Socially Fair Matching: Exact and Approximation Algorithms

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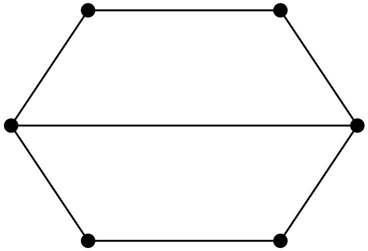
Matching



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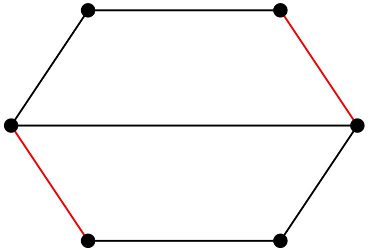

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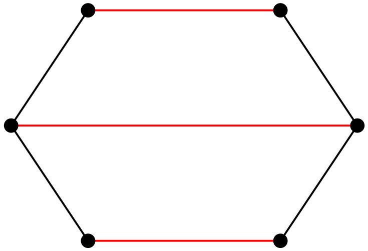
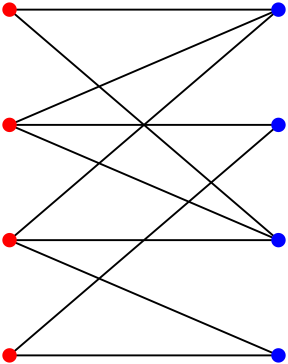


Bipartite Matching



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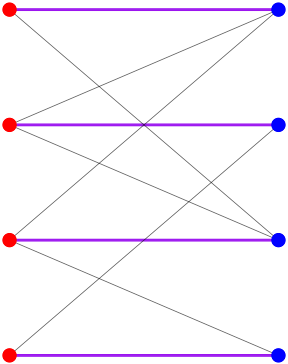

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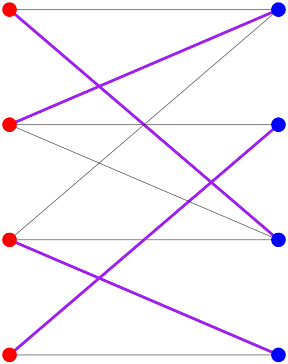

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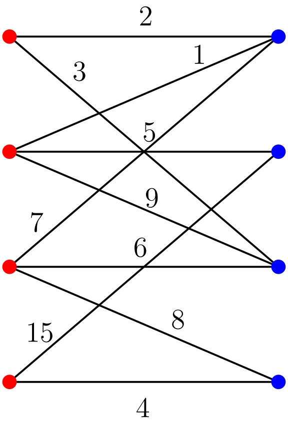

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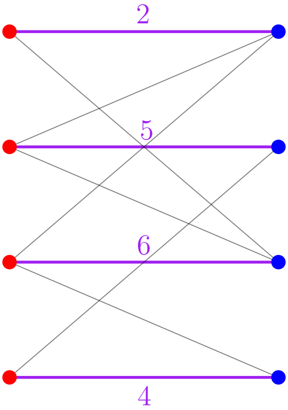

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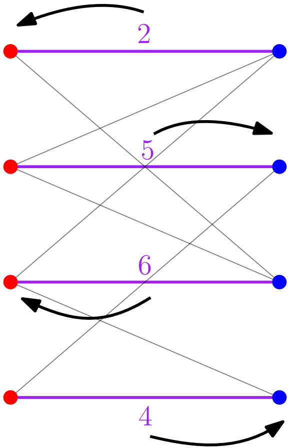

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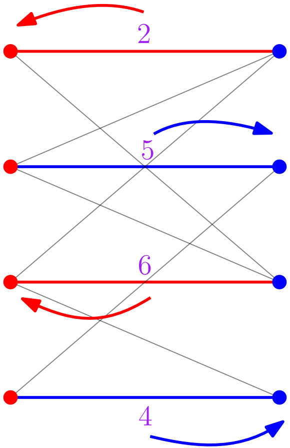

