From Curves to Words and Back Again: Geometric Computation of Minimum-Area Homotopy

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Bradley McCoy*, David L. Millman, Carola Wenk

Montana State University
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Montreal, Canada
WADS

Collaborator Appreciation

Hsien-Chih Chang

Brittany Fasy

David Millman

Carola Wenk

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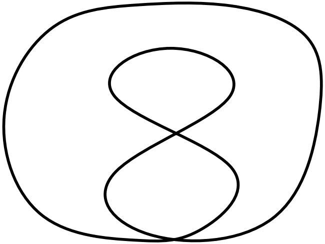
Input to Our Problem

Generic Curves



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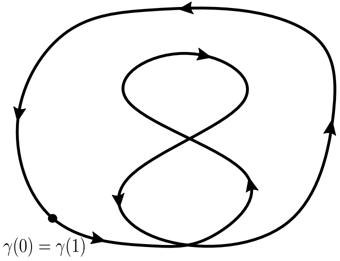

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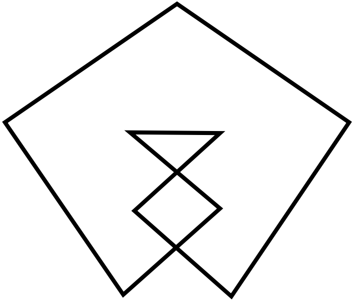

n - number of segments, $|F|$ - number of faces, $|V|$ - number of self-intersections

Input to Our Problem

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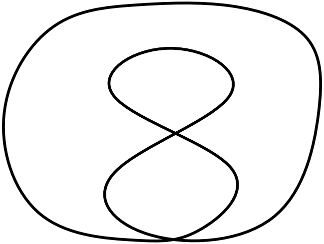

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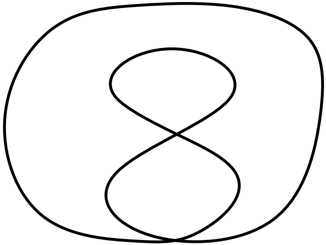
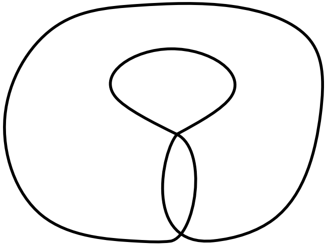
Input to Our Problem

Minimum Area Homotopy



Input to Our Problem

Minimum Area Homotopy



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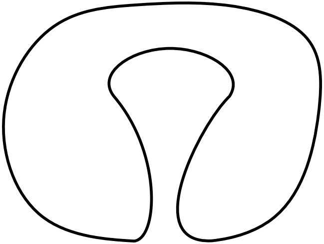
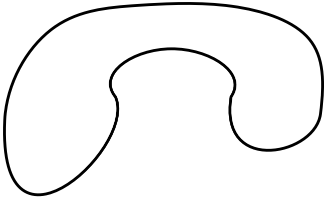


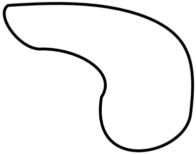
Input to Our Problem

Minimum Area Homotopy



Input to Our Problem

Minimum Area Homotopy



Input to Our Problem

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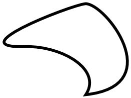

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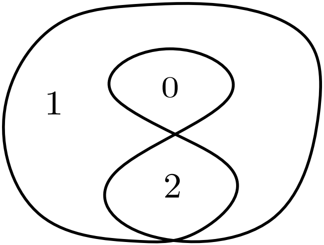


Agenda

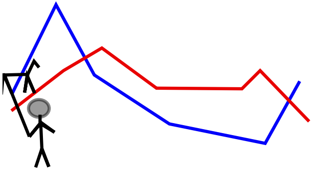
Talk Outline

- Data Analysis
- Self-Overlapping Curves
- Minimum Area Homotopy
- Word Equivalence
- Mapping Class Groups
- An Open Inverse Problem

Data Analysis Tool

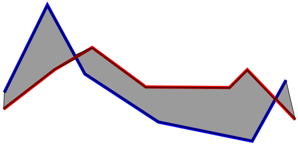
Measure Curve Similarity

Fréchet Distance



$\mathcal{O}(n^2 \log n)$

Minimum Homotopy Area



$\mathcal{O}(n^2 + |F|^6)$

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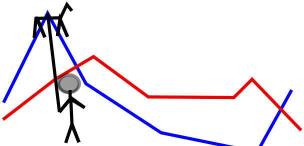

