Colored Constrained Spanning Tree on Directed Graphs

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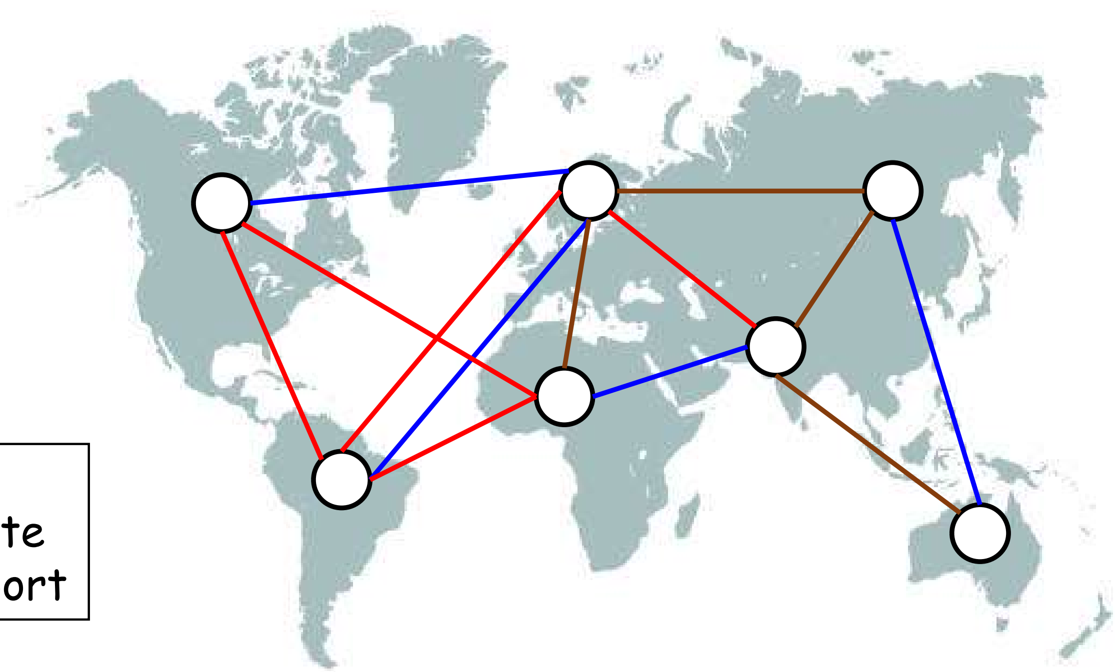
Introduction

Airline A

Airline B

Airline C



Constraint:
Each airline can only operate at most κ flights in an airport

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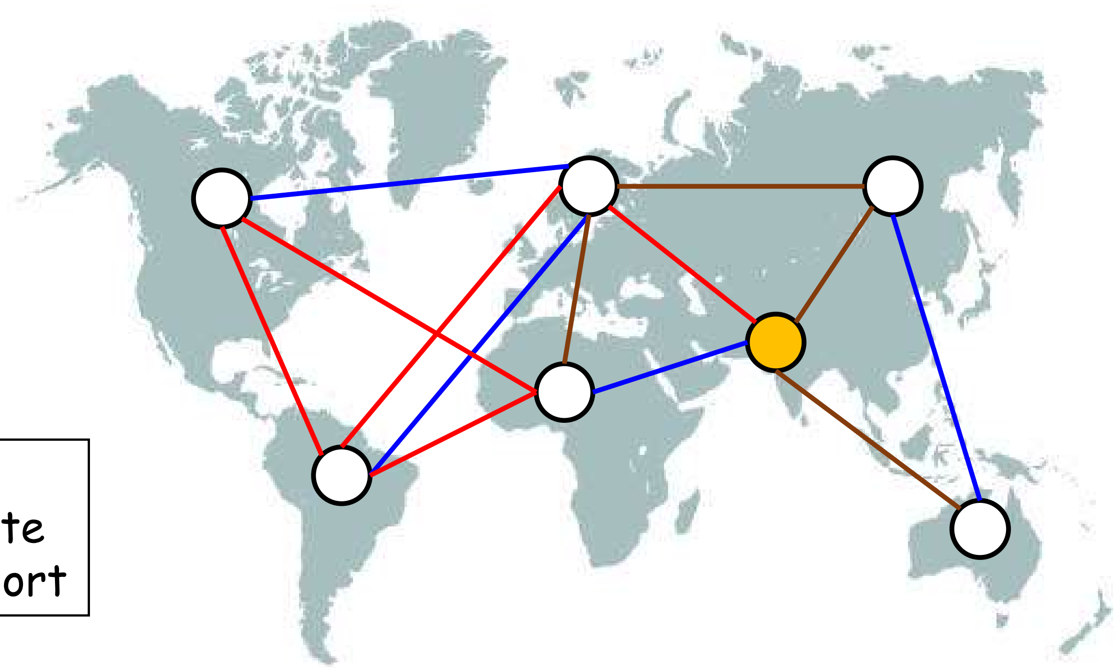

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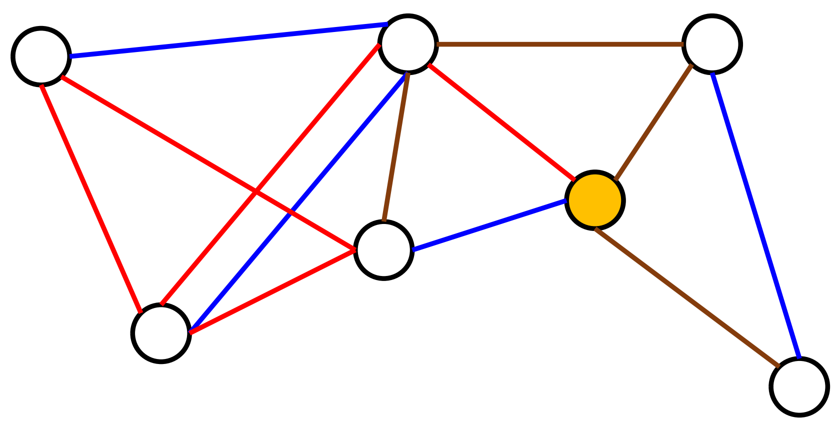
κ-Colored-Constrained Spanning Tree (κ-CCST)

- κ-Colored-Constraint:
  the number of incident edges of the same color $\leq \kappa$



