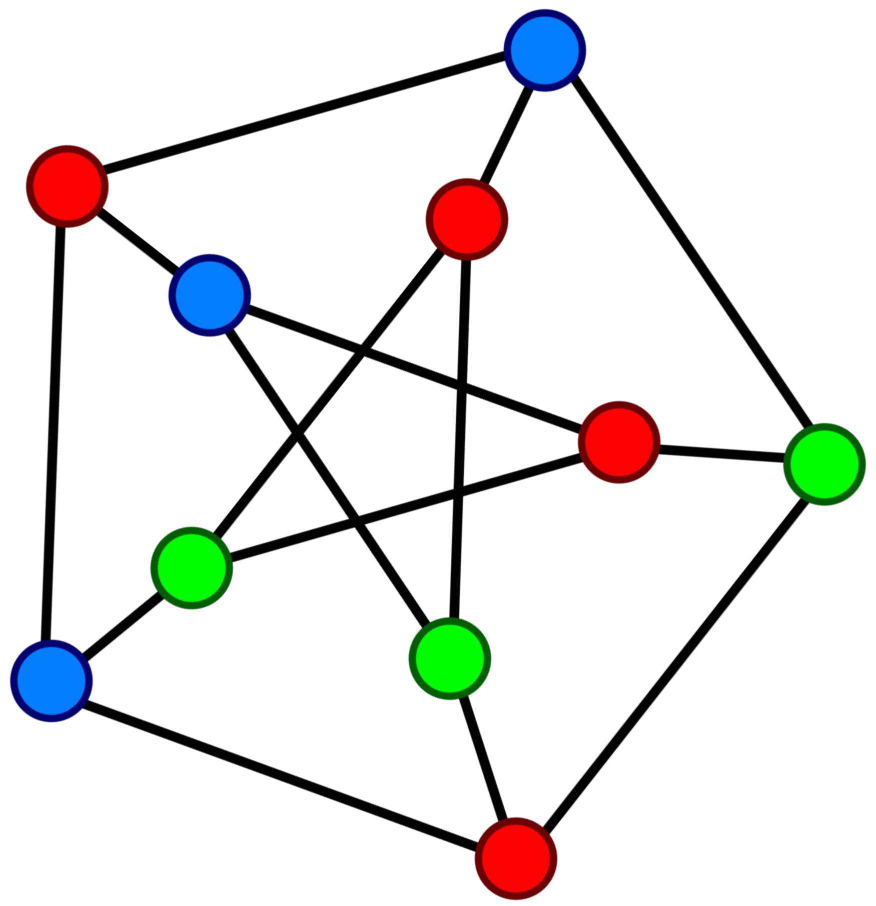
3-Coloring (C_4, C_s)-free Diameter Two Graphs



Presented by: Prahlad Narasimhan Kasthurirangan

Authors: Tereza Klimošová (Charles University), Vibha Sahlot (University of Cologne)

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Given: An undirected diameter two graph $G(V, E)$ with no induced C_4 and C_s for some fixed $s \geq 10$.

Aim: Decide if G is 3-colorable?

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Given: An undirected diameter two graph $G(V, E)$ with no induced C_4 and C_s for some fixed $s \geq 10$.

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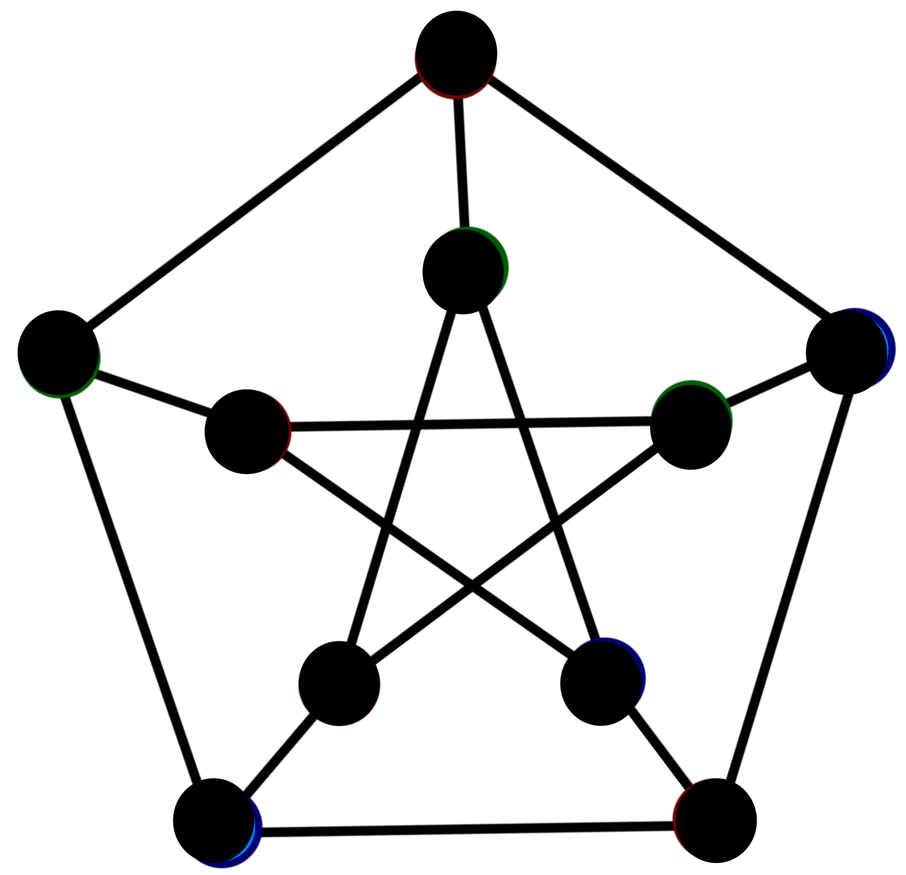
- Distance between two vertices is the length of the shortest path between then (number of edges in the shortest path).
- Diameter of a graph is the distance between any two most distanced vertices.



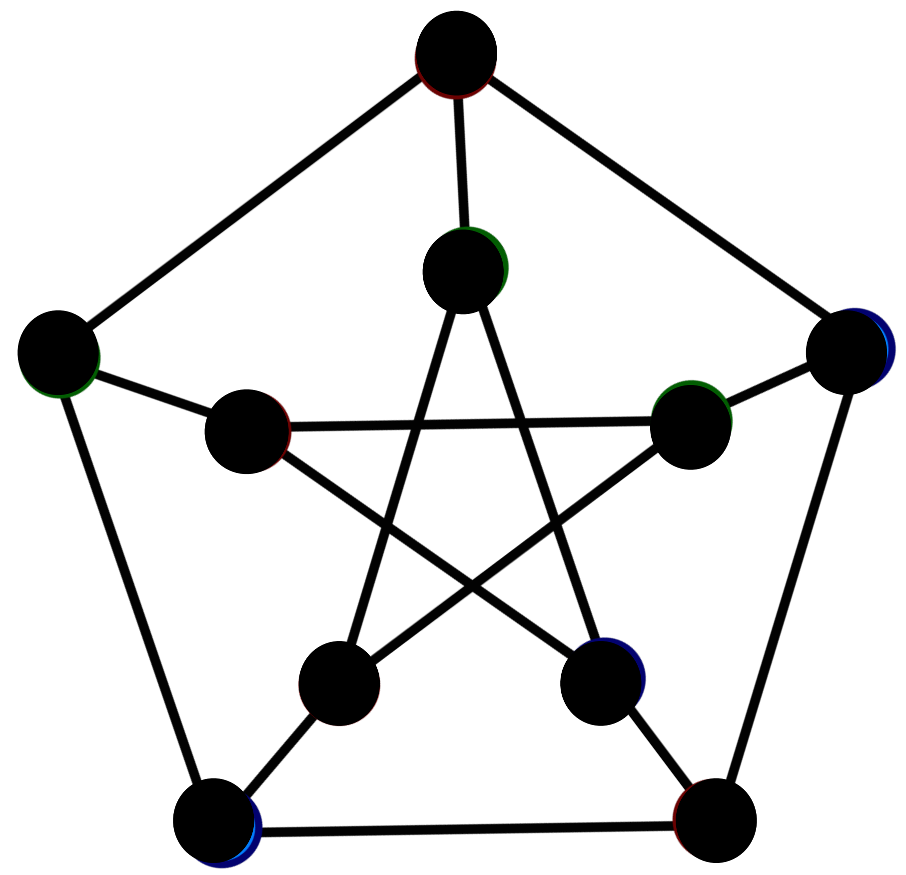
Graph with diameter two

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Given: An undirected diameter two (C_4, C_s)-free graph $G(V, E)$ for some fixed $s \geq 10$.

Aim: Decide if G is 3-colorable?

– A graph G is H-free if it does not contain H as an induced subgraph



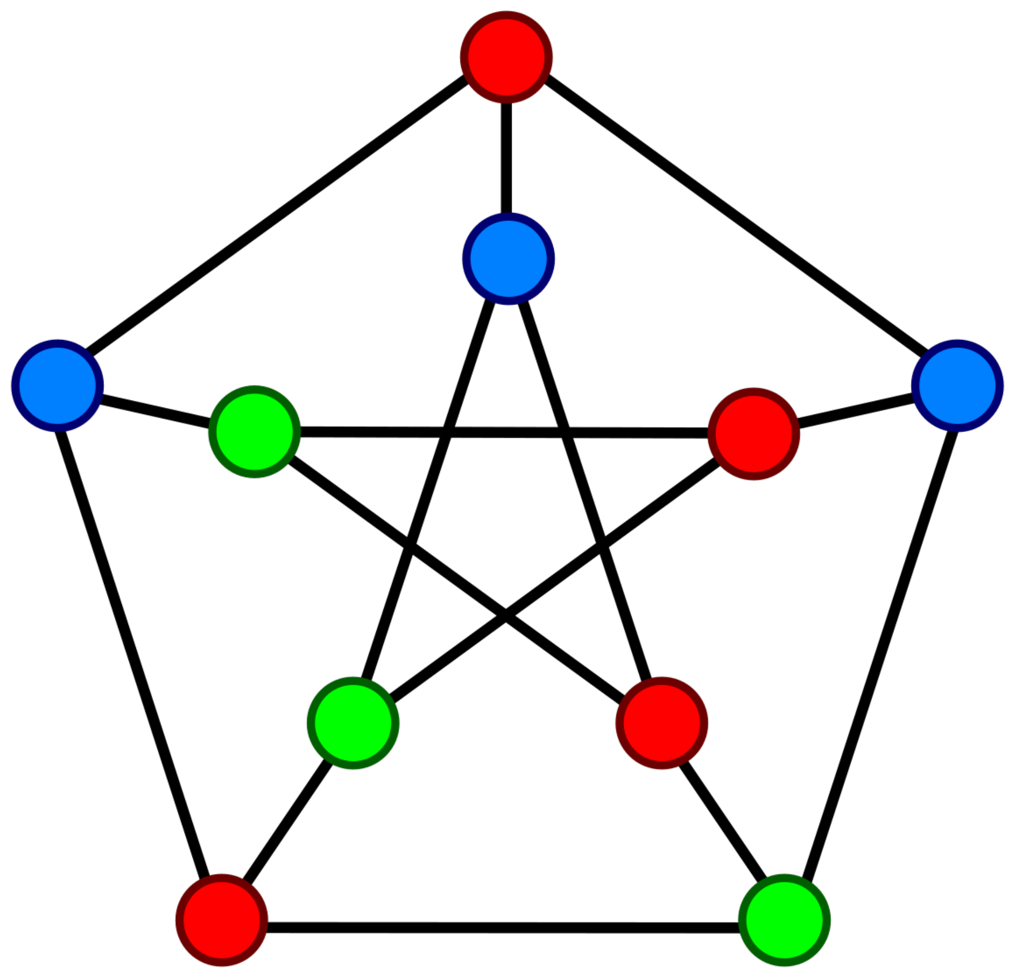
Graph with no induced C_4

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Given: An undirected diameter two (C_4, C_s)-free graph $G(V, E)$ for some fixed $s \geq 10$.

Aim: Decide if G is 3-colorable?

– A graph G is k-colorable if we can assign k colors to the vertices of G such that endpoints of every edge is coloured distinctly.



A 3-colorable graph

3-Coloring (C_4, C_s)-free Diameter Two Graphs

- The 3-Coloring is NP-hard even on planar graphs [Garey,Jonson,Stockmeyer 1976]

- Lots of research has been done on hereditary classes of graphs, i.e., classes that are closed under vertex deletion.

- However, many natural classes of graphs are not hereditary, for example, graphs with bounded diameter.

3-Coloring (C_4, C_s)-free Diameter Two Graphs

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- However, many natural classes of graphs are not hereditary, for example, graphs with bounded diameter.

- 3-Coloring on graphs with diameter two has been posed as an open problem in several papers.

Previous Results

- 3-Coloring is NP-complete for the class of graphs with diam. 3, even for triangle-free graphs [Mertzios and Spiraki 2016].

Subexponential algorithm for 3-Coloring diam.2 graph with runtime $2^{\mathcal{O}(\sqrt{nlogn})}$.

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- Improved algorithm for 3-Coloring diam. 2 graph with runtime $2^{\mathcal{O}(n^{\frac{1}{3}}log^2 n)}$ [Debski et al.2022].
- 3-Coloring diam. 2 graph is solvable in polynomial time for:
  - graphs that have at least one articulation neighborhood [Mertzios,Spirakis 2016].
  - (C3, C4)-free graphs [Martin et al. 2019].
  - C_5 -free or C_6-free graphs , (C_4, C_s)-free graphs where $s \in \{7,8,9\}$ [Martin et al. 2021].
  - $K^2_{1,r}$-free or $S_{1,2,2}$-free graphs, where r ≥ 1 [Martin et al. 2019]

Our Results

Theorem 1. List 3-Coloring is solvable in polynomial time on (C_4, C_s)-free graphs with diameter two for any constant s ≥ 10.

Theorem 2. List 3-Coloring is solvable in polynomial time on (C_3, C_7)-free graphs with diameter two.

Our Results

Theorem 1. List 3-Coloring is solvable in polynomial time on (C_4, C_s)-free graphs with diameter two for any constant $s \geq 10$.

This Presentation

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Given: An undirected diameter two graph $G(V, E)$ with no induced C_4 and C_s for some fixed $s \geq 10$.

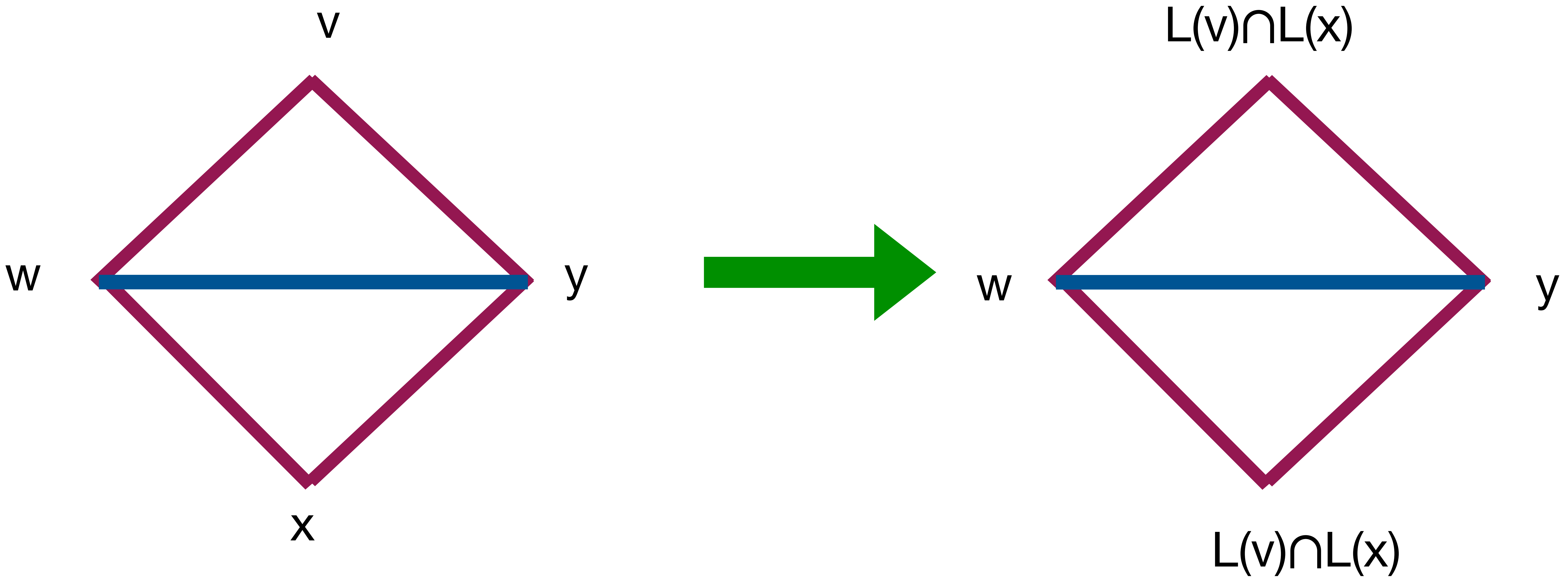
Aim: Decide if G is 3-colorable?