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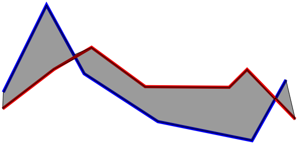

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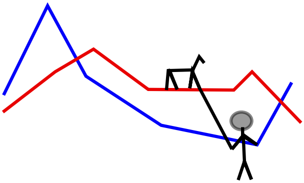
Fréchet Distance



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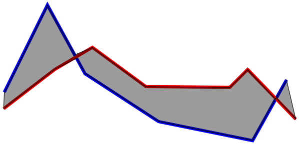

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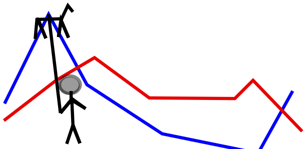

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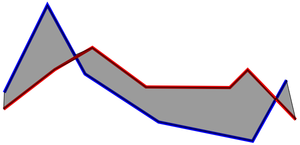
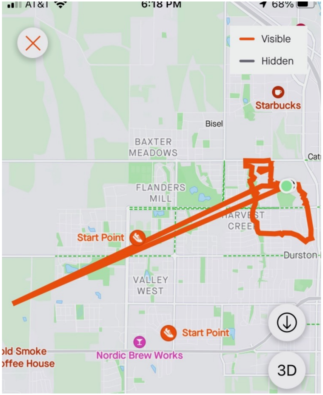


$\mathcal{O}(n^2 + |F|^6)$

Data Analysis Tool

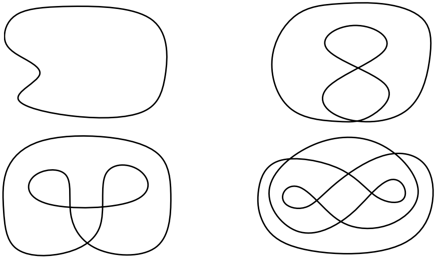
Robust to Single Data Errors



Is a Curve Self-Overlapping?

Definition

A curve γ is **self-overlapping** if there exists an immersion $F : D \to \mathbb{R}^2$ with $\gamma = F|_{\partial(D)}$.



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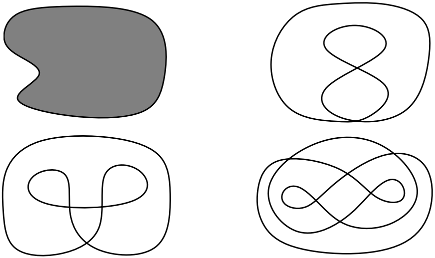

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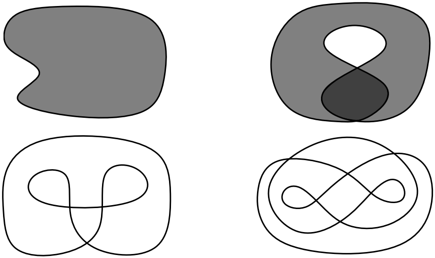

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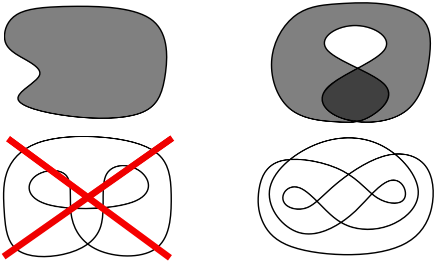

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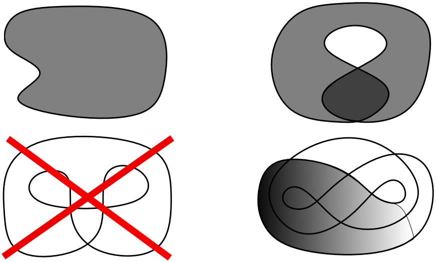

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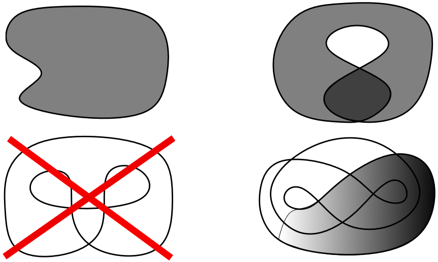
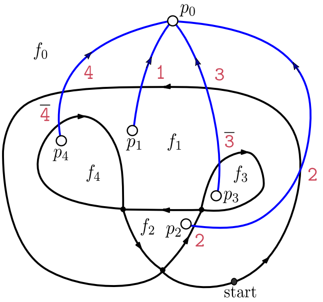