Example: 1-CCST

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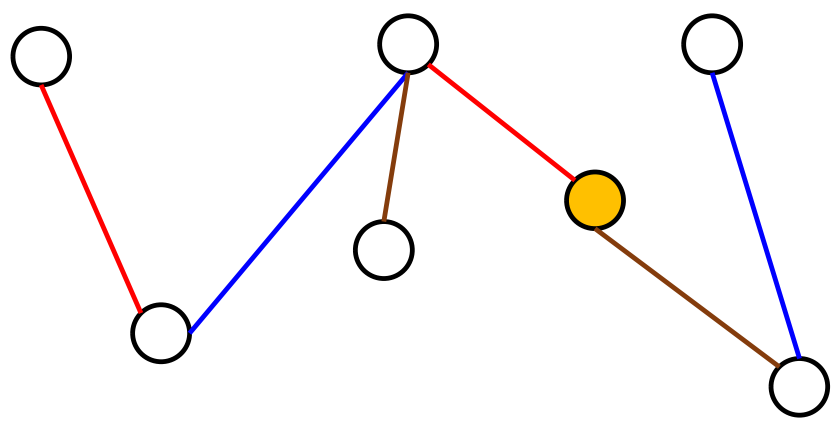
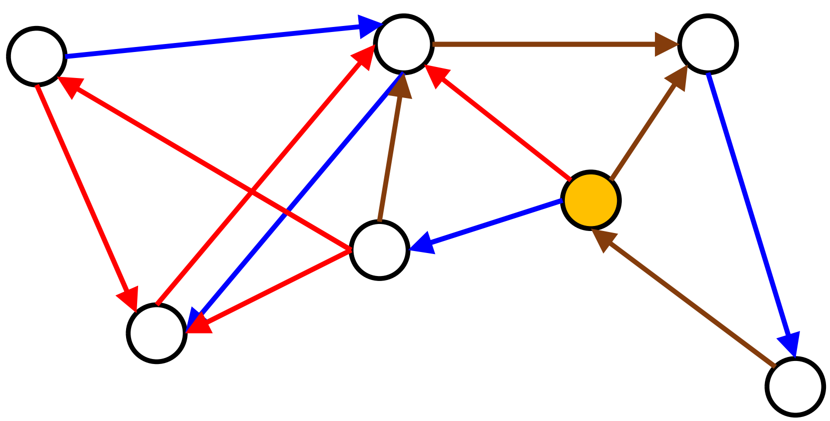


Example: 1-CCST

κ-Colored-Constrained Spanning Tree (κ-CCST)

- Directed κ-CCST:
  A spanning out-tree that follows the κ-colored-constraint
- Both incoming and outgoing edges are counted



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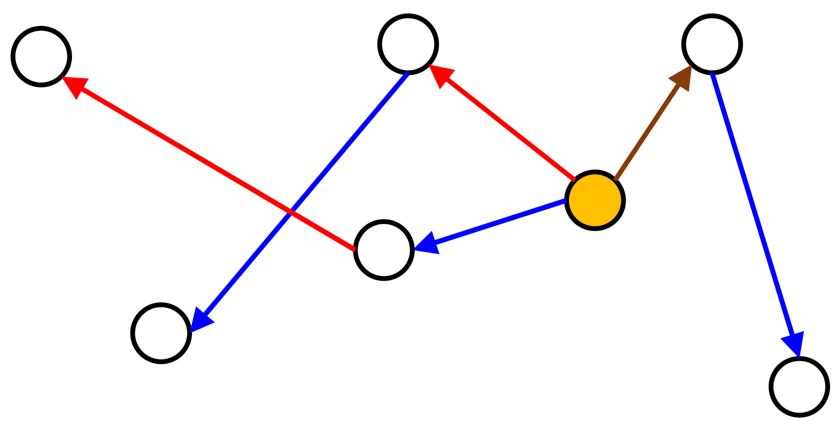

Example: directed 1-CCST

κ-Colored-Out-Constrained Spanning Tree (κ-COCST)

- κ-Colored-Out-Constraint:
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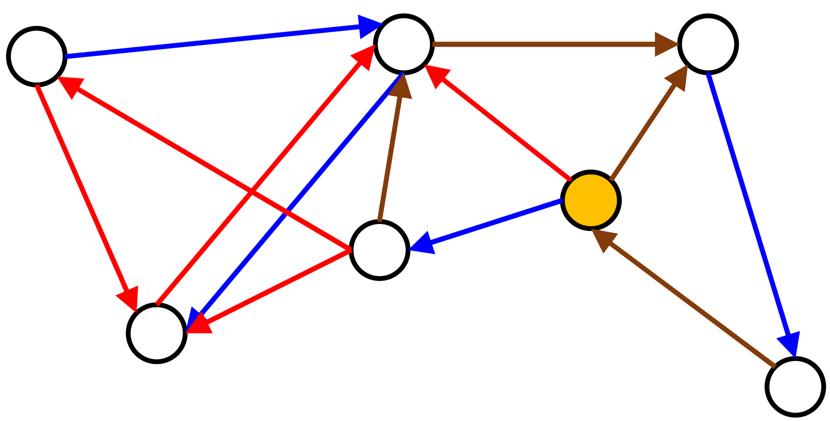

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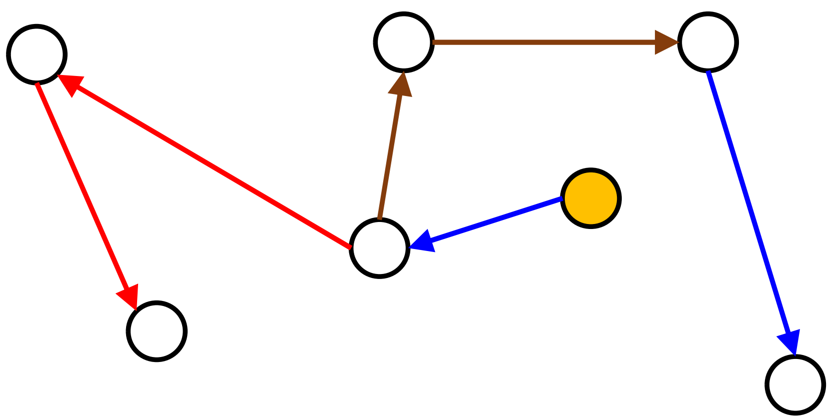

Example: 1-COCST

Previous Works

- 1-CCST on edge-colored undirected graphs [Borozan et al. (Eur. J. Comb. 19)]
    - NP-hard
    - Tractable in special cases
- Ramsey-type problems: [Kano et al. (Discrete Math. 20) …]
    - How large should the graph be to guarantee a 1-CCST

Results

- κ-CCST

Graph Class	λ	κ	Hardness
Directed	≥ 2	≥ 1	NP-Hard
DAG	≥ 3	≥ 1	NP-Hard
DAG	$= 2$	$= 1$	P

- κ-COCST

Graph Class	λ	κ	Hardness
Directed	≥ 4	≥ 1	NP-Hard
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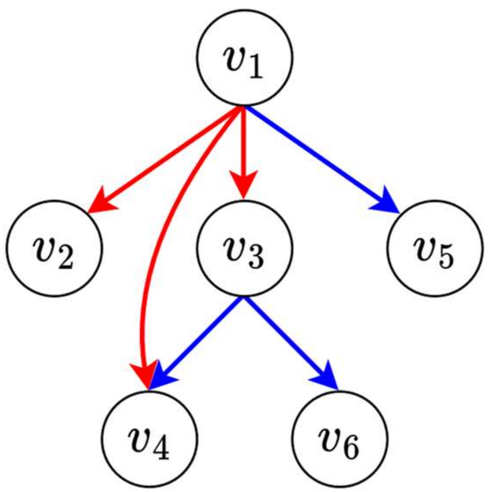
Tractable Cases

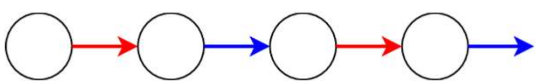
Rooted DAG

- Root: A vertex that can reach all vertices
- The root must be the only zero-indegree vertex on a DAG