Technique

- 2-List Coloring is polynomial time solvable[Edward 1986].
- Convert given instance into instance of List-3-Coloring.
- Use polynomial time reductions to convert G into polynomially many instances of 2-List Coloring.

Some simple reductions

– If G contains a diamond (v,w,x,y) then assign L(v)∩L(x) to both x and v.



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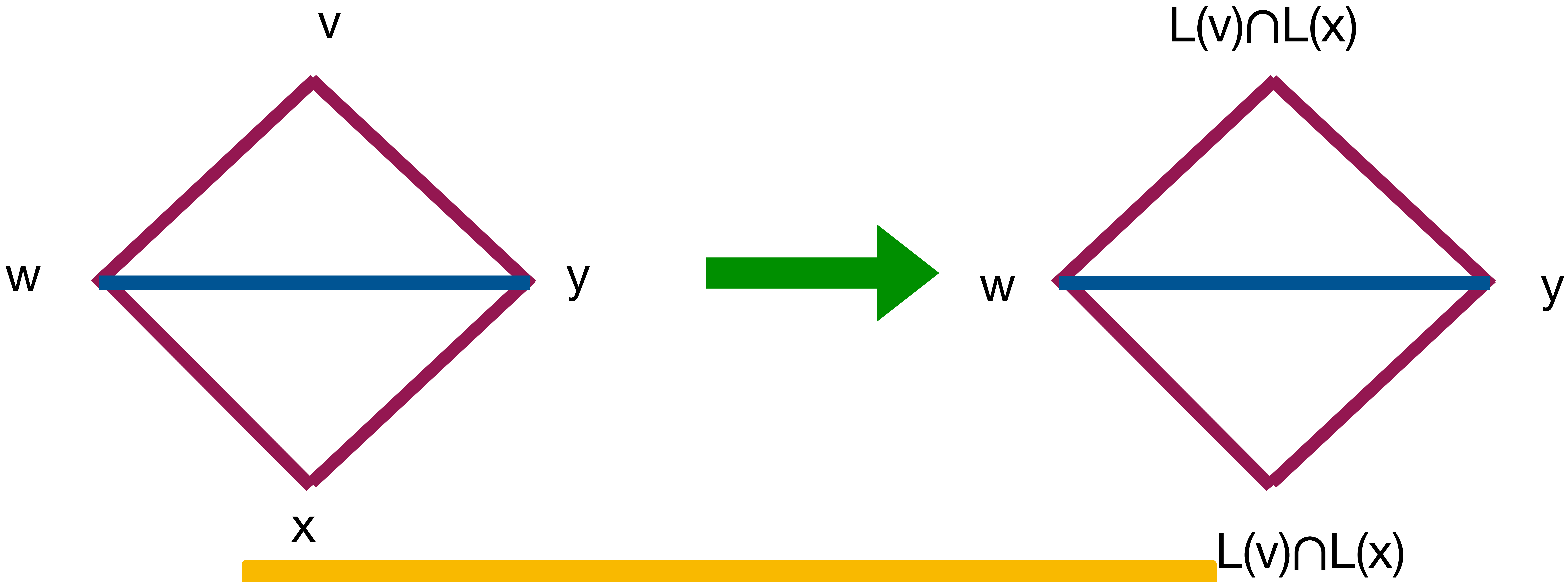
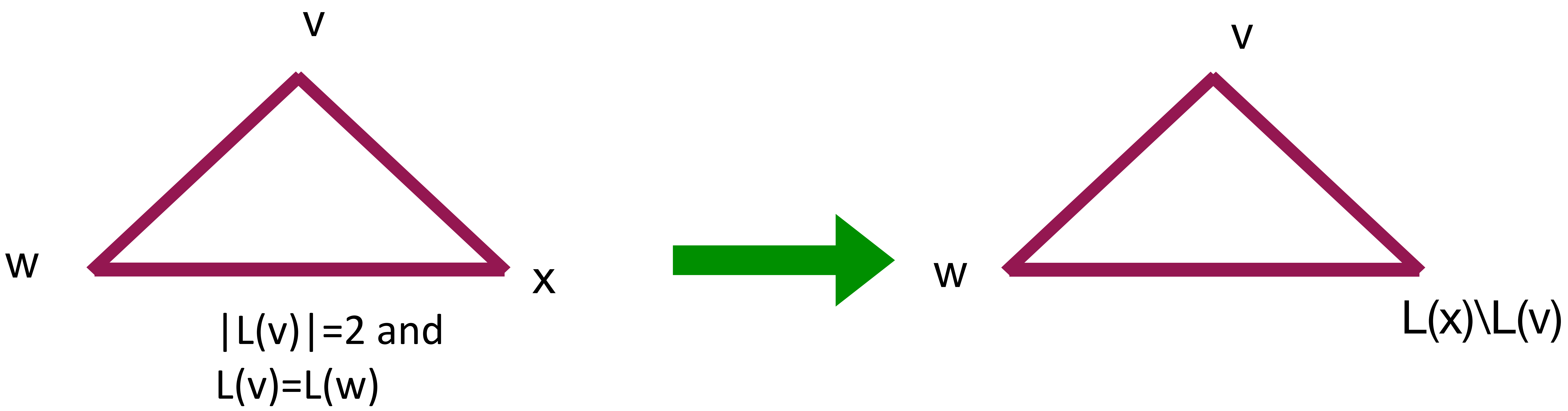


w and y are coloured differently from v and x

Some simple reductions

– If G contains a triangle (v,w,x) where |L(v)|=2 and L(v)=L(w), then assign L(x)\L(v) to x.

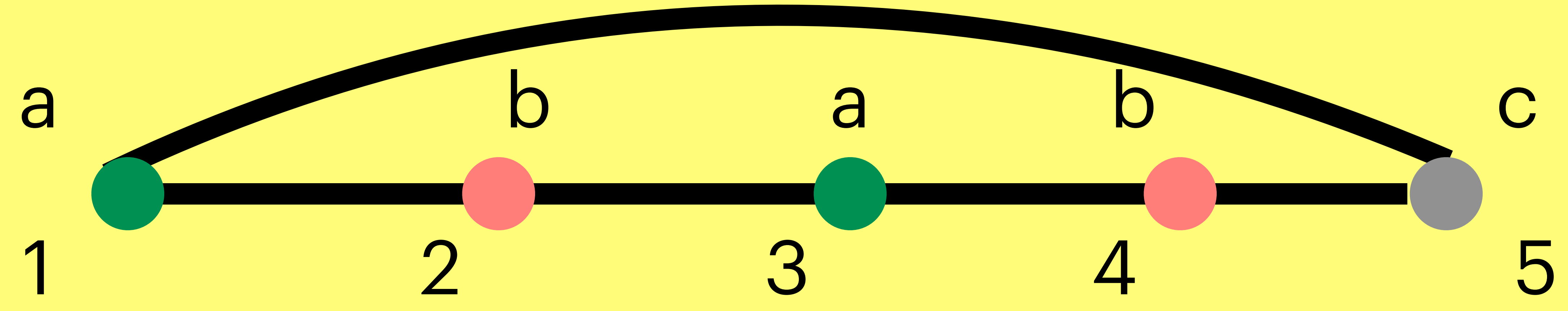


Colors in L(v) are used to color v and w

3-Coloring (C_4, C_s)-free Diameter Two Graphs

– If G contains contains an induced C_5 colour it as follows (upto cyclic ordering). Otherwise polynomial time solvable.

C_5



N_1

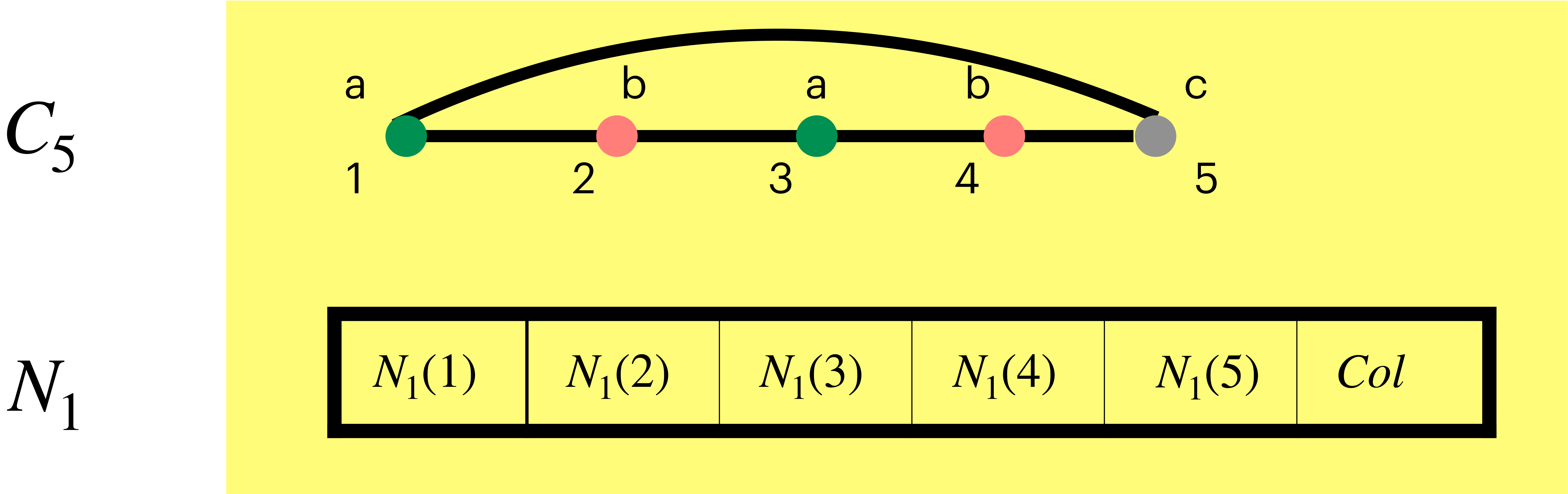
$N_1(1)$	$N_1(2)$	$N_1(3)$	$N_1(4)$	$N_1(5)$	Col

N_2

L_3	L_2	Col

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. If there are at most k connected components in some $N_1(i)$ for $i \in [5]$, then polynomial time by resolving 2^k instances of 2-List Coloring.



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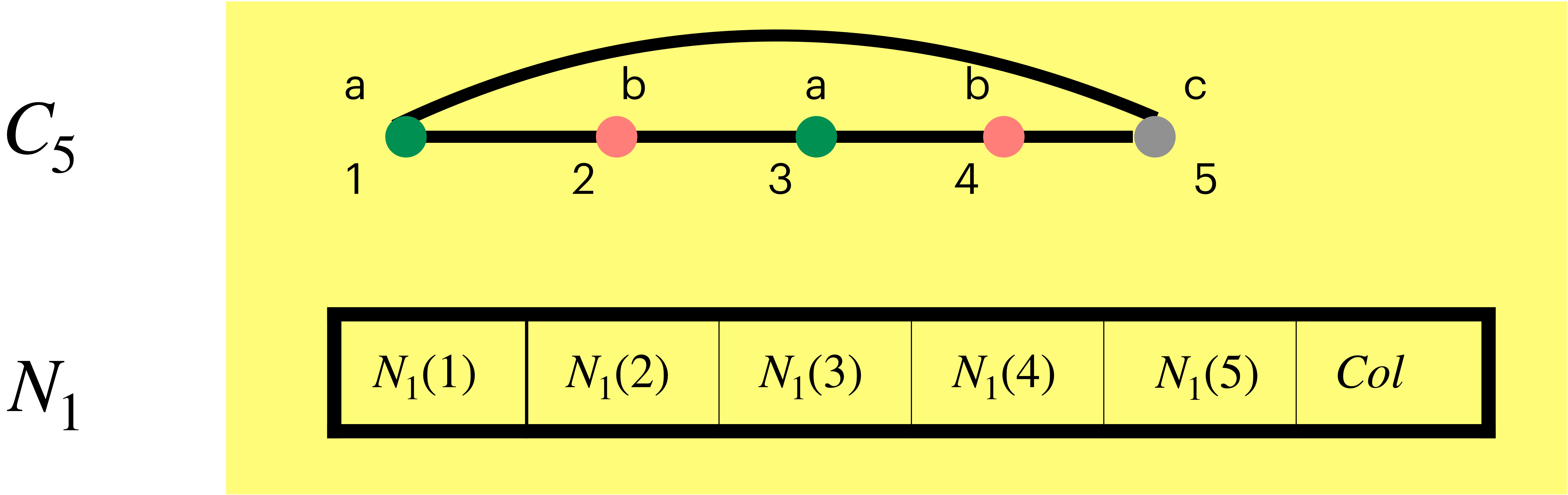
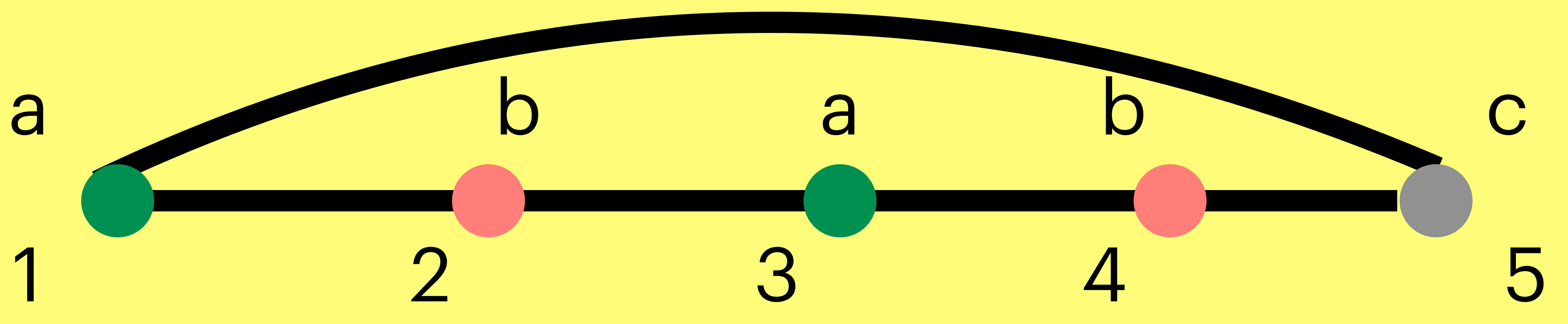


No odd cycle in $N_1(i)$ and thus all connected components are bipartite

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. Each vertex in $N_1(i)$ is not adjacent to any $j \neq i$ for $i, j \in [5]$ in C_5.

