\delta+1}, c^{\delta+2}, \ldots$

- Sample a $\lfloor k \rfloor$-colorable subset S from the distribution $\rho \cdot z_S$.
- Randomly permute the coloring.
- Concatenate this coloring to the coloring of G so far.

Next Steps: Min-Sum Coloring on Perfect Graphs

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The "latency constant" $c \approx 3.591$ shows up in the current best approximation ratio for interval, chordal, and perfect graphs.

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Question:

Is c a fundamental lower bound, or can these approximations be improved?

Thank you!