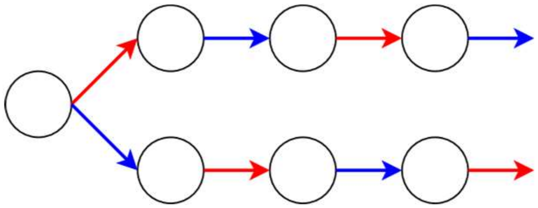


1-CCST on a 2-edge-colored DAGs

- Observe that a 1-CCST on a 2-edge-colored DAG must be either an *alternating path* or a *V-shaped tree*
- Dynamic Programming



alternating path

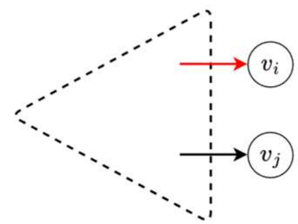


V-shaped Tree

The Dynamic Programming

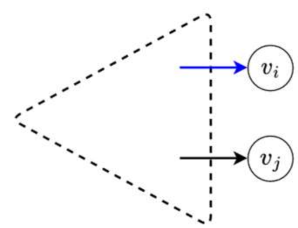
- Maintain the leaves and their incoming edge colors on trees spanning $v_1, v_2, \ldots, v_i$ for all $i = 1, 2, \ldots, n$

$f(v_i^{\text{red}}) = \{v_j^c : \text{tree}$



$\text{exists}\}$

$f(v_i^{\text{blue}}) = \{v_j^c : \text{tree}$



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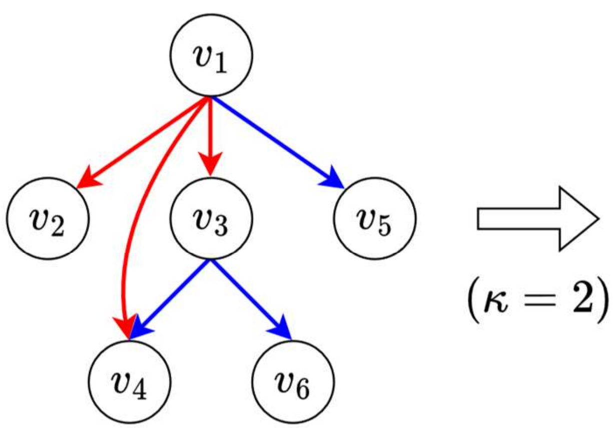
- Allows a linear time algorithm using stack and bit vector

κ-COCST on λ-edge-colored DAGs

- Match each non-root vertex to one of its incoming edge, such that colored-out-constraint holds
- B-matching → Maximum Flow



G G'

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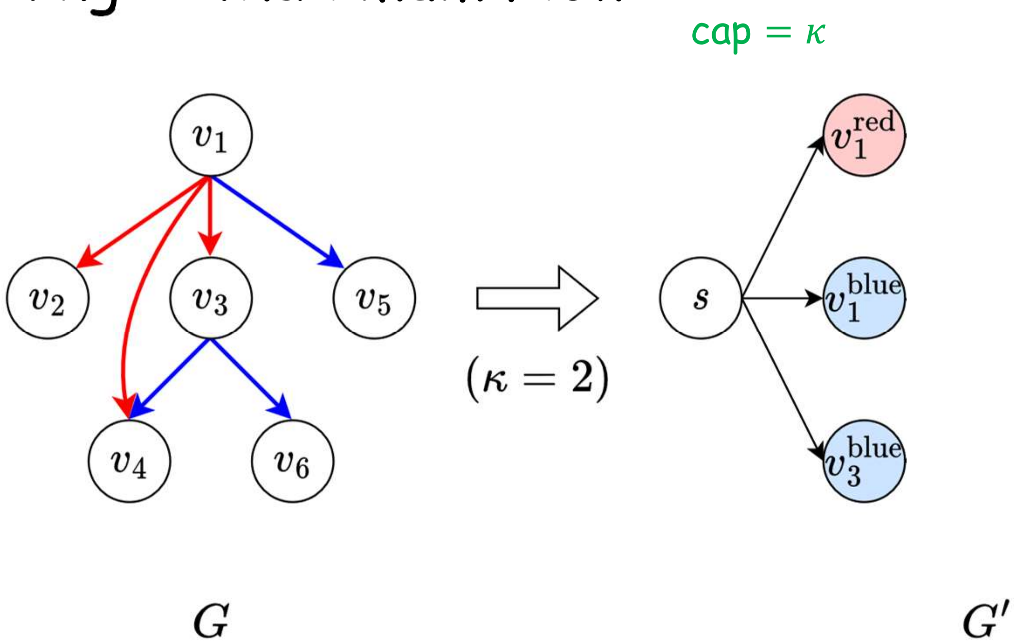

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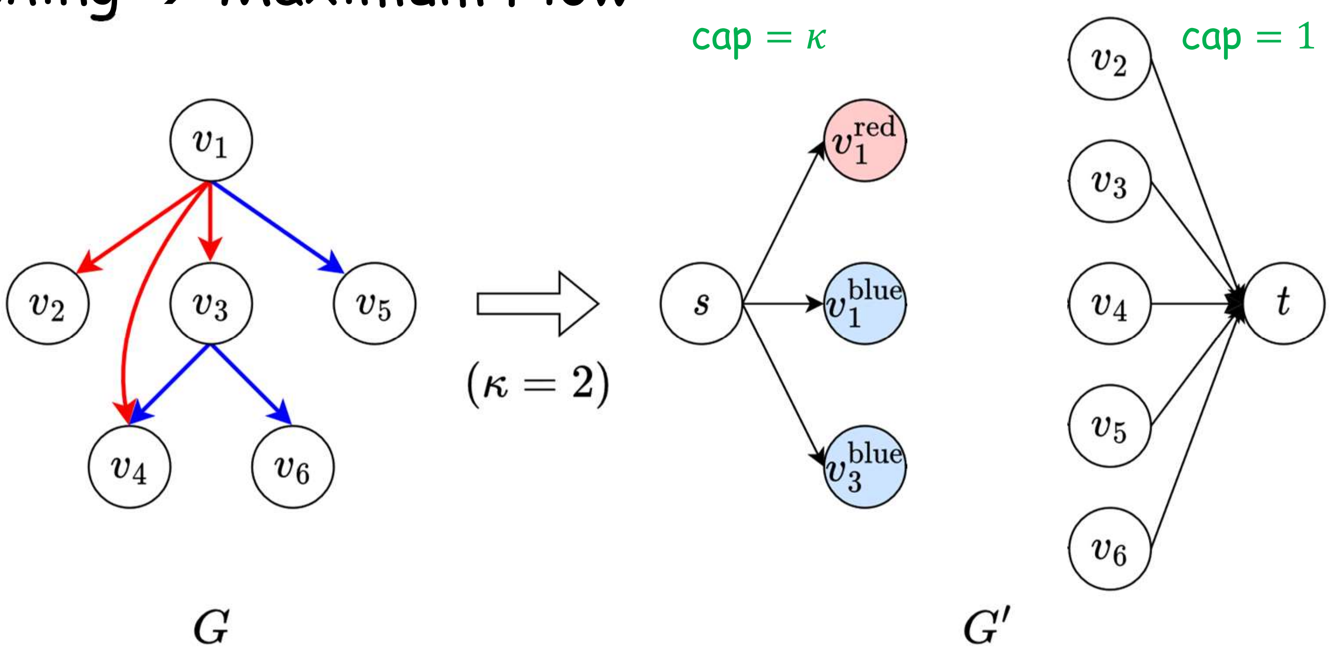