C_5



N_1

$N_1(1)$	$N_1(2)$	$N_1(3)$	$N_1(4)$	$N_1(5)$	Col

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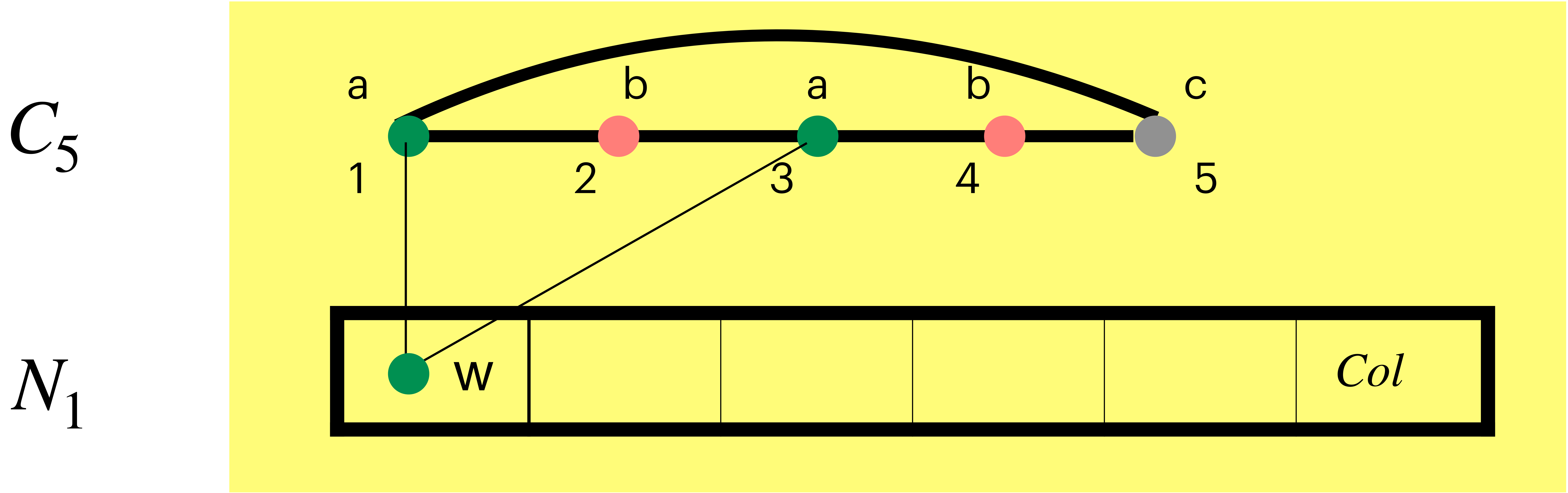
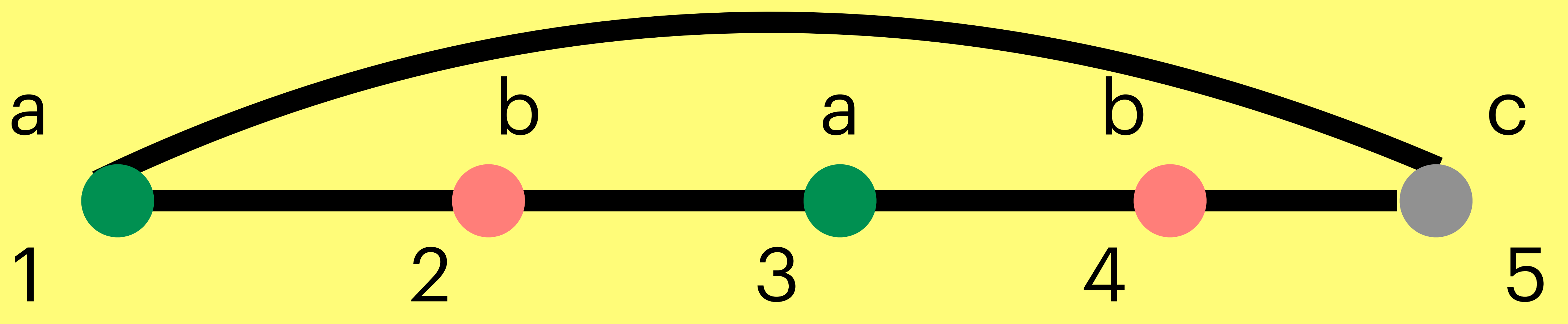


No induced C_4 (by assumption) and $(w,2), (w,4), (w,5) \notin E(G)$ (otherwise $w \in Col$)

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. Each vertex in $N_1(i)$ has at most one neighbor in $N_1(j)$ for $i, j \in [5]$.

C_5



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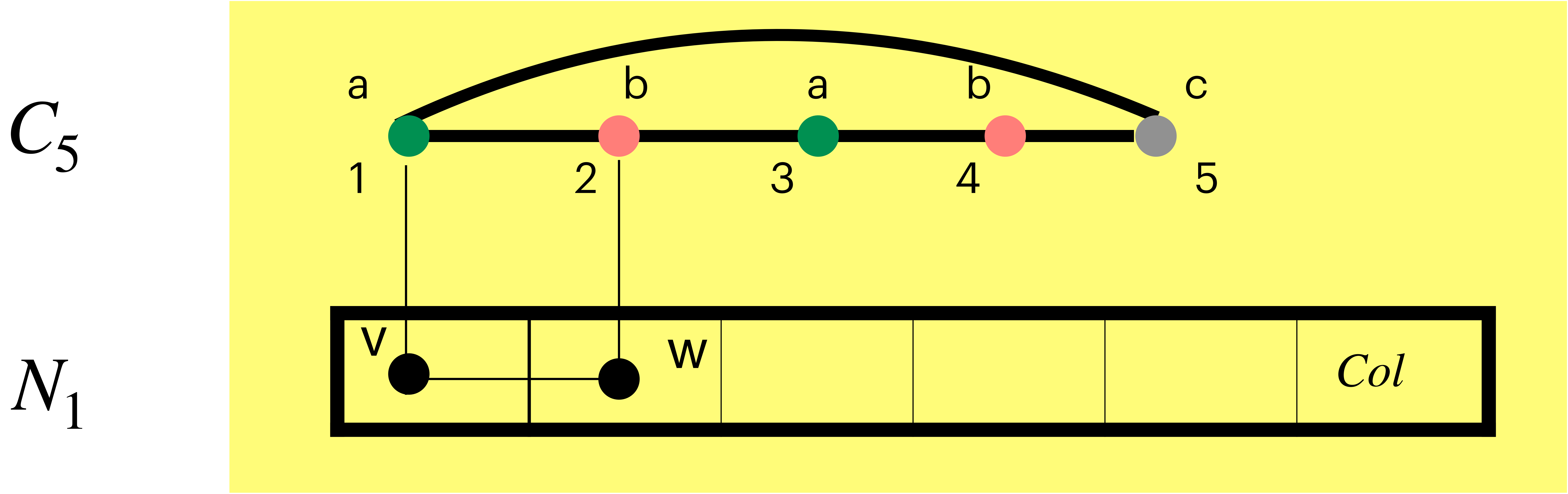

Vertex in $N_1(i)$ *is not adjacent to vertex in* $N_1(i+1)$

3-Coloring (C_4, C_s)-free Diameter Two Graphs

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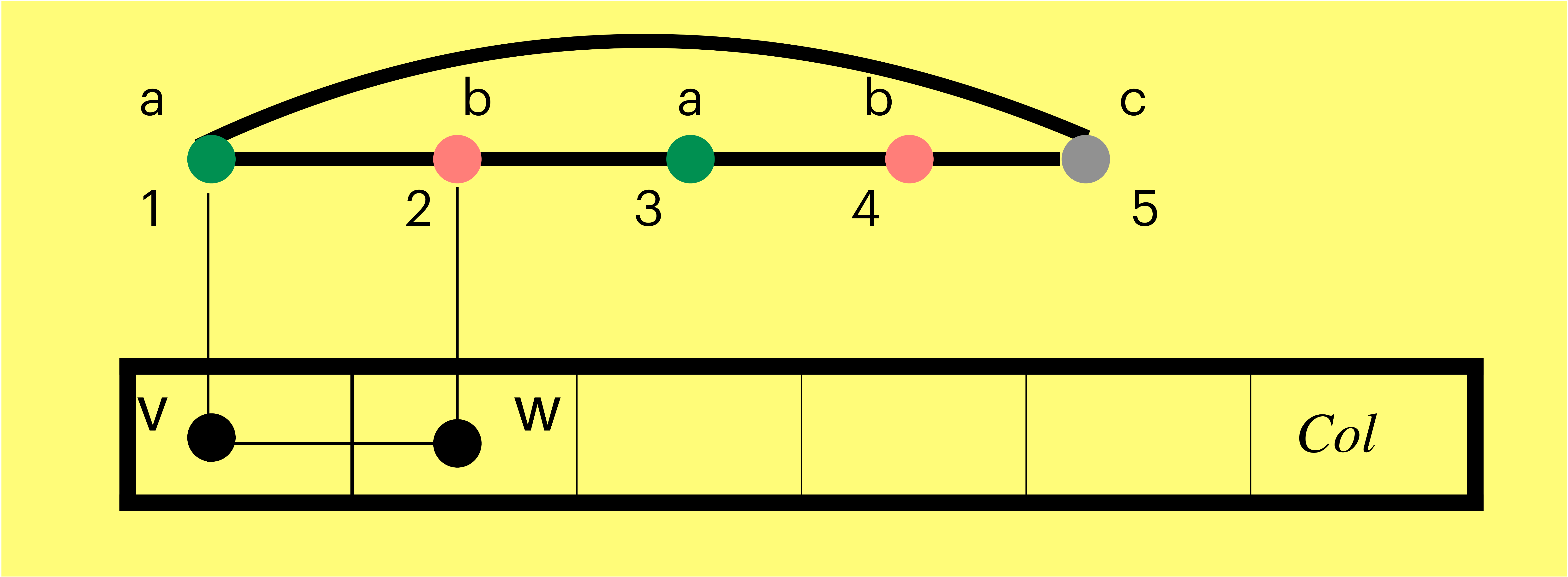
C_5

N_1



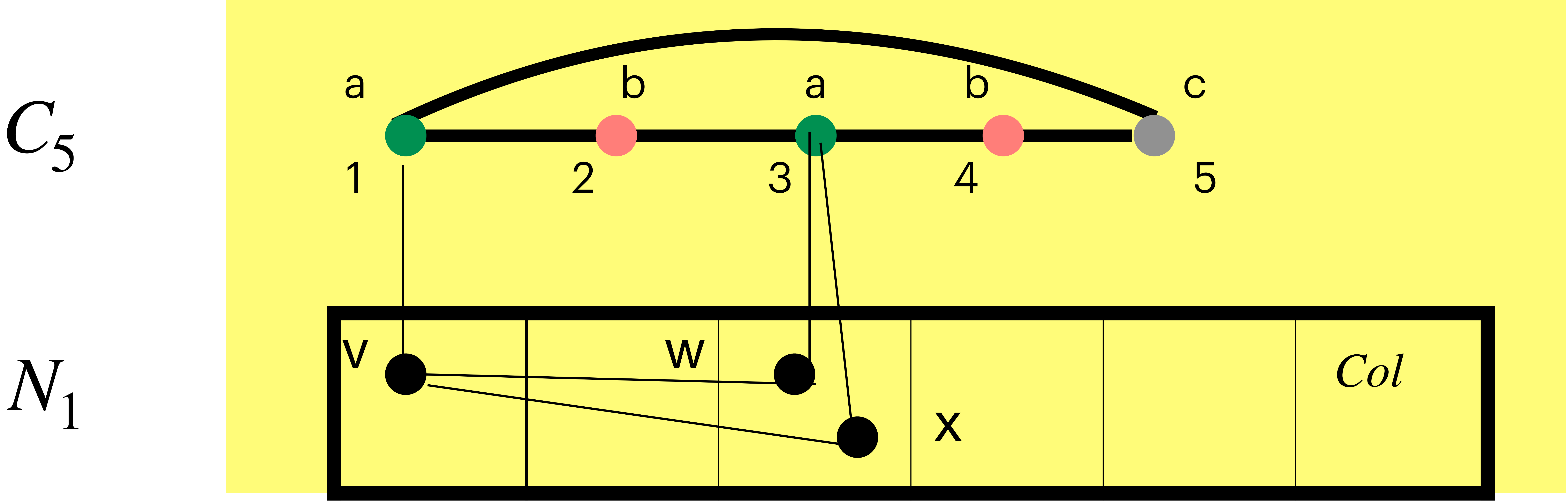
Vertex in $N_1(i)$ *is not adjacent to vertex in* $N_1(i+1)$

No induced C_4 (by assumption) and no K_4 (otherwise not 3-colorable)

No **diamond** as otherwise v or w belongs to Col

3-Coloring (C_4, C_5)-free Diameter Two Graphs

Lemma. Eachvertex in $N_1(i)$ has at most one neighbor in $N_1(j)$ for $i, j \in [5]$.



Vertex in $N_1(i)$ *is adjacent to at most one vertex in* $N_1(i+2)$ *and* $N_1(i+3)$

3-Coloring (C_4, C_s)-free Diameter Two Graphs

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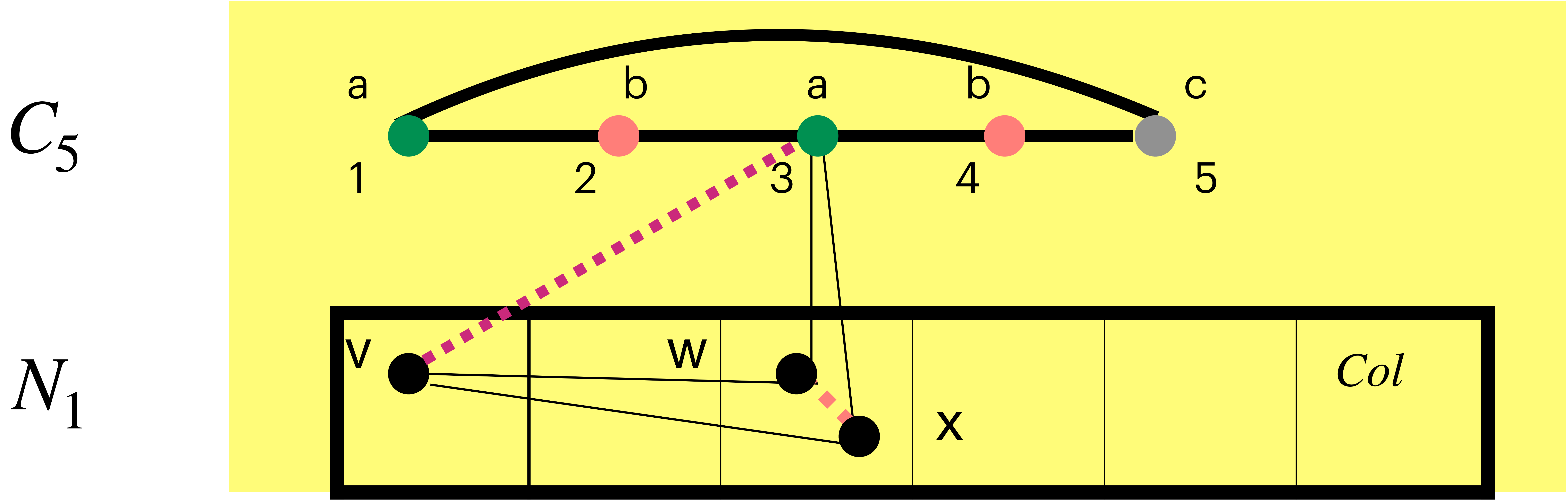

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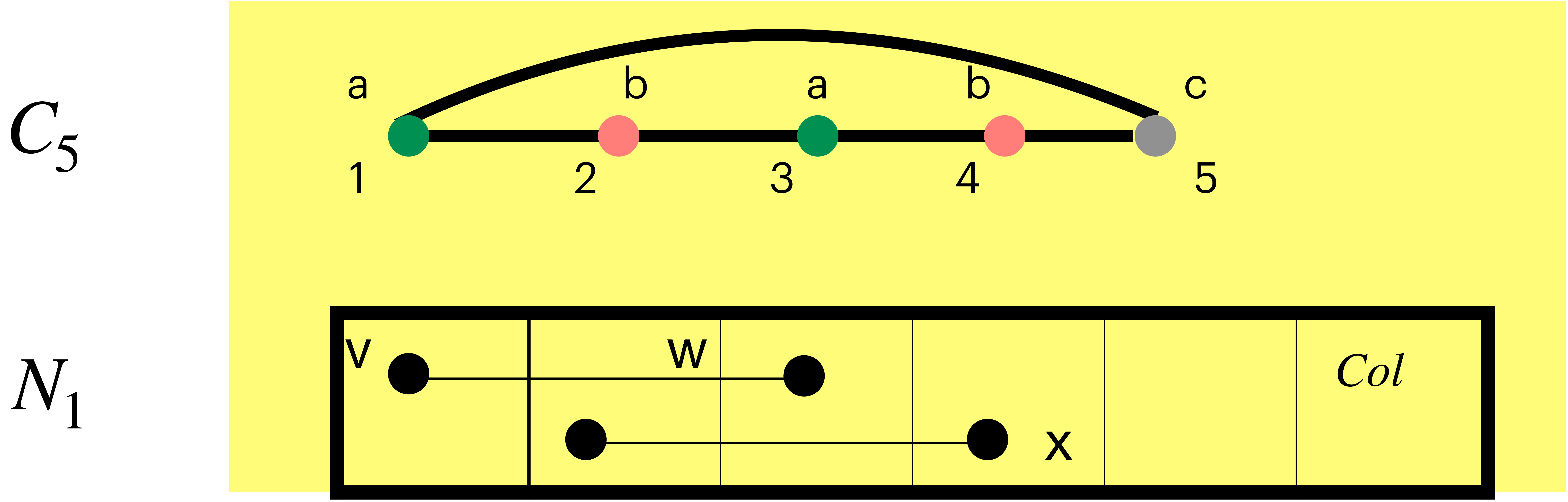
$(v,3) \notin E(G)$ (previous lemma) and $(w, x) \notin E(G)$ (otherwise $(v) \notin Col$)

Similarly for (I+3)

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. $|N_1(1)| = |N_1(3)|, |N_1(2)| = |N_1(4)|$ and $G[N_1(1), N_1(3)]$, $G[N_1(2), N_1(4)]$, are perfect matchings.