Blank's Algorithm

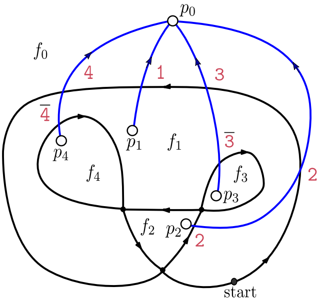
Blank Word Construction



$W = []$

Blank's Algorithm

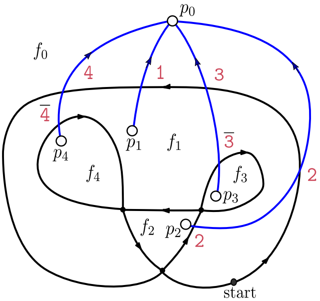
Blank Word Construction



$$W = [2]$$

Blank's Algorithm

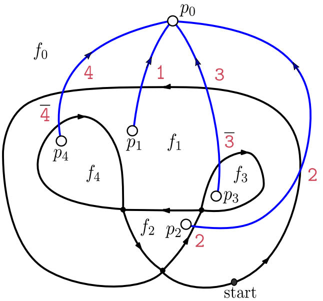
Blank Word Construction



$W = [23]$

Blank's Algorithm

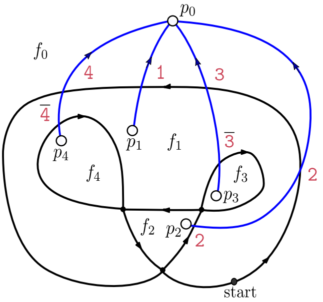
Blank Word Construction



$W = [231]$

Blank's Algorithm

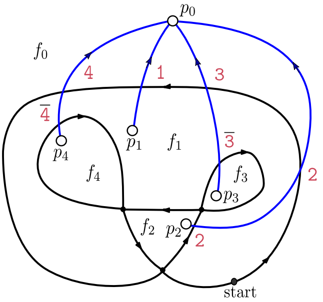
Blank Word Construction



$W = [2314]$

Blank's Algorithm

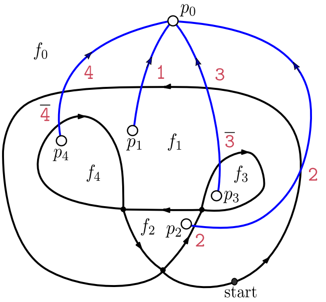
Blank Word Construction



$W = [23142]$

Blank's Algorithm

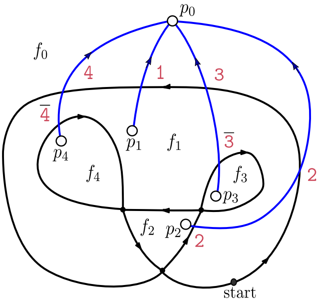
Blank Word Construction



$$W = [23142\overline{3}]$$

Blank's Algorithm

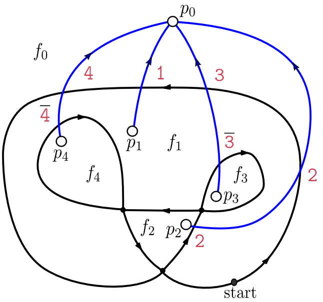
Blank Word Construction



$$W = [231423\overline{3}\overline{4}]$$

Blank's Algorithm

Blank Cuts and Groupings

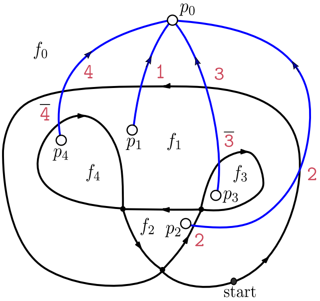


$w = [23142\overline{3}\overline{4}]$

- *Positive* subword $\sigma = f_1 f_2 \dots f_k$ where each f_i is positive

Blank's Algorithm

Blank Cuts and Groupings

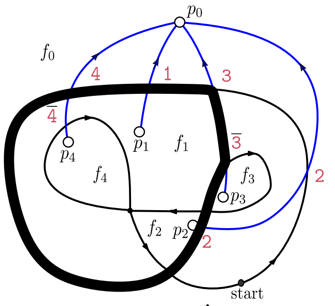


$w = [231\,\overbrace{42\overline{3}\overline{4}}]$

- *Positive* subword $\sigma = f_1 f_2 \ldots f_k$ where each f_i is positive
- Positive pairing $(f, \bar{f})$, $w = [fp\bar{f}w']$ p positive word