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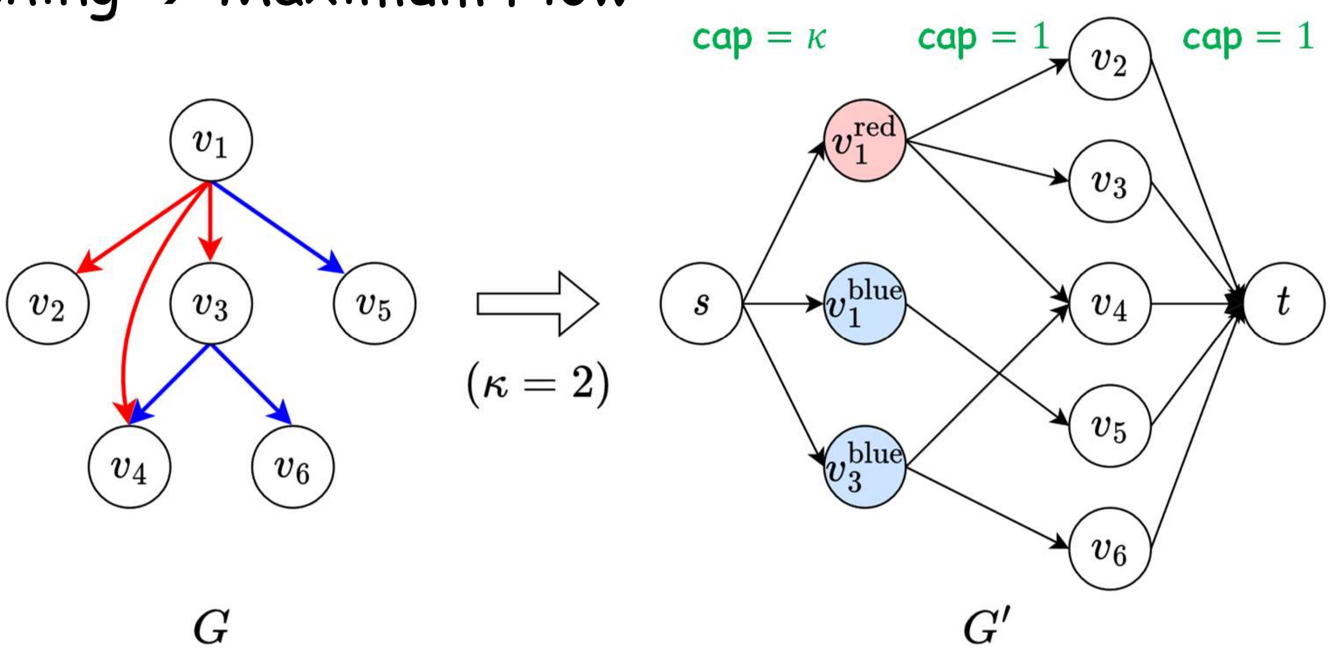

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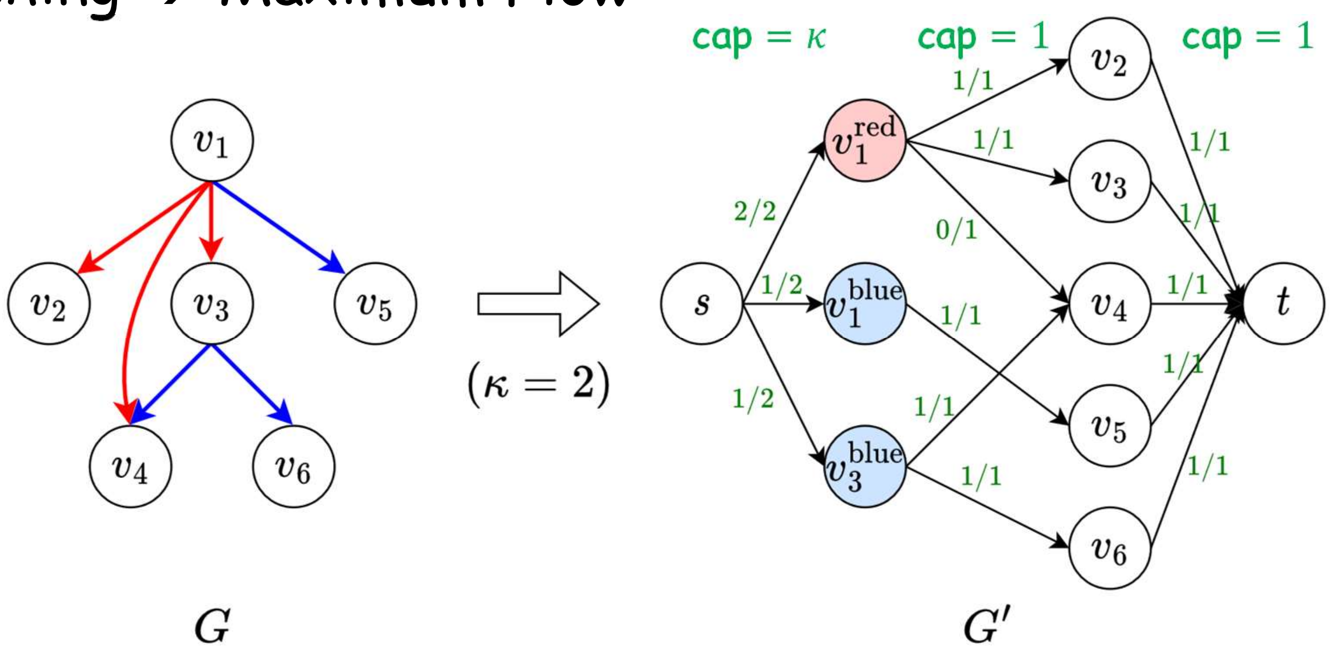