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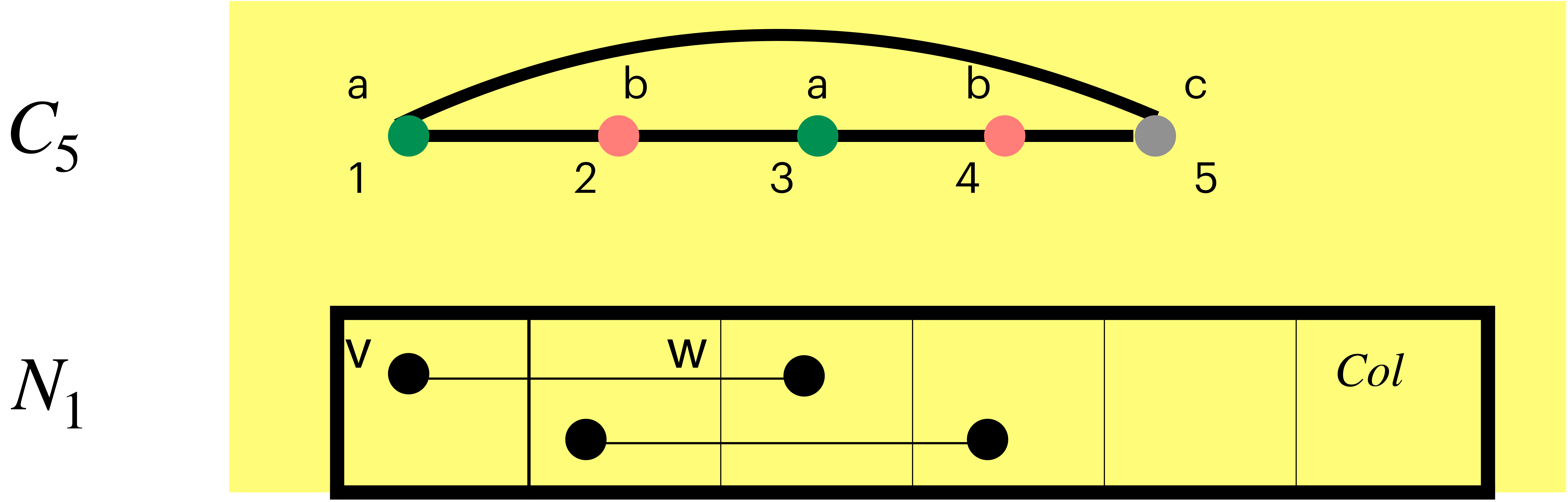
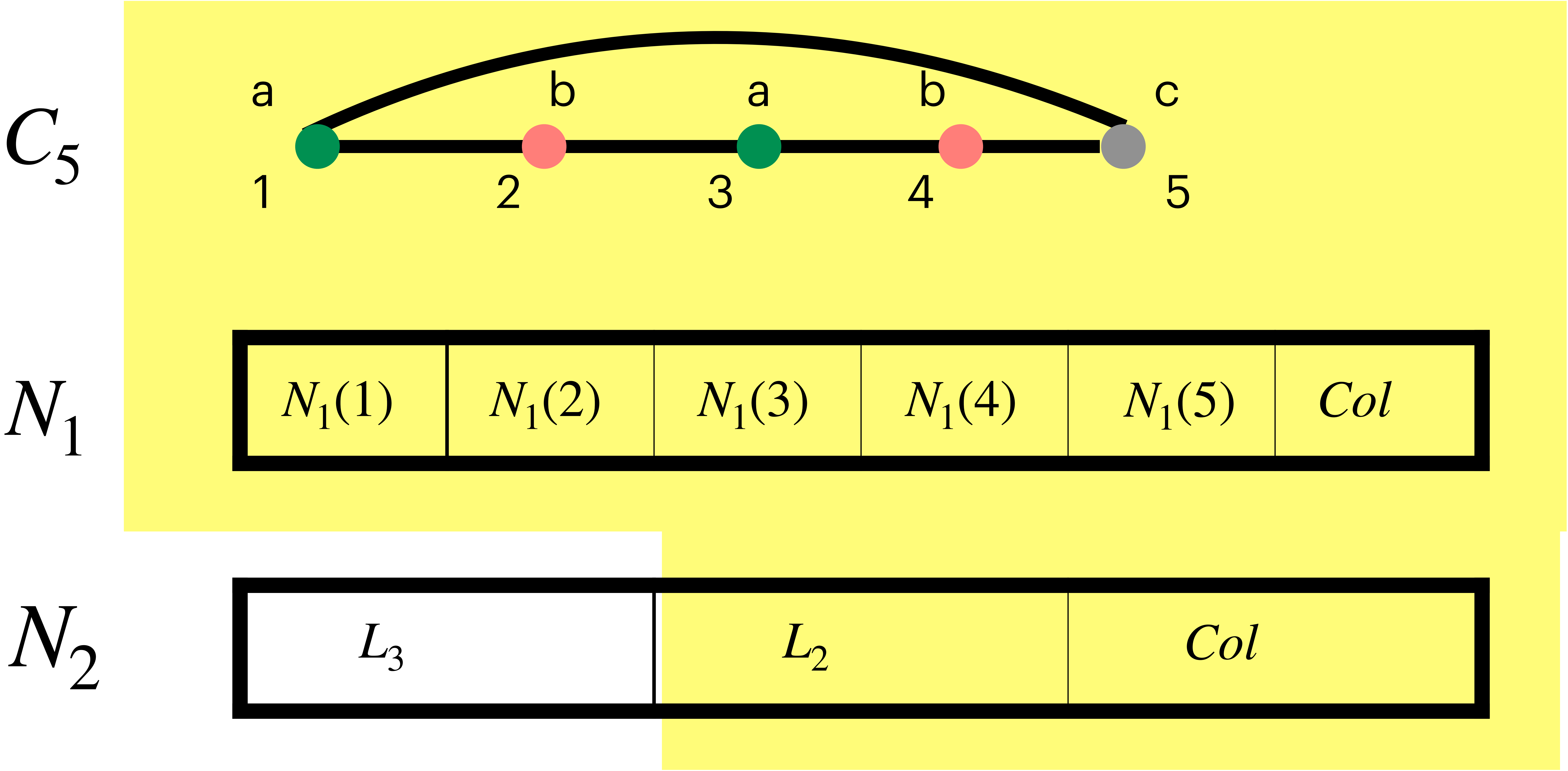


If v is not adjacent to any vertex in $N_1(3)$, then distance between v and 3 is more than 2 (Contrad.) and v is adjacent to at most one vertex in $N_1(3)$ (by previous lemma)

3-Coloring (C_4, C_5)-free Diameter Two Graphs

Lemma. All $v \in N_2 \backslash Col$ has at most one neighbor in each $N_1(i)$, $i \in [5]$ and all $v \in L_3$ has exactly one neighbor in each $N_1(i)$, $i \in [5]$.



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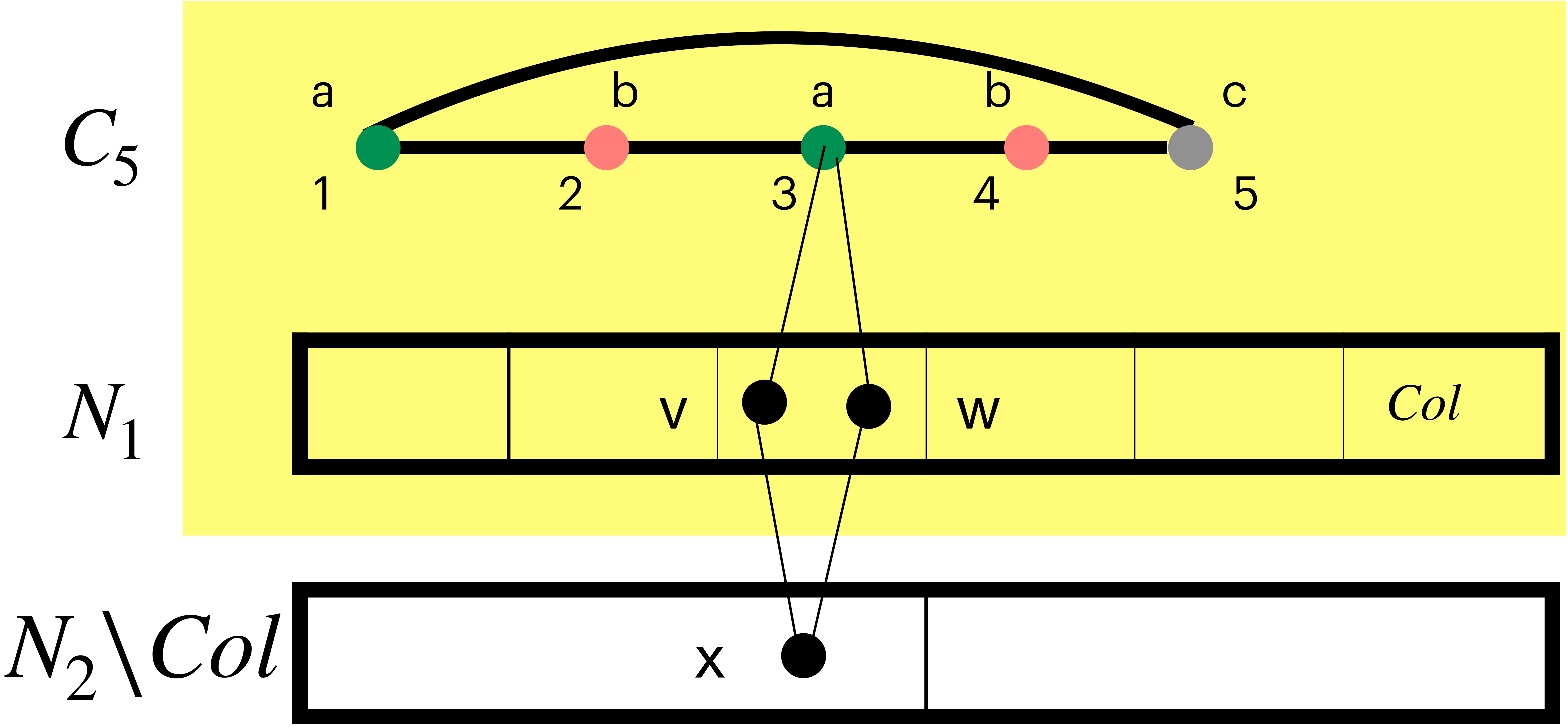
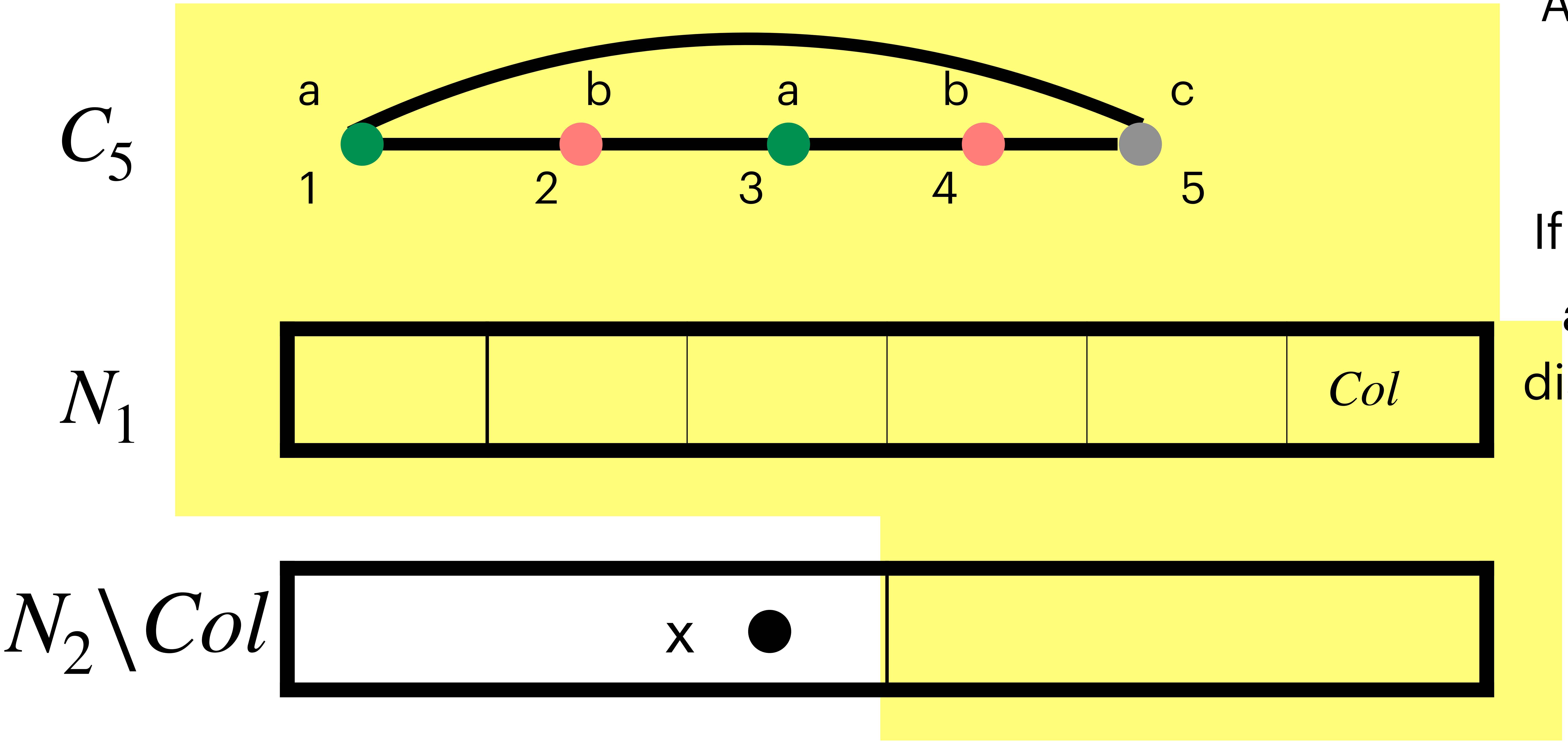


All $v \in N_2 \backslash Col$ has at most one neighbor in each $N_1(i), i \in [5]$

$(v, w) \notin E(G)$ (otherwise $x \in Col$) and $(a, x) \notin E(G)$ (by contruction)

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. All $v \in N_2 \backslash Col$ has at most one neighbor in each $N_1(i), i \in [5]$ and all $v \in L_3$ has exactly one neighbor in each $N_1(i), i \in [5]$.