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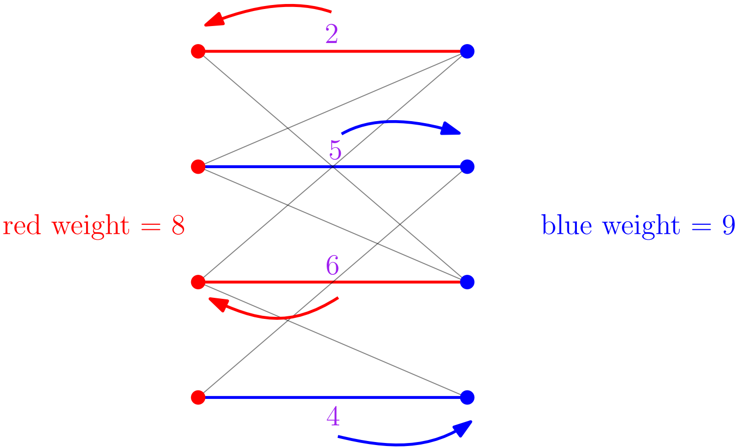

Socially Fair Matching

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Input: Complete bipartite graph $G = (R \uplus B, E)$ and weights $w : E \to \mathbb{R}^+$
Task: Find a perfect matching M and a partition of M into (M_R, M_B)

$$\max \{w(M_R), w(M_B)\}$$

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A perfect matching M and a corresponding partition of M into (M_R, M_B) with $w(M_R) = r$ and $w(M_B) = b$ will be called **an** (r, b) **perfect matching**

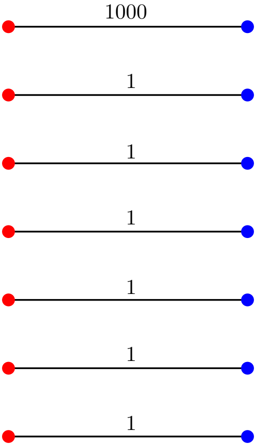
Our Results

- **Deterministic PTAS**: $(1+\epsilon)$-approximation in $n^{O(1/\epsilon)}$ time
- **Randomized** (Monte-Carlo) algorithms:
  - **Polynomial time exact algorithm** when weights are integral and polynomially bounded
  - **Randomized EPTAS** for general weights: $(1+\epsilon)$-approximation in randomized $(n/\epsilon)^{O(1)}$ time

Socially Fair Matching: Starting point

- Goal: find an (r, b) perfect matching with minimum $\max\{r, b\}$
  Minimize maximum of two parts
- So finding minimum-weight perfect matching should be the first step, right?
- Not quite...

Socially Fair Matching vs Min Weight Perfect Matching



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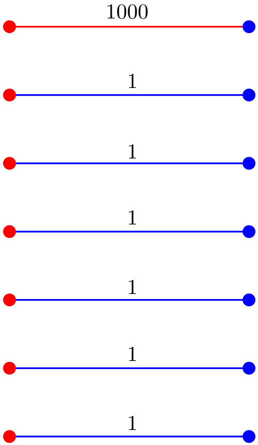

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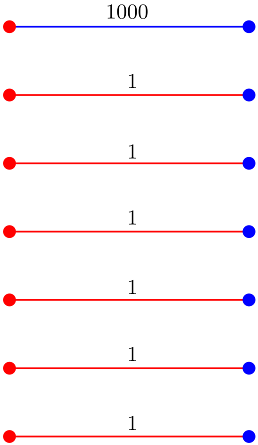

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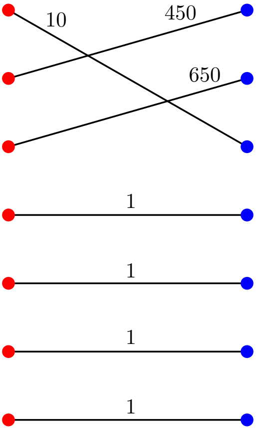

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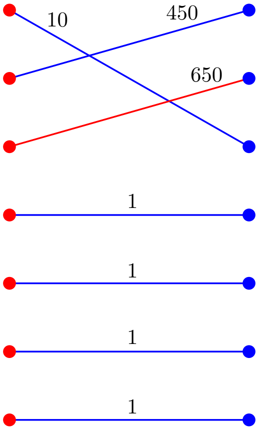