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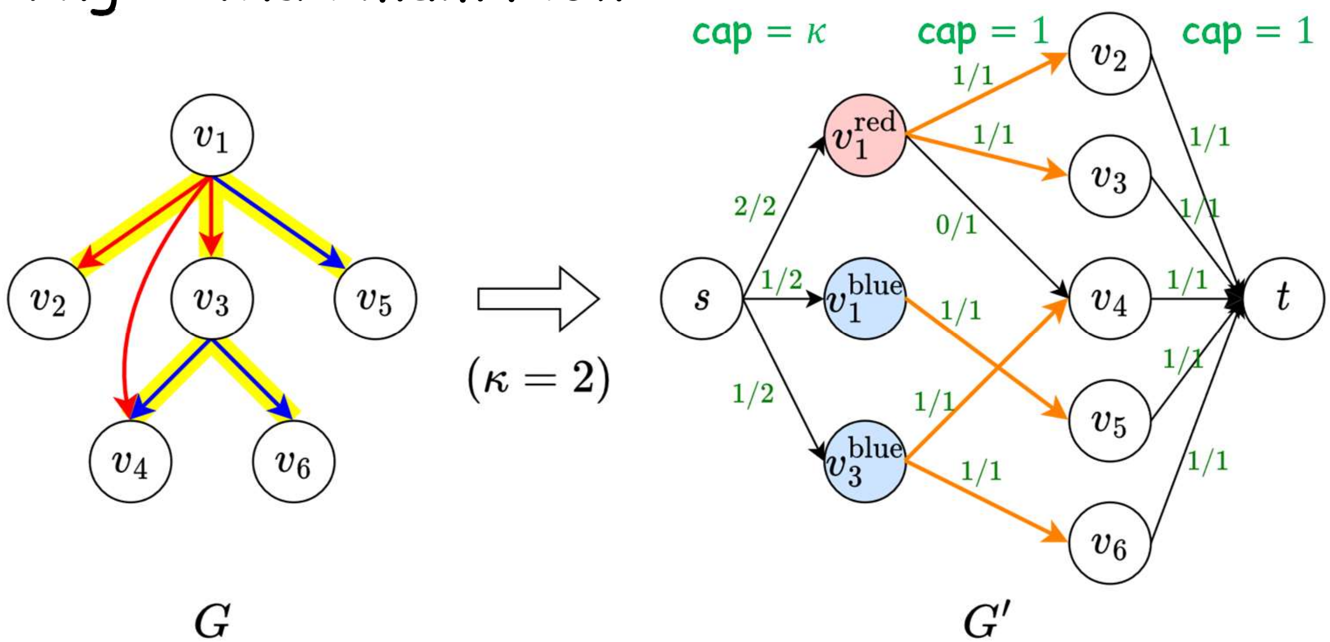
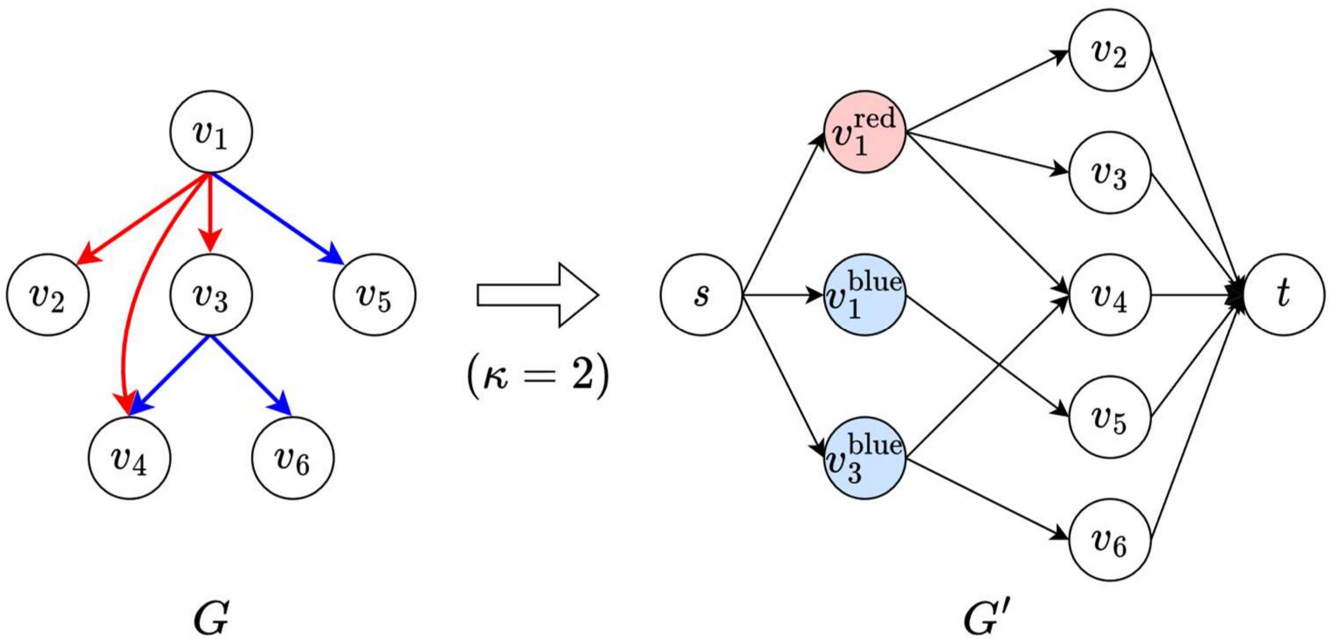


κ-COCST on λ-edge-colored DAGs

- Suppose the edge-colored graph has m edges and n vertices
- Flow graph has at most $2m + n$ edges and $m + n - 1$ vertices



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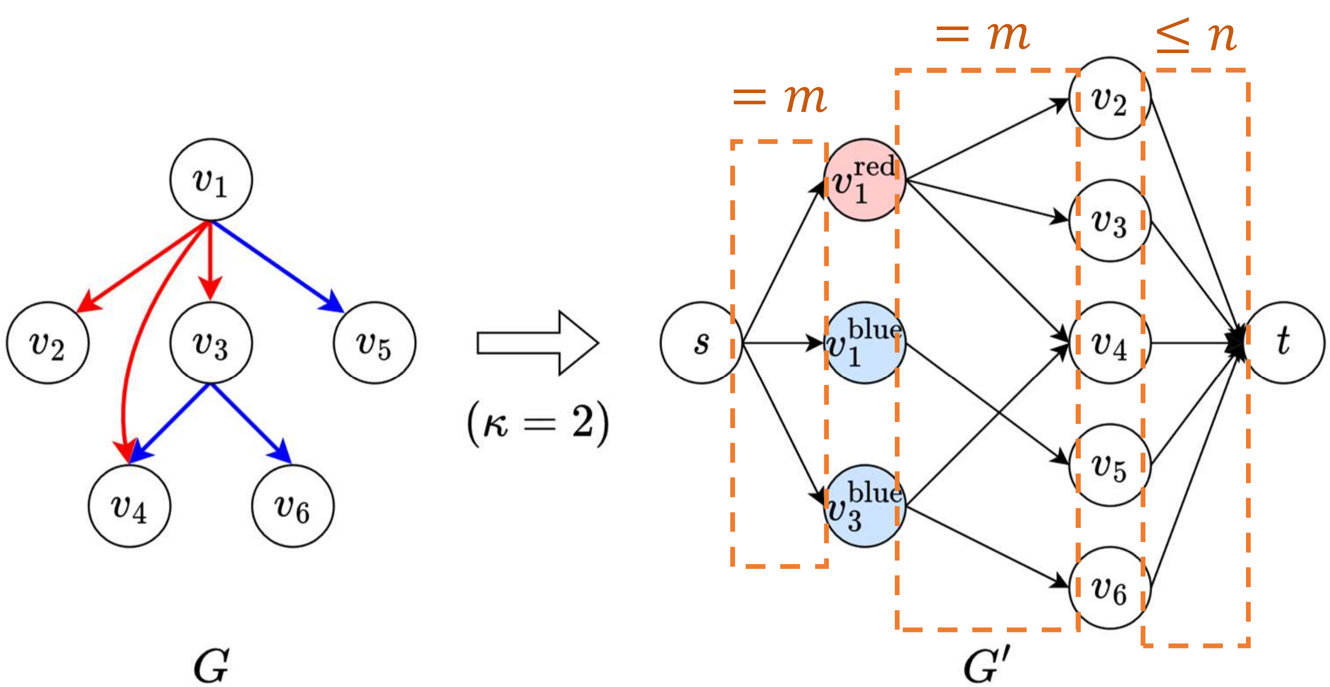