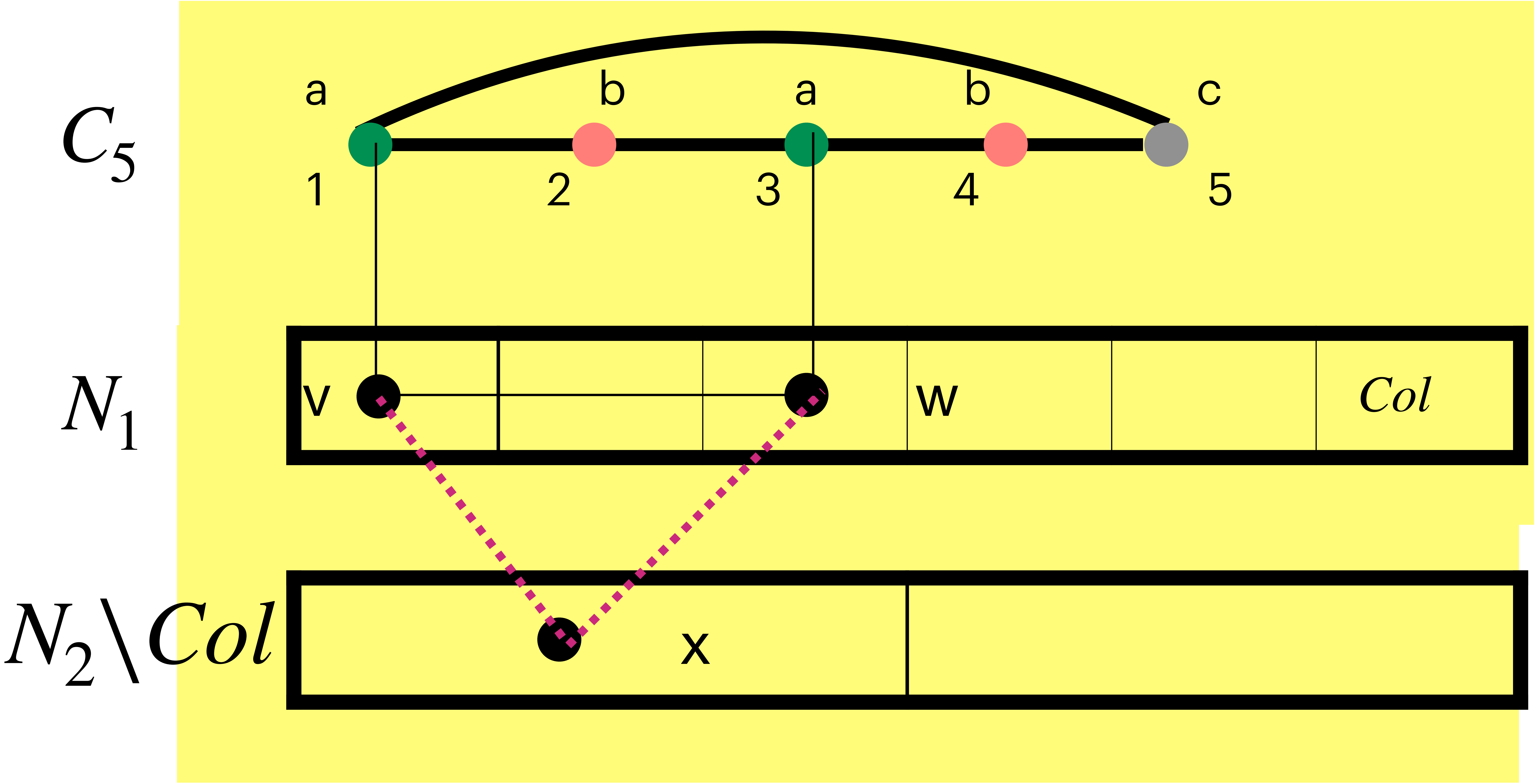


All $v \in L_3$ has exactly one neighbor in each $N_1(i), i \in [5]$

If $x \in L_3$ is not adjacent to any vertex in $N_1(1)$ then distance between x and 1 is more than 2(Contrad.)

3-Coloring (C_4, C_5)-free Diameter Two Graphs

Lemma. Any $v \in N_1(1)$ and $w \in N_1(3)$ such that $(v, w) \in E(G)$, then v and w don't share common neighbor in L_2 or L_3. Similarly for vertices in $N_1(2)$ and $N_1(4)$.



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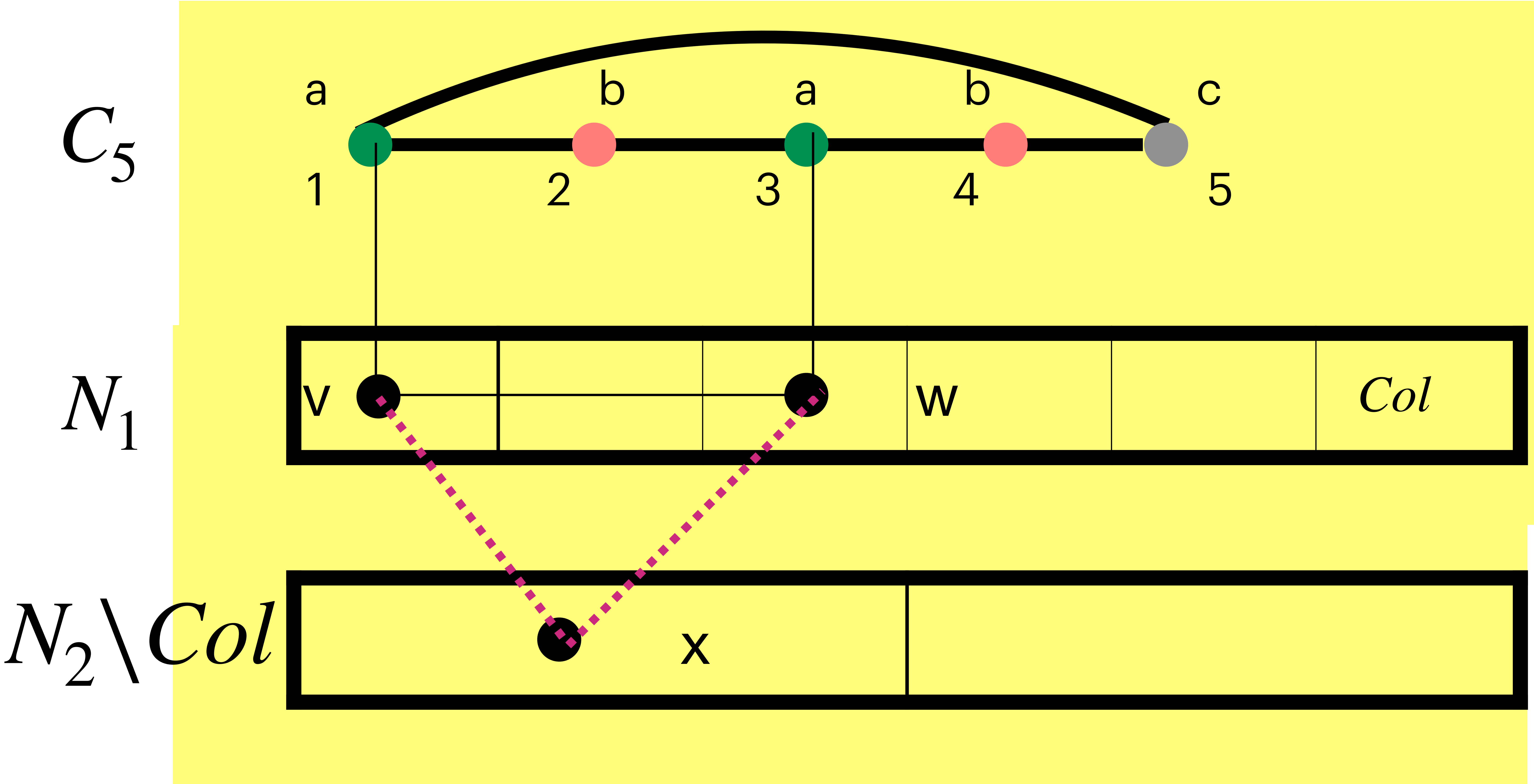
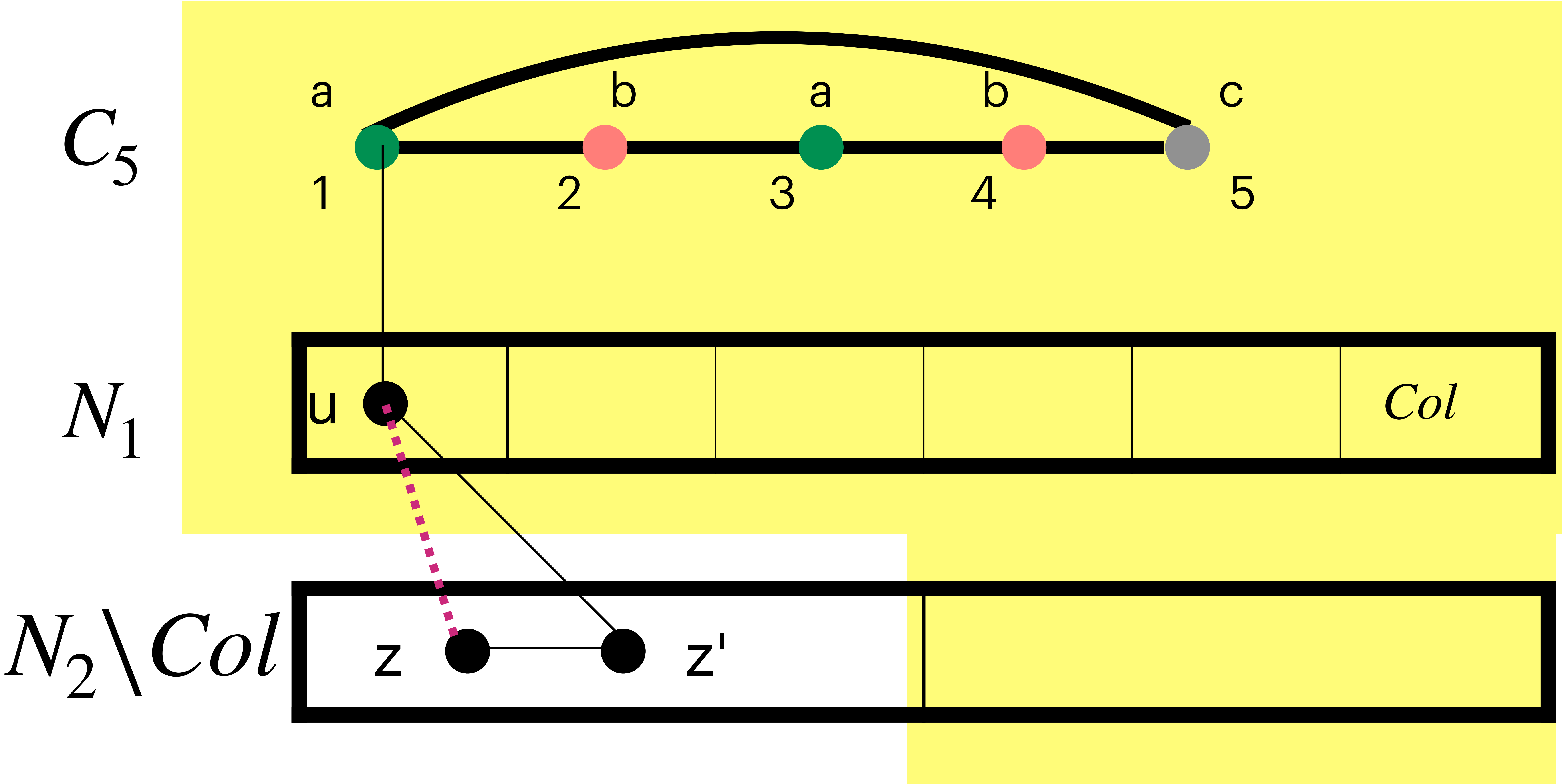


If $x \in L_3 \cup L_2$ is adjacent to both v and w, then x should be coloured c and thus $x \notin L_3 \cup L_2$

3-Coloring (C_4, C_5)-free Diameter Two Graphs

Lemma. If $z \in L_3$ and $u \in N_1(i), i \in [5]$ such that $(z, u) \notin E(G)$, then there is at most one vertex $z' \in L_3 \cup L_2, z \neq z'$ such that $(z, z'), (u, z') \in E(G)$.