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  Given integers $\{a_1, a_2, \ldots, a_n\}$ which we want to divide into two subsets of equal sum,
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  Given integers $\{a_1, a_2, \ldots, a_n\}$ which we want to divide into two subsets of equal sum,
  create a perfect matching with $w(e_i) = a_i$ and set other weights to ∞
- $\implies$ Socially Fair Matching is **weakly NP-hard**
  (i.e., NP-hard if weights are arbitrary)

Socially Fair Matching: Starting point

- Min Weight Perfect Matching may be “bad” if we cannot divide it in a balanced way
- But it already gives a 2-approximation
- If the edge-weights are “small” compared to OPT, then we can balance easily

Socially Fair Matching: Deterministic PTAS

- Heavy edges: weight more than ϵ OPT
  Light edges: weight at most ϵ OPT

Observation

Optimal solution uses no more than $\frac{2}{\epsilon}$ heavy edges

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Proof.

Suppose not.
Then wlog red side uses more than $1/\epsilon$ heavy edges
Then the total red weight is more than ϵ OPT $\cdot 1/\epsilon =$ OPT, a contradiction. □

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3. Remaining bipartite graph only contains light edges
4. Find a minimum-weight perfect matching for remaining vertices containing only light edges
   which can be approximately balanced to any ratio
5. Results in $(1 + \epsilon)$-approximation

Randomized Exact Algorithm

- Edge-weights are integers from $[0, N]$ for some integer N
- Algebraic tools: DeMillo–Lipton–Schwartz–Zippel Lemma and Polynomial Identity Testing (PIT)
- Reduce the problem to checking whether certain polynomials are not identically zero
- Polynomials are implicitly defined: no explicit representation, but can be efficiently evaluated at given points
- Polynomial $P_{r,b}(x, y, Z)$ is not identically zero iff there exists a perfect matching M that can be partitioned into (M_R, M_B) where $w(M_R) = r$ and $w(M_B) = b$.

Randomized Exact Algorithm

- Let $\mathbb{F}$ be a field of characteristic 2 and let $\mathbb{F}[U]$ be the ring of polynomials where $U = \{x, y, z_{1,1}, z_{1,2}, \ldots, z_{n,n}\}$ is a set of variables
- Define matrix $A = (a_{ij})$ where $a_{ij} = (x^{w_{ij}} + y^{w_{ij}}) \cdot z_{ij}$.
- $\det(A) = Q(x, y, Z)$ is a polynomial in $\mathbb{F}[U]$, where $Z = \{z_{1,1}, \ldots z_{n,n}\}$

$$Q(x, y, Z) = \sum_{r=0}^{N} \sum_{b=0}^{N} x^r y^b \cdot P_{r,b}(Z)$$

- Degree of $Q(x, y, Z)$ is at most $n \cdot N$

Observation

$P_{r,b}(Z)$ is not identically 0 iff there exists a (r, b) perfect matching.

Polynomial Identity Testing

- Goal is to find $0 \leq r, b \leq N$ with minimum $\{r, b\}$ such that, there exists a (r, b) perfect matching
- Equivalently, iterate over all $0 \leq r, b \leq N$ and check whether $P_{r,b}(Z)$ is identically zero
- DLSZ Lemma implies
  If we evaluate $P_{r,b}(Z)$ at randomly sampled $k \gg n^2$ values, with high probability,
  all evaluations are zero iff $P_{r,b}(Z)$ is identically zero
- Each evaluation takes time polynomial in n and N
- Can solve Socially Fair Matching exactly in time $(n+N)^{O(1)}$ with high probability

Randomized EPTAS

- We assumed that weights are integral and from range $[0, N]$
- Reduce general case to this case (heavily inspired from knapsack-type EPTAS)
- Weights can be made integers in the range $[0, n/\epsilon]$, at the expense of $(1+\epsilon)$-loss
- Then solve exactly (via PIT)
- Randomized $(n/\epsilon)^{O(1)}$ time algorithm that computes a $(1+\epsilon)$-approximation with high probability

Summary and Open Problems

- Deterministic PTAS for general weights
  Randomized EPTAS (exact if weights are integer and polynomially bounded)
- NP-hard for general weights (reduction from Partition)
- Can we get deterministic EPTAS?
- Problem naturally extends to 3 colors and more, but algorithms unclear
  3-Dimensional Matching is NP-hard
  Approximations? Beyond polynomial-time?