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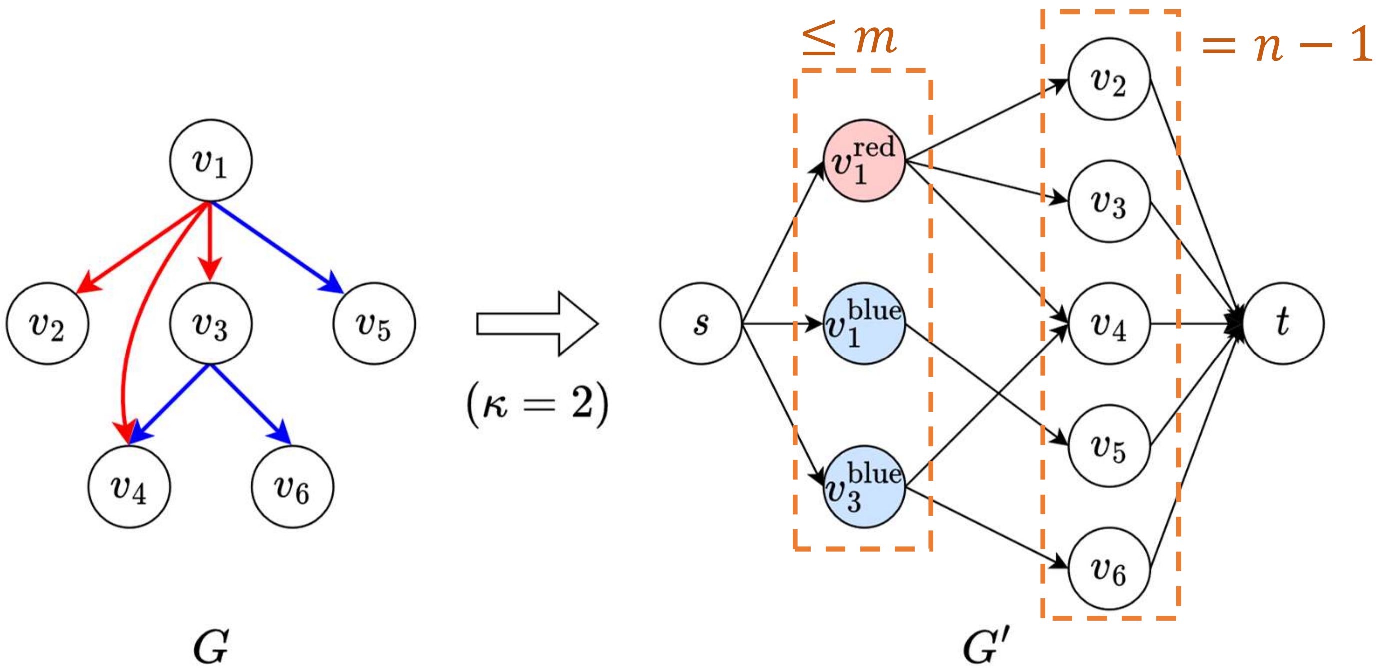

Hardness results

κ-CCST on DAGs

Borozan et al. (Eur. J. Comb. 2019)

κ-CCST
on undirected graphs
NP-Hard

reduction →

κ-CCST
on directed graphs

κ-CCST on DAGs

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→ reduction →

κ-CCST
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degree-constrained
spanning tree
problem

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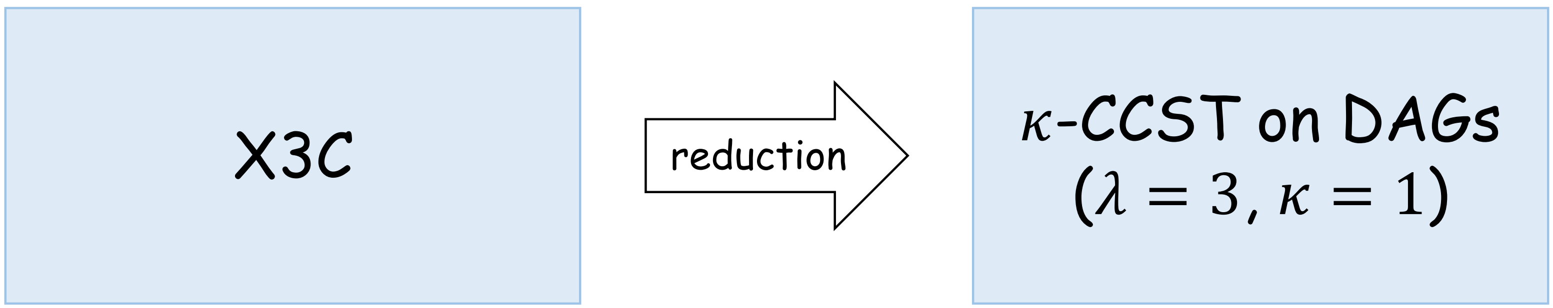
reduction

κ-CCST on DAGs

polynomial time solvable on DAGs

κ-CCST on DAGs



- NP-Hard problem X3C:
  - Input: $X = \{x_1, x_2, \dots, x_{3n}\}$ and $\mathrm{C} = \{c_1, c_2, \dots, c_m\}$.
  - Output: Are there n elements of C whose union is X?
  - Ex: $(m = 5, n = 2)$
    $X = \{1, 2, 3, 4, 5, 6\}$,
    $C = \{\{1,2,3\}, \{2,3,4\}, \{1,2,5\}, \{2,5,6\}, \{1,5,6\}\}$.

κ-CCST on DAGs

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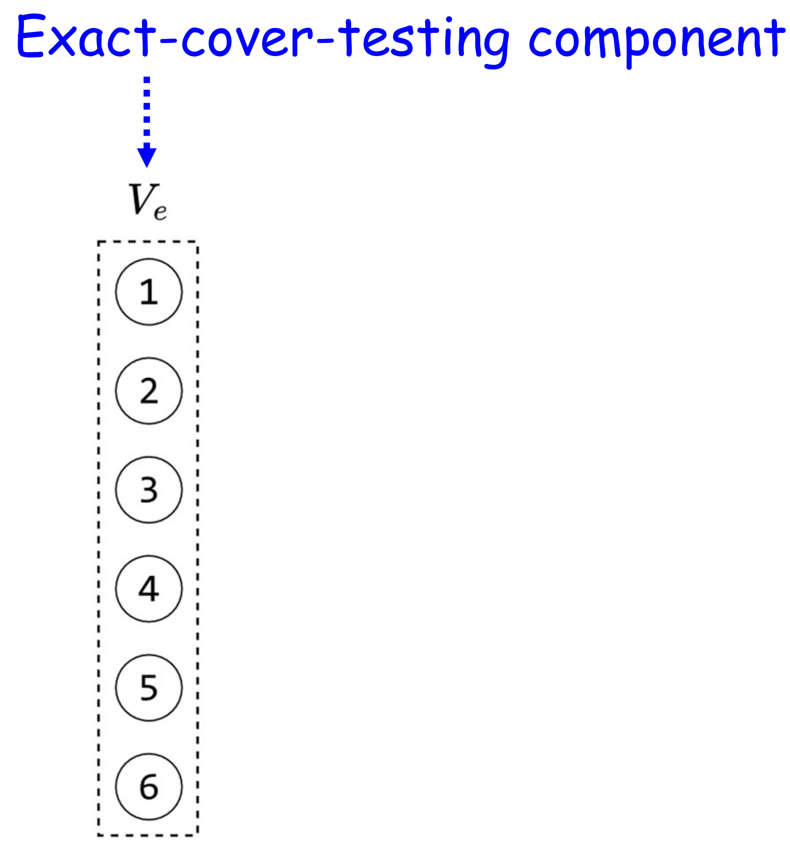