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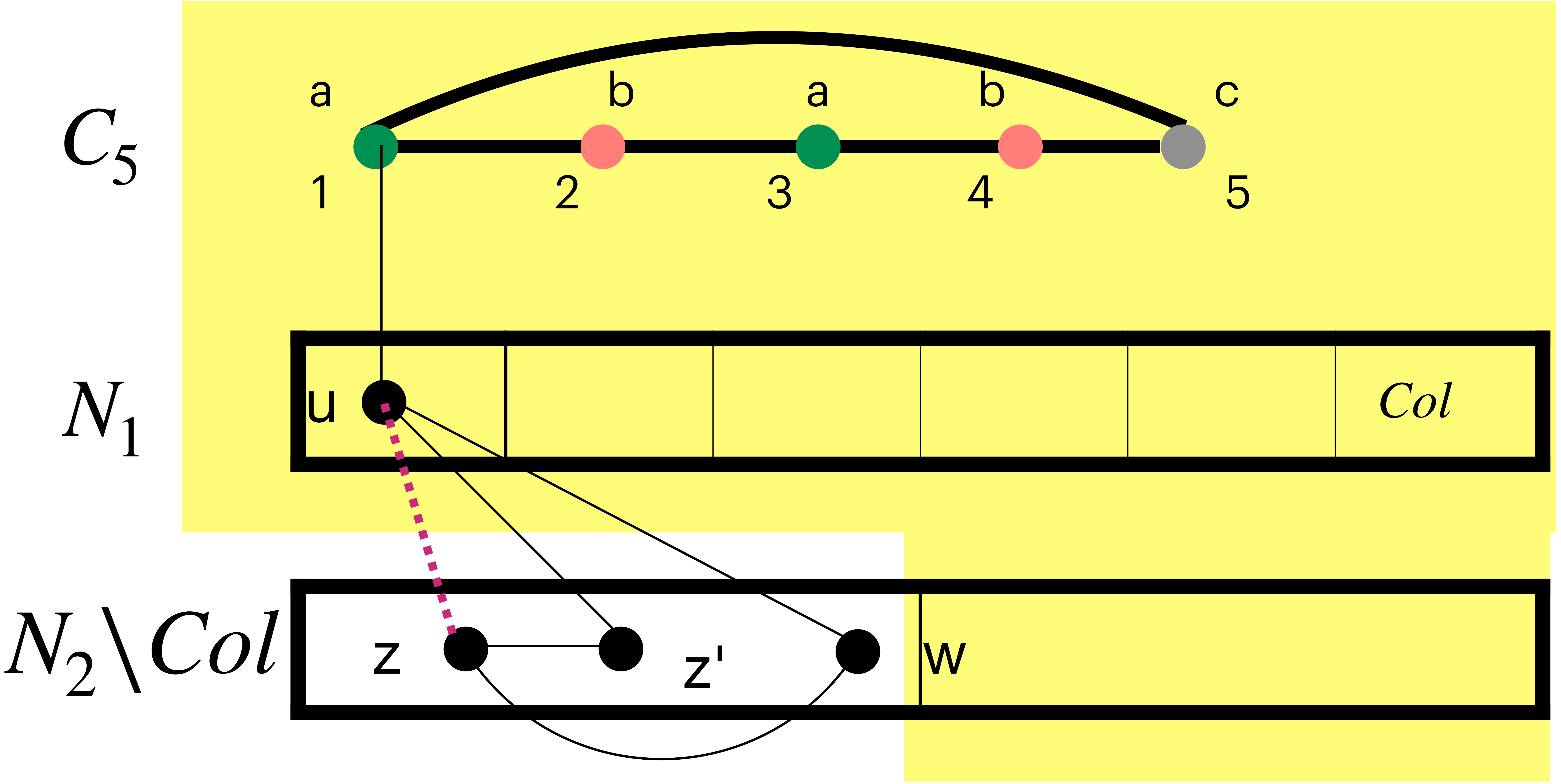
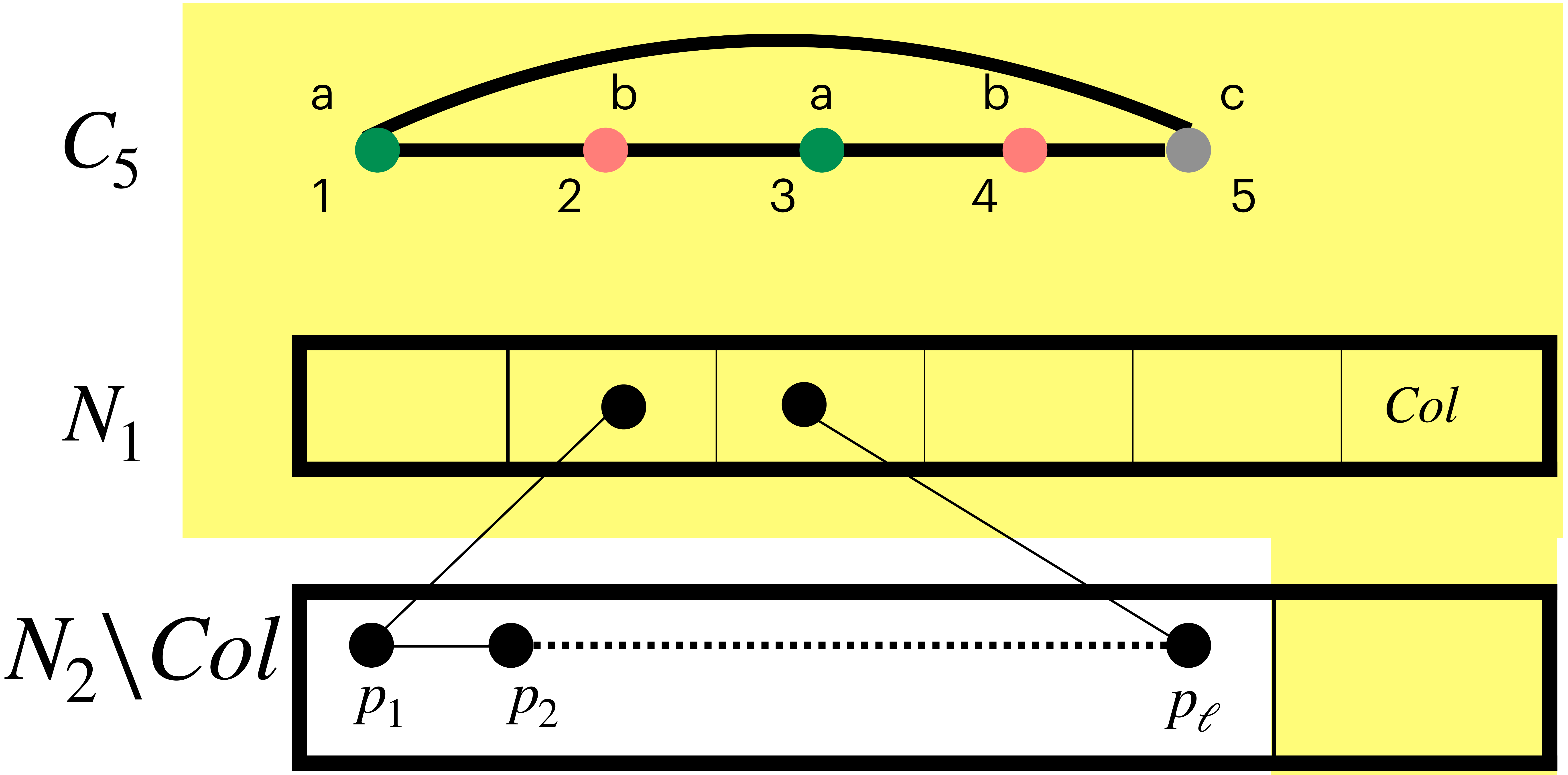


If $w \in L_3$ s.t. $(w, z), (w, u) \in E(G)$, then as G is C_4-free, $(z', w) \in E(G)$ and L(u)=L(z), thus $z \in L_2$ (contrad.)

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Lemma. Either $G[L_2 \cup L_3]$ contains an induced path $P_{\ell*} = (p_1, p_2, \ldots p_\ell)$ {neighbors of p_1 and p_ℓ in N_1 is disjoint from neighbors of $p_2, p_3, \ldots p_{\ell-1}$}, or 3-Coloring can be decided by solving at most $\mathcal{O}(3^{6\ell})$ 2-List Coloring instances.



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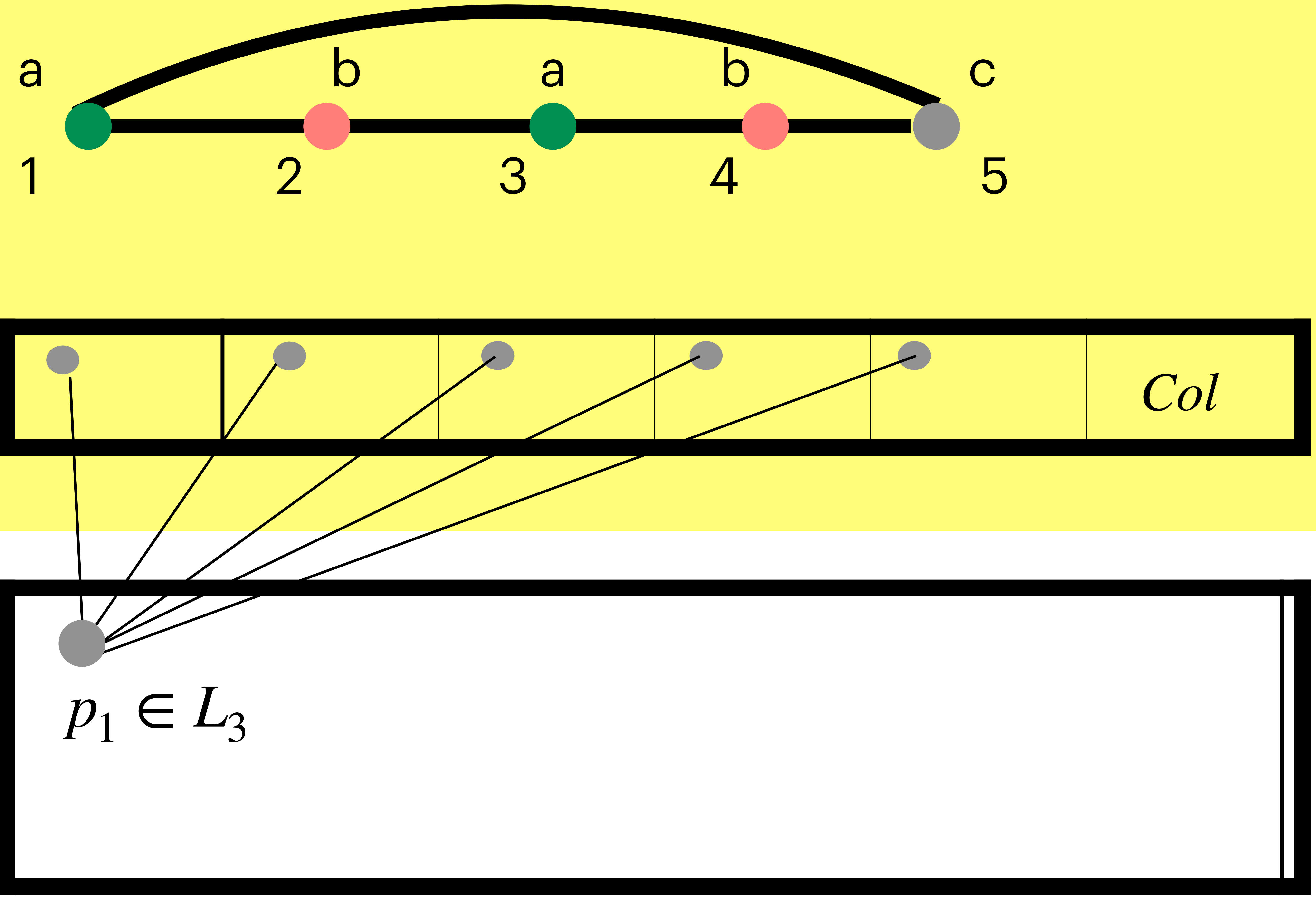


We are not modifying N_1, L_3, etc

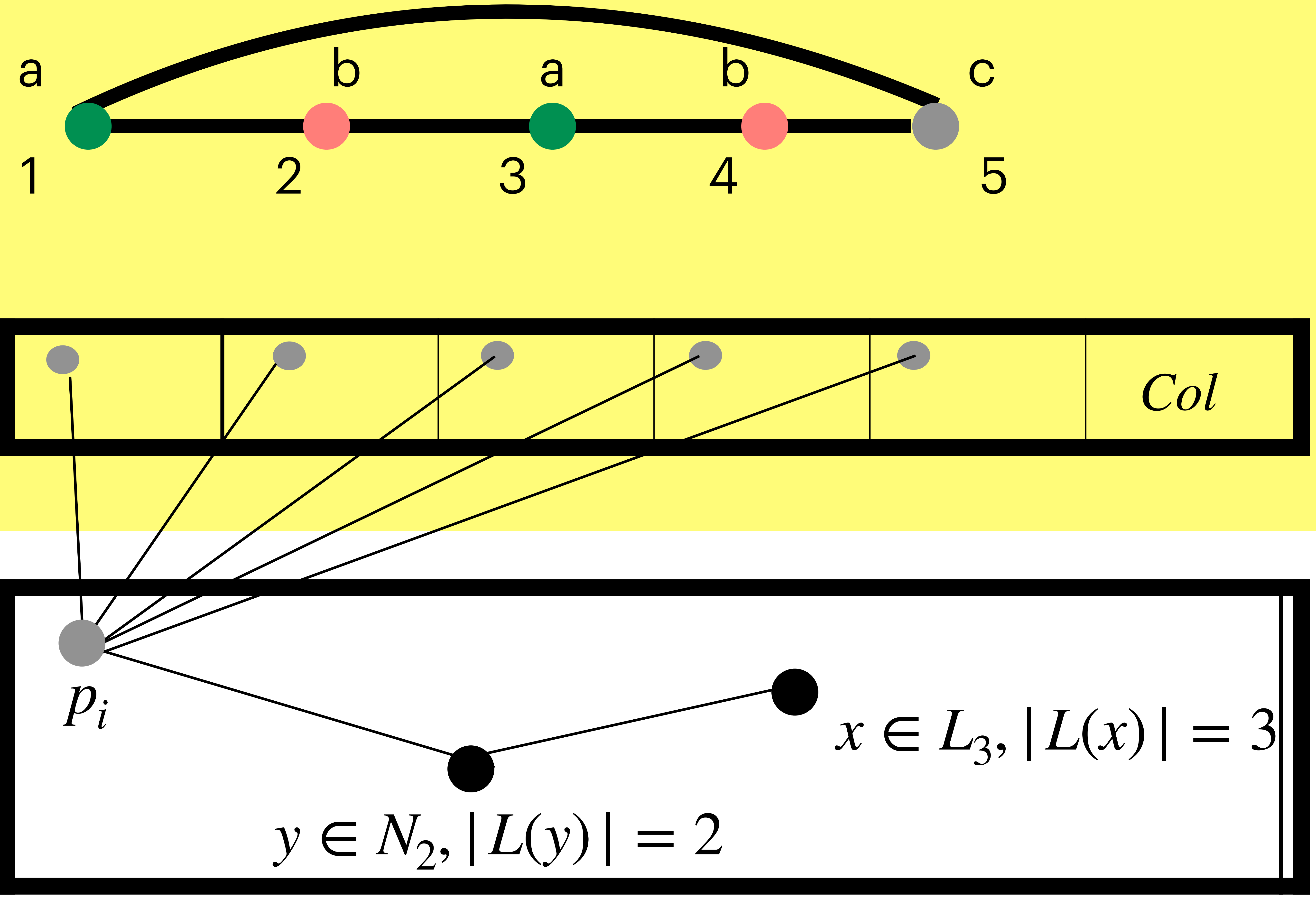
Contruction of $P_{\ell *}$

Else $\mathcal{O}(3^{6\ell})$ 2-List Coloring instances

Pick $p_1 \in L_3$. Set j=0. For i=2j+1

1. Color p_i and its 5 neighbors in N_1

3-Coloring (C_4, C_s)-free Diameter Two Graphs



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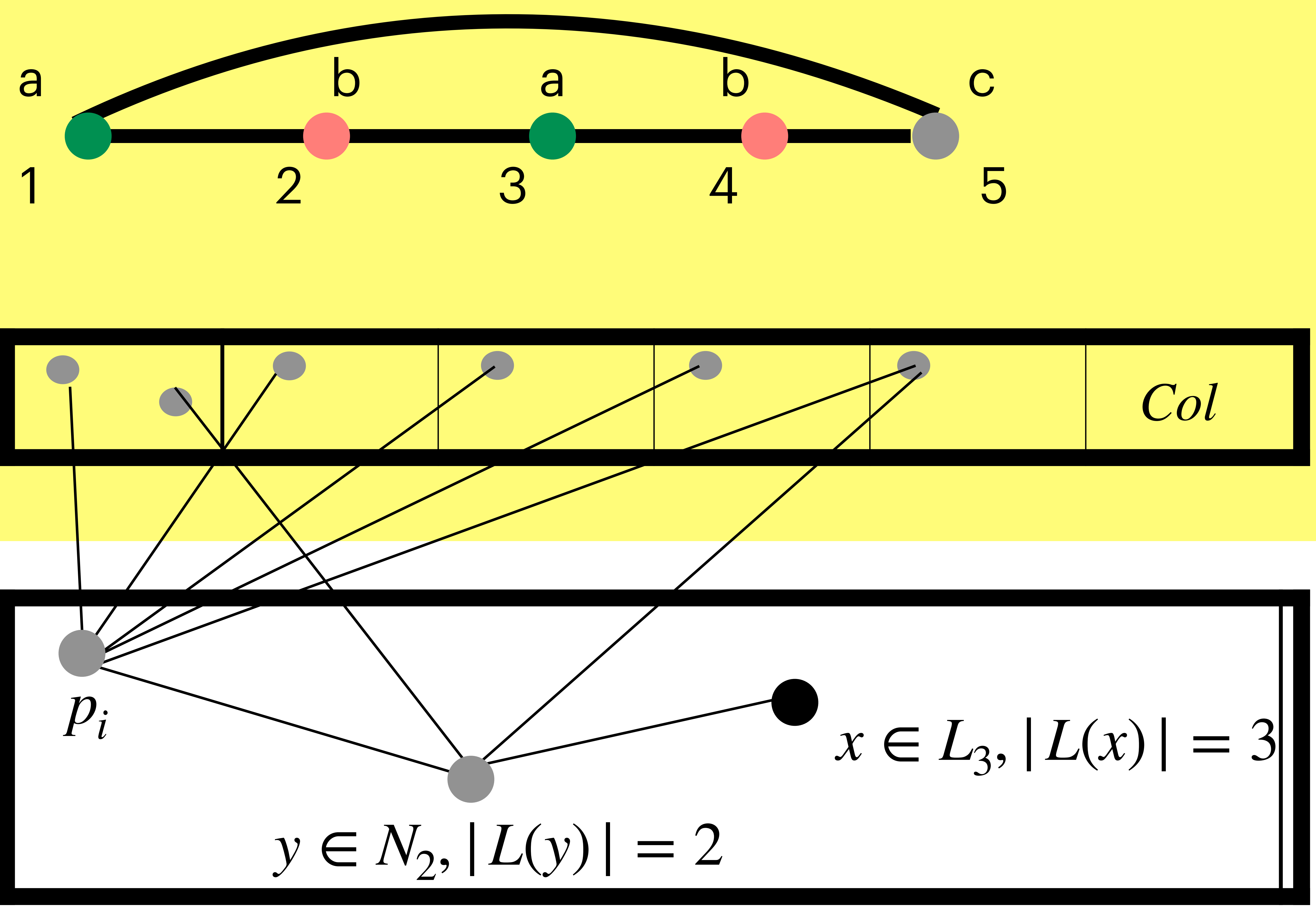
1. Color p_i and its 5 neighbors in N_1
2. If $\exists x \in L_3, y \in N_2$ s.t.
   i. |L(x)|=3
   ii. $y \in N(x) \cap p_i$ and $|L(y)| = 2$
   iii. $N(y) \cap (N_1 \setminus Col)$ not adjacent to p_1 .