Blank's Algorithm

Blank Cuts and Groupings

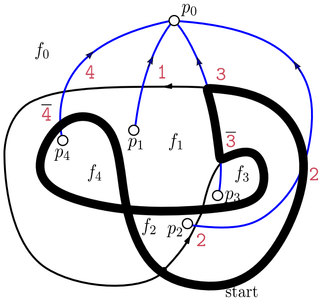


$$w = [2\,\overbrace{3142\bar{3}}\,\bar{4}]$$

- *Positive* subword $\sigma = f_1 f_2 \dots f_k$ where each f_i is positive
- Positive pairing $(f, \bar{f})$, $w = [fp\bar{f}w']$ p positive word
- Blank cut - replace positive pairing with identity

Blank's Algorithm

Blank Cuts and Groupings



$w = [2\,\overbrace{31423\overline{3}}\,\overline{4}]$

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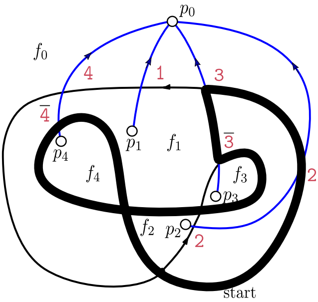
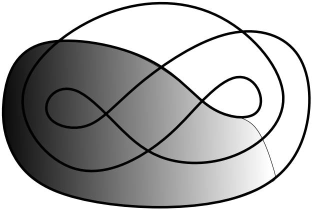


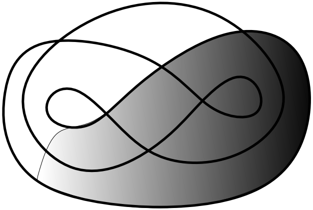
- *Positive* subword $\sigma = f_1 f_2 \dots f_k$ where each f_i is positive
- Positive pairing $(f, \bar{f})$, $w = [fp\bar{f}w']$ p positive word
- Blank cut - replace positive pairing with identity
- Groupable - remove positive pairings until positive word

Blank's Algorithm

Count the Number of Extensions



$w = a_2 a_5 a_1 a_3 a_4 a_6 a_5^{-1} a_3 a_4 a_6 a_7 a_2 a_5 a_3 a_6^{-1}$



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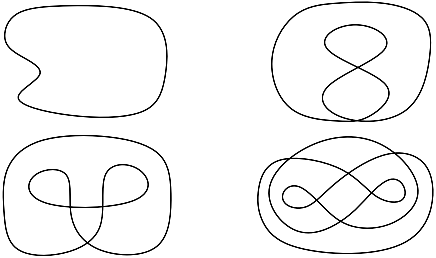
Examples

Definition

A **minimum area homotopy** is a continuous deformation from a curve γ to a point that sweeps over the minimum area in the plane.



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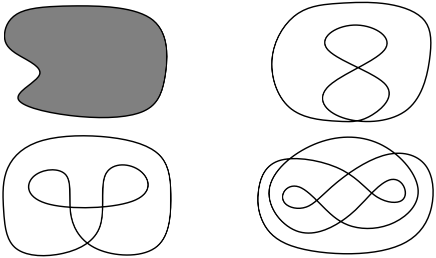

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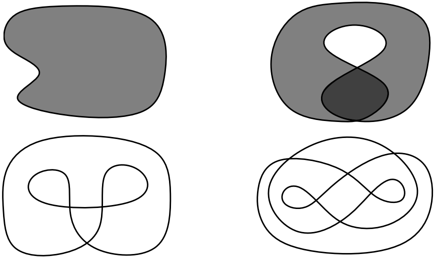

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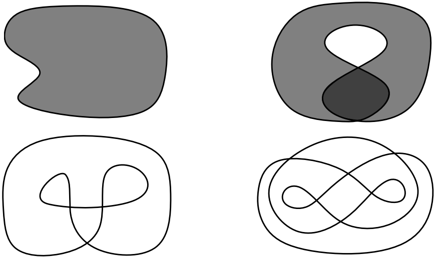

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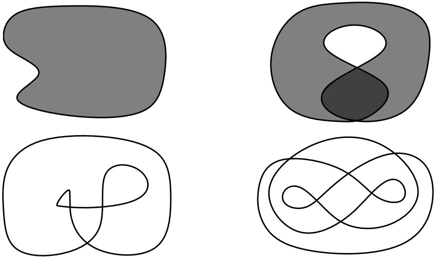

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