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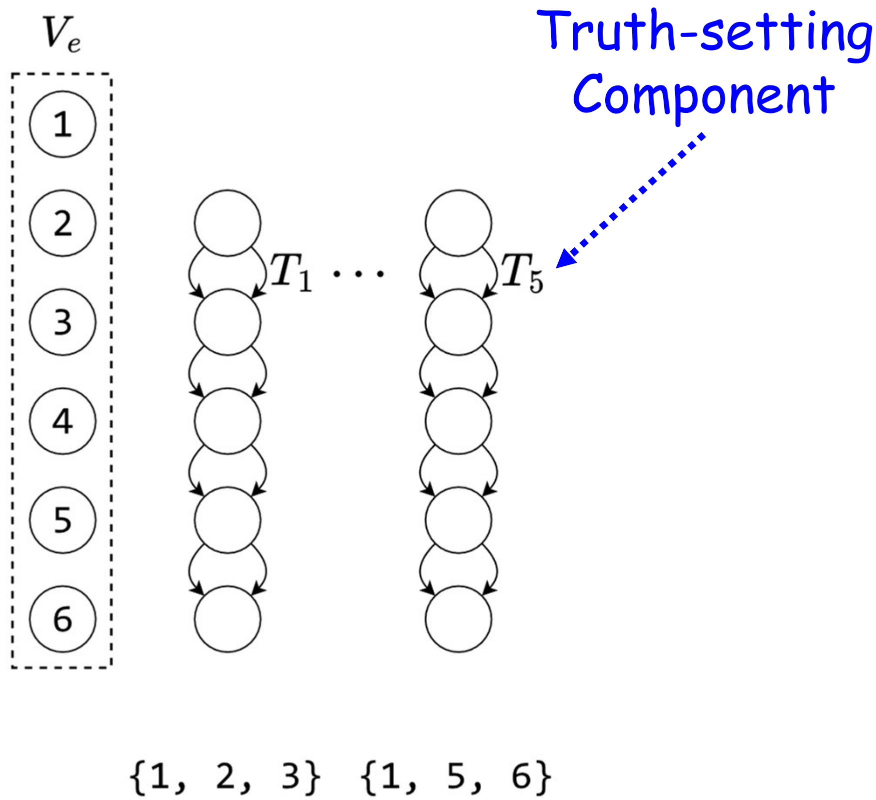

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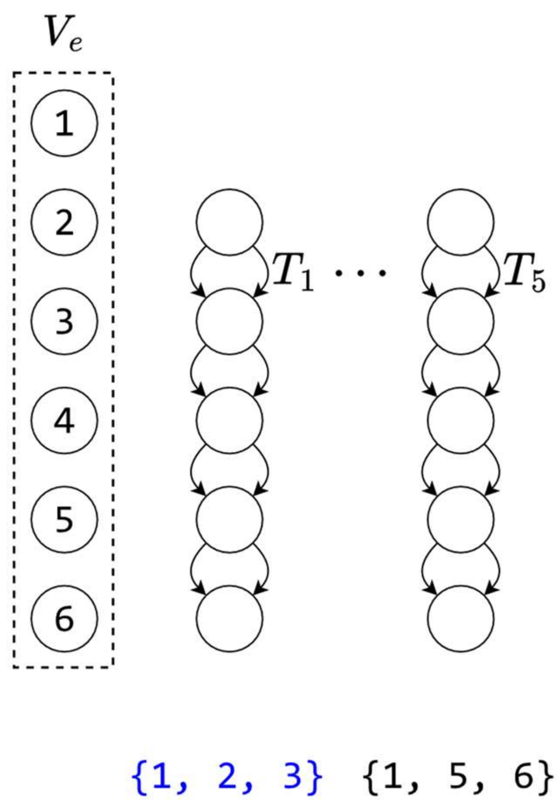

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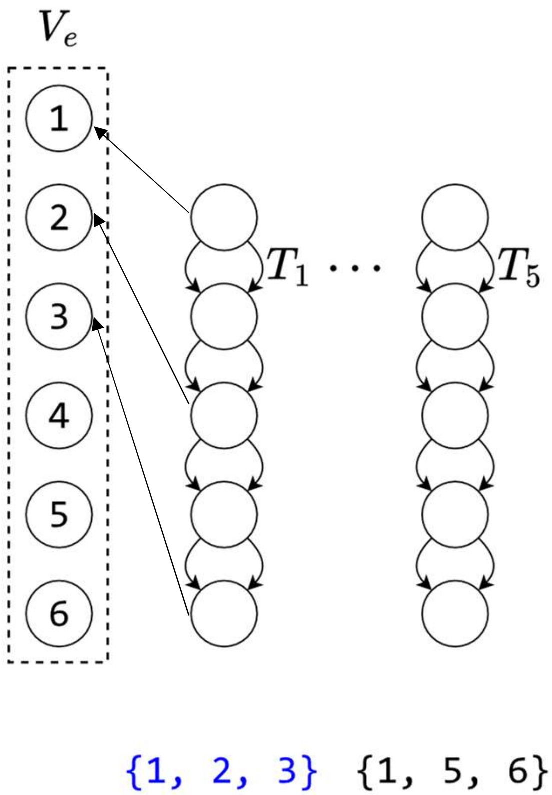

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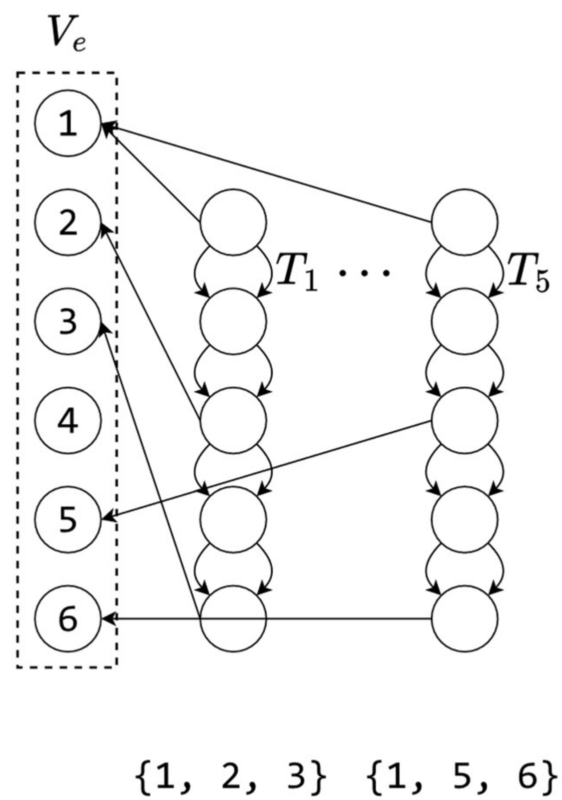