Set $p_{i+1} = y, p_{i+2} = x$.

3-Coloring (C_4, C_s)-free Diameter Two Graphs



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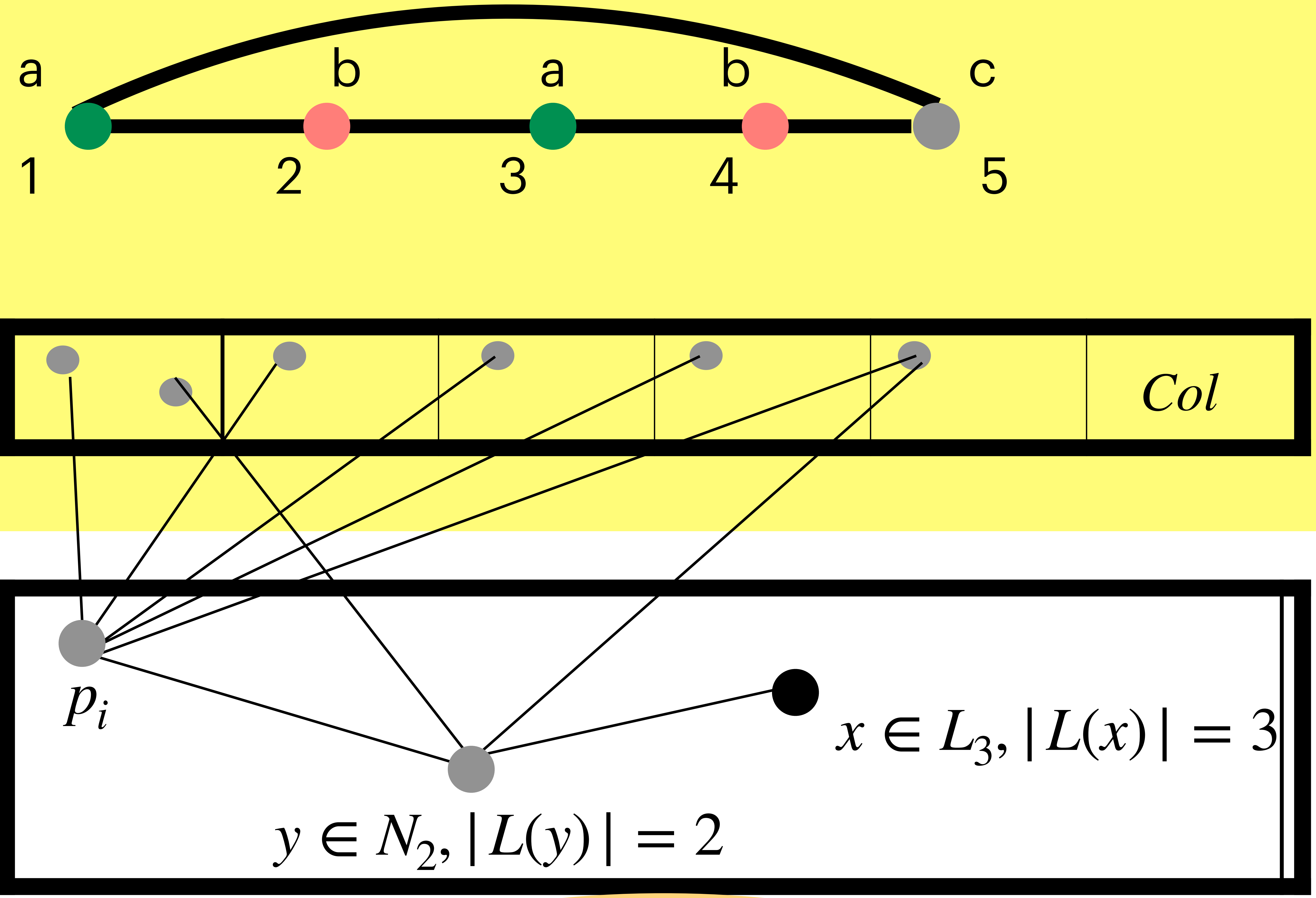
Set $p_{i+1} = y, p_{i+2} = x$. Color p_{i+1} and its at most 5 neighbours in N_1 . $j \leftarrow j + 1$.

3-Coloring (C_4, C_s)-free Diameter Two Graphs



$\mathcal{O}(3^{6\ell})$ instances

Else $\mathcal{O}(3^{6\ell})$ 2-List Coloring instances

Contruction of P_{ℓ^*}

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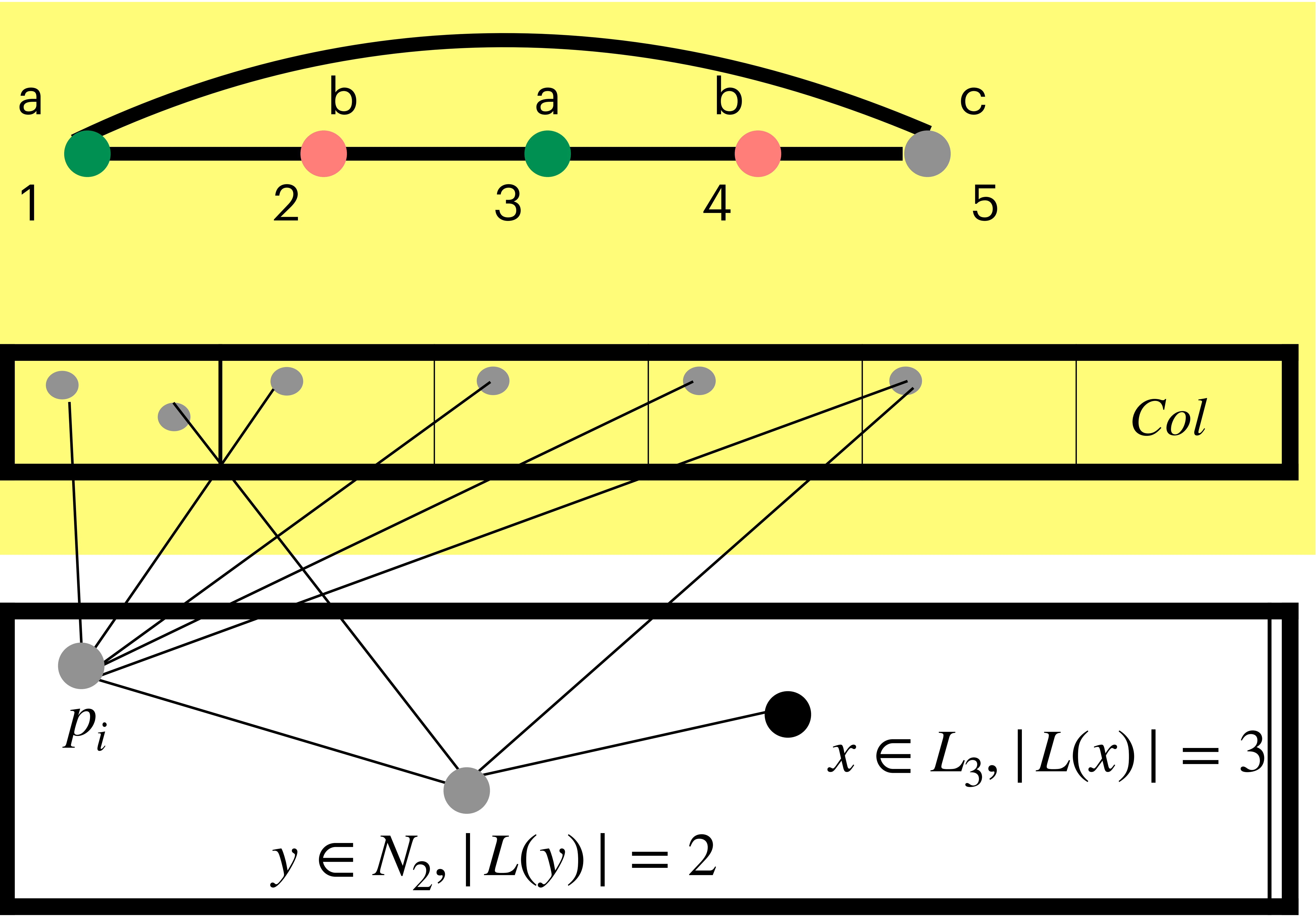
If fails then 2-List Coloring instances

Set $p_{i+1} = y, p_{i+2} = x$. Color p_{i+1} and its at most 5 neighbours in N_1 .$j \leftarrow j + 1$.

3-Coloring (C_4, C_s)-free Diameter Two Graphs



Else $\mathcal{O}(3^{6\ell})$ 2-List Coloring instances

Contruction of $P_{\ell*}$

Pick $p_1 \in L_3$. Set j=0. For i=2j+1

1. Color p_i and its 5 neighbors in N_1

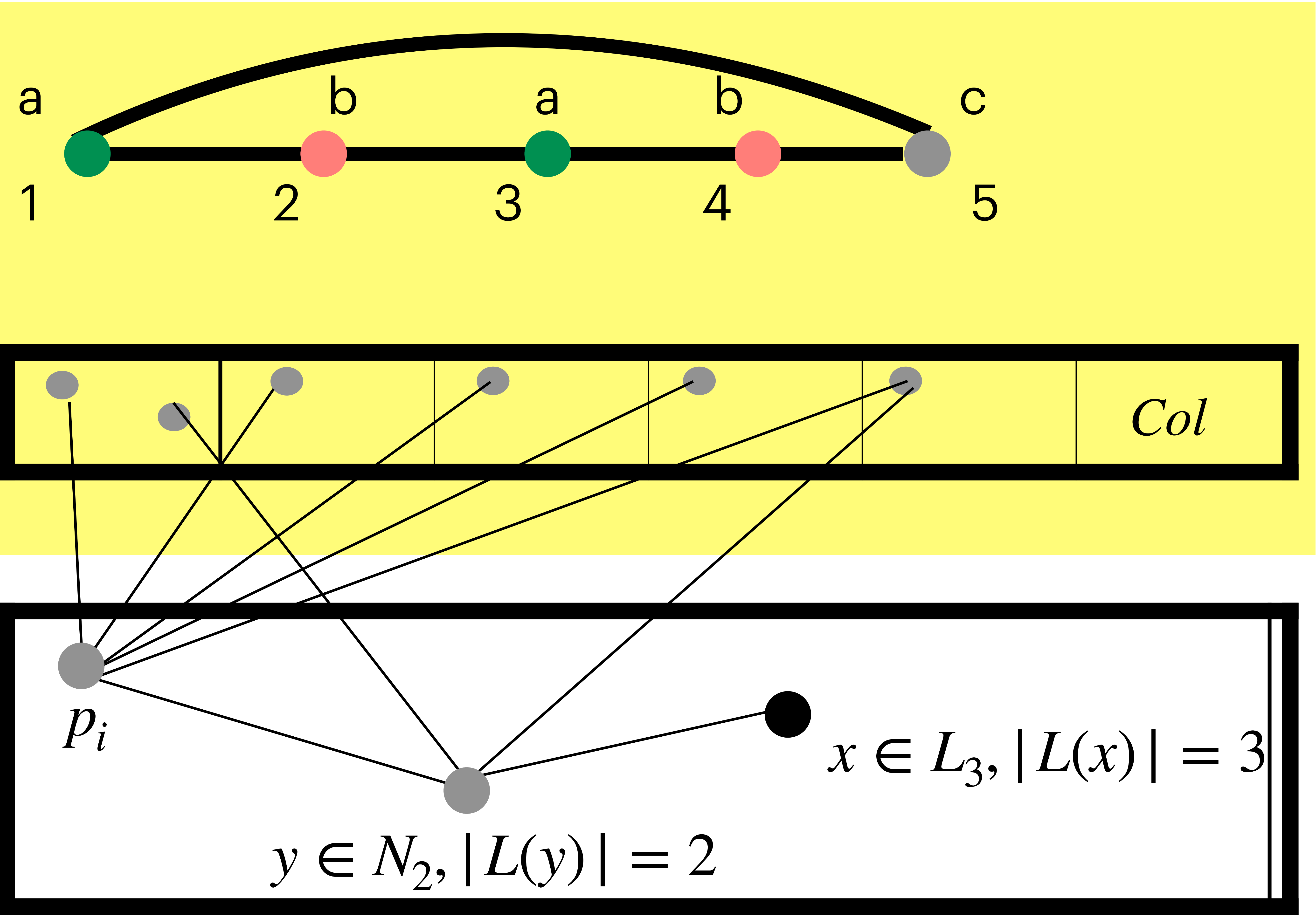
2. If $\exists x \in L_3, y \in N_2$ s.t.

   i. |L(x)|=3

   ii. $y \in N(x) \cap p_i$ and $|L(y)| = 2$

If $\exists x$, then $\exists y$ (diam 2).

3-Coloring (C_4, C_s)-free Diameter Two Graphs



Contruction of $P_{\ell*}$

Else $\mathcal{O}(3^{6\ell})$ 2-List Coloring instances

Pick $p_1 \in L_3$. Set j=0. For i=2j+1

1. Color p_i and its 5 neighbors in N_1

2. If $\exists x \in L_3, y \in N_2$ s.t.

   i. |L(x)|=3

   ii. $y \in N(x) \cap p_i$ and $|L(y)| = 2$

   iii

p_i **and its neighbours in** N_1 **are coloured, thus** $y \in N_2,\ |L(y)| = 2$

3-Coloring (C_4, C_s)-free Diameter Two Graphs

p_{2j+3} is chosen s.t.
$|L(p_{2j+3})| = 3$: not adjacent to
$p_1, p_2 \ldots p_{2j+1}$ and their neighbours in
N_1

Contruction of $P_{\ell *}$

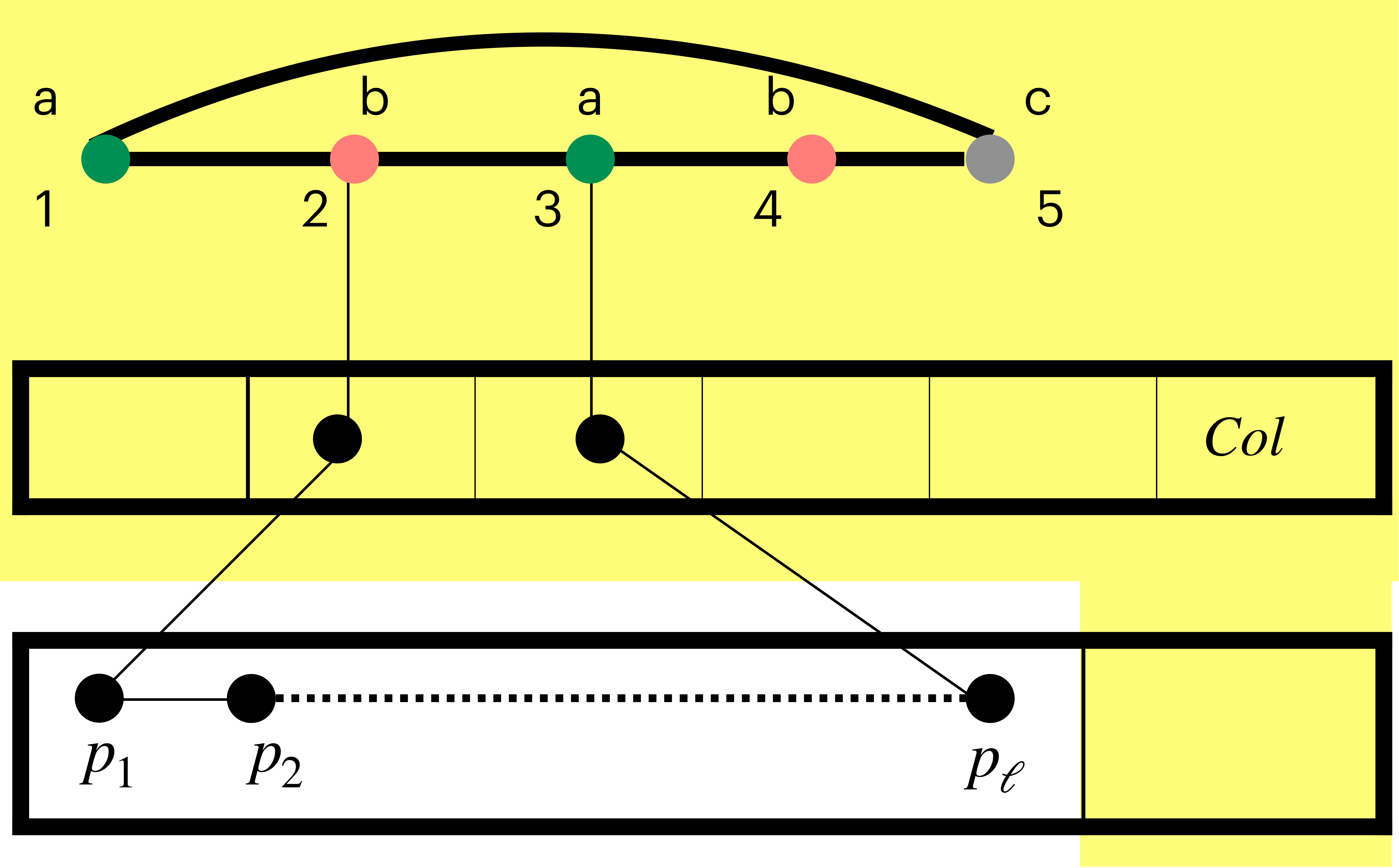
Pick $p_1 \in L_3$. Set j=0. For i=2j+1

1. Color p_i and its 5 neighbors in N_1
2. If $\exists x \in L_3, y \in N_2$ s.t.
   i. |L(x)|=3
   ii. $y \in N(x) \cap p_i$ and $|L(y)| = 2$
   iii. $N(y) \cap (N_1 \setminus Col)$ not adjacent to p_1 .

Set $p_{i+1} = y, p_{i+2} = x$.

3-Coloring (C_4, C_s)-free Diameter Two Graphs

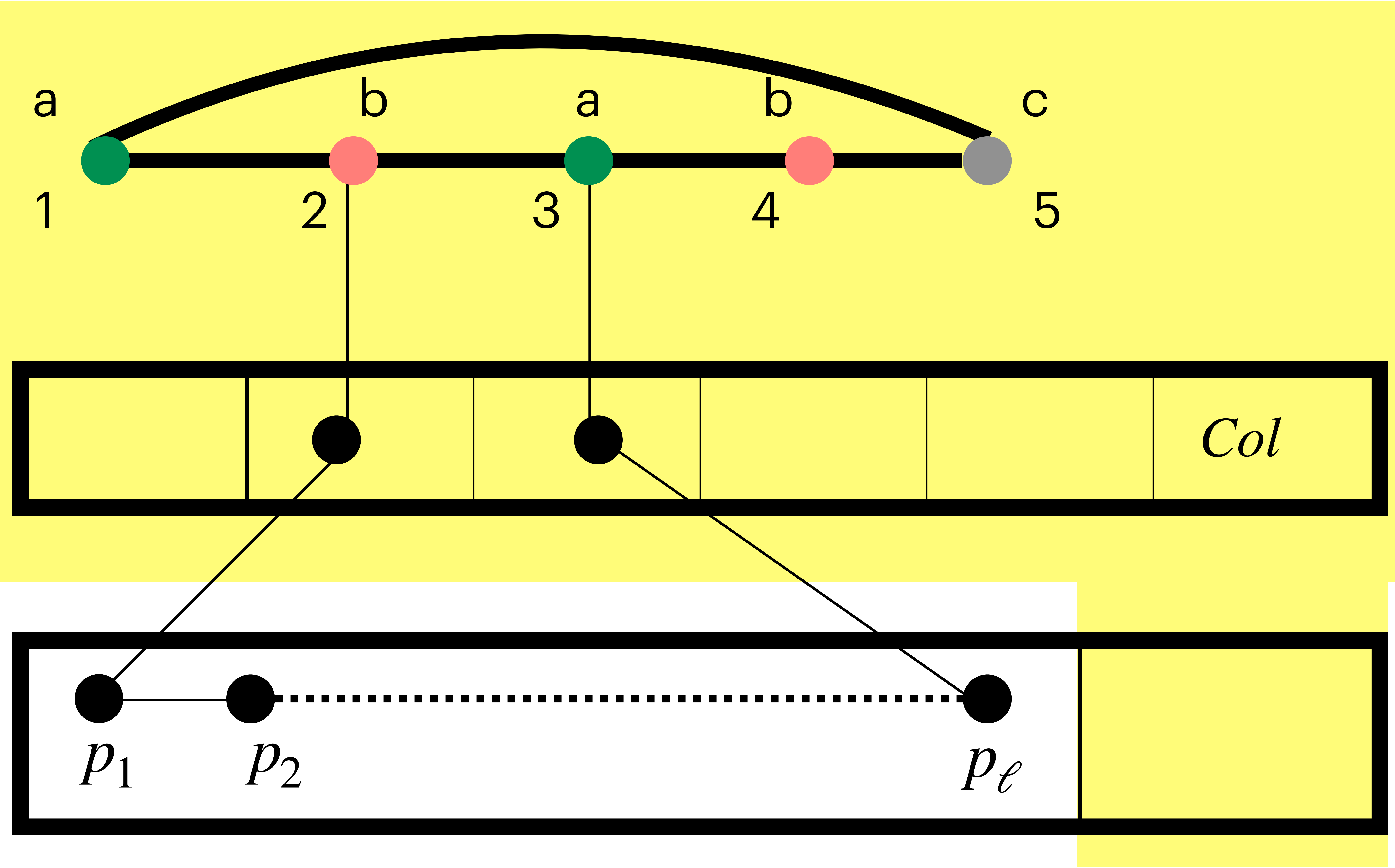
Theorem 1. List 3-Coloring is solvable in polynomial time on (C_4, C_s)-free graphs with diameter two for any constant s ≥ 10.

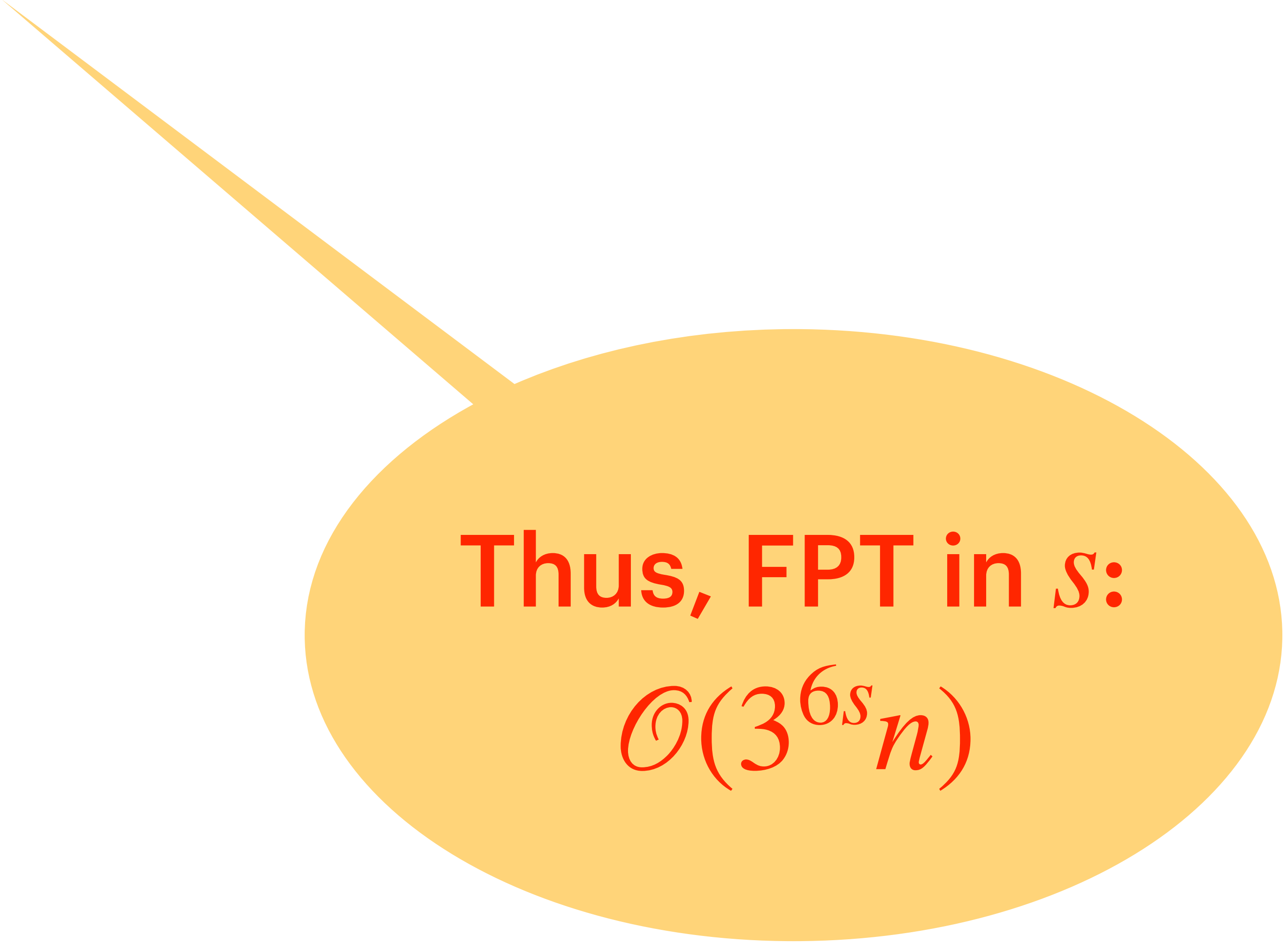


If G contains $P_{\ell*}$, for $\ell = s - 4$, then we can construct C_s which is a contradiction.

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Theorem 1. List 3-Coloring is solvable in polynomial time on (C_4, C_s)-free graphs with diameter two for any constant s ≥ 5.





3-Coloring (C_4, C_s)-free Diameter Two Graphs

Some open problems:

- 3-Coloring C_4-free diameter two graphs.

3-Coloring (C_4, C_s)-free Diameter Two Graphs

Some open problems:

- 3-Coloring C_4-free diameter two graphs.
- 3-Coloring C_3-free diameter two graphs.

3-Coloring (C_4, C_s)-free Diameter Two Graphs

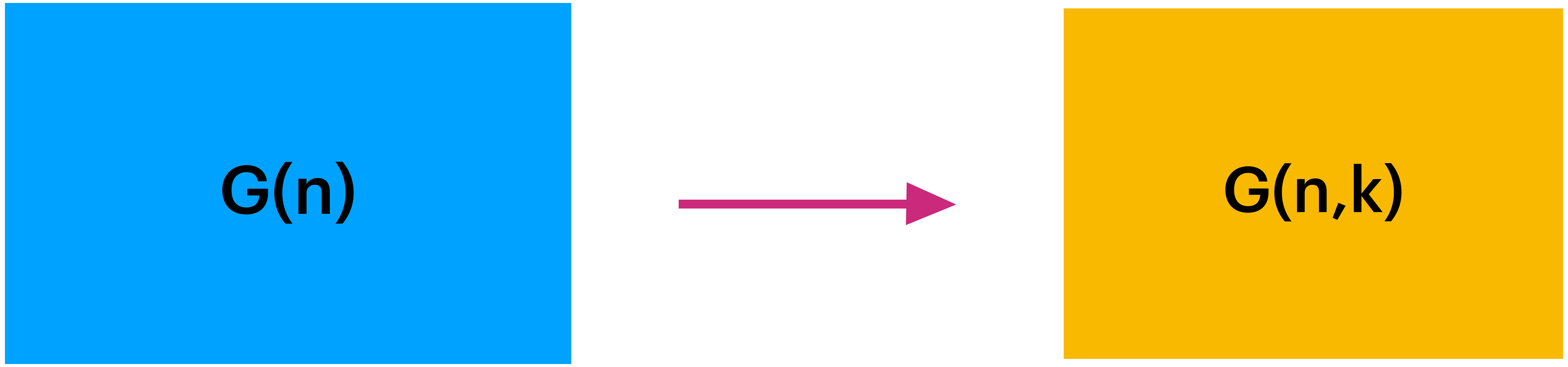
<u>Some open problems:</u>

- 3-Coloring C_4-free diameter two graphs.
- 3-Coloring C_3-free diameter two graphs.
- 3-Coloring diameter two graphs.

Thank You

Parameterized problems

- A parameterized problem is a decision problem where we associate an integer parameter to each instance
  - The parameter measures some aspect of the instance
  - Formally, $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$ contains the YES-instances (x, k)



Problem Instance

Parameterized problem Instance

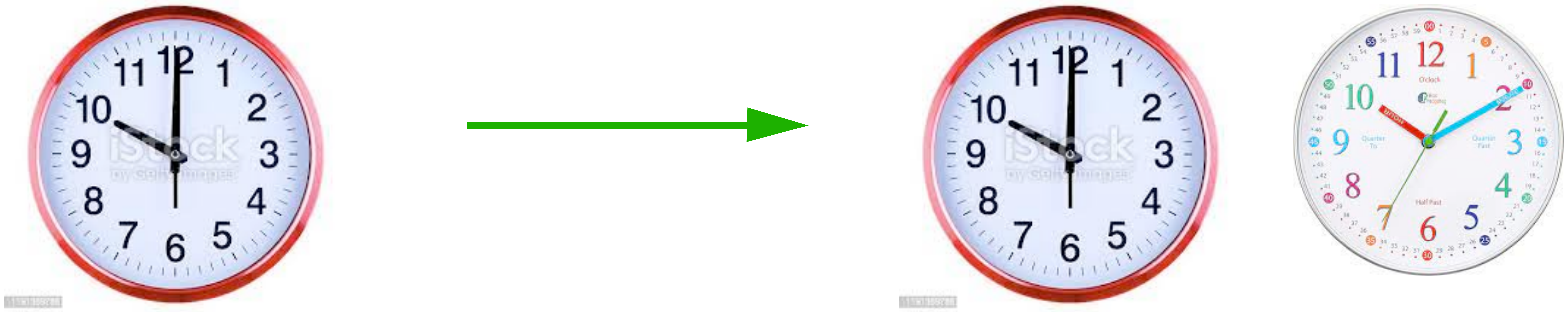
Fixed-parameter tractability

- A parameterized problem is fixed-parameter tractable (FPT) if:

  there is an algorithm that decides
  - inputs of size n,
  - with parameter value k,

  in time $f(k) \cdot n^c$ for some constant c and function f

FPT: $f(k) \cdot n^c$



Time complexity: f(n)

Parameterized time complexity: g(n,k)