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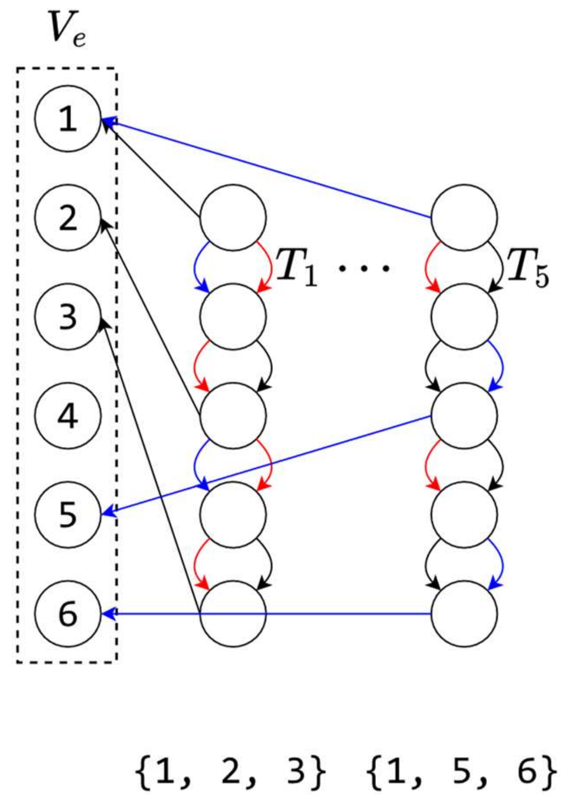

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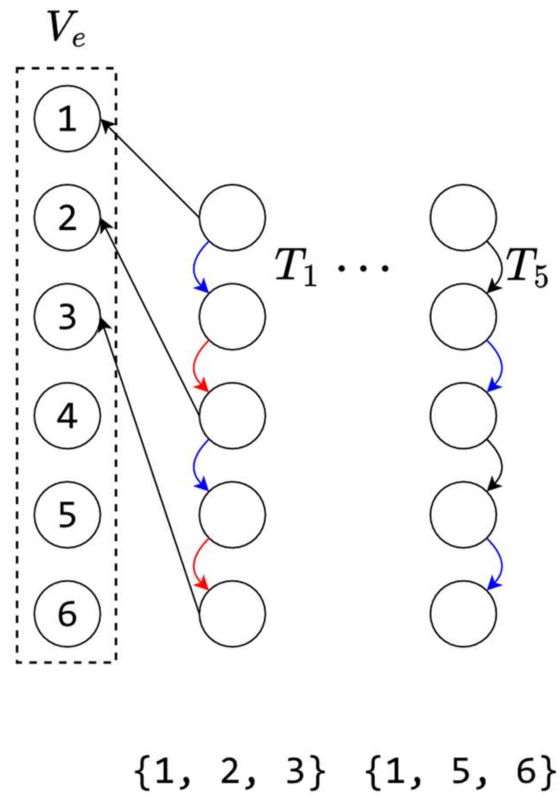


chosen v.s. not chosen

κ-COCST on directed graphs

κ-CCST
on directed graphs
$(\lambda \geq 2, \kappa \geq 1)$

reduction

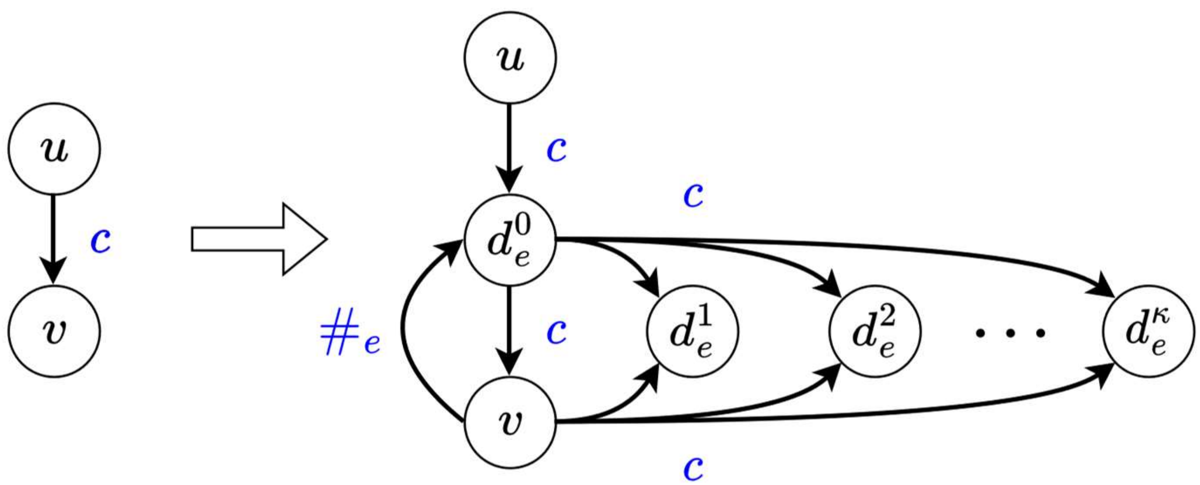
κ-COCST
on directed graphs
$(\lambda \geq 4, \kappa \geq 1)$

NP-Hard

κ-COCST on directed graphs

- Replace each edge $e = (u, v, c)$ with a gadget G_e.

edge e



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κ-COCST on directed graphs

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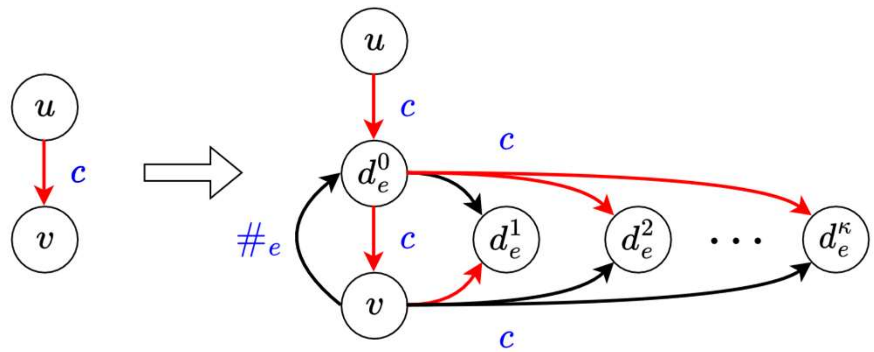

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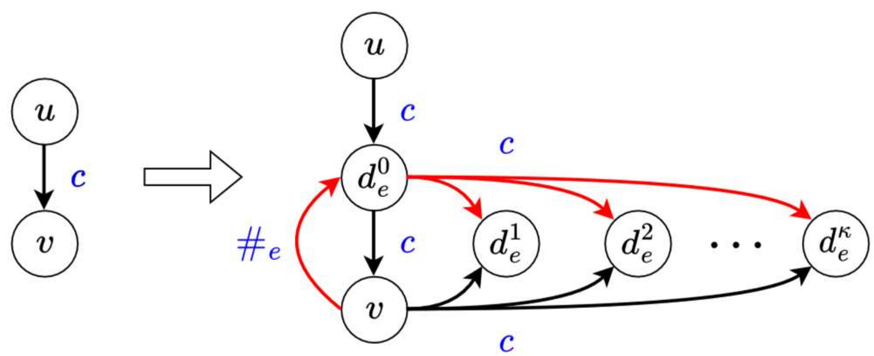

Conclusion

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DAG	≥ 3	≥ 1	NP-Hard
DAG	$= 2$	≥ 2	Open
DAG	$= 2$	$= 1$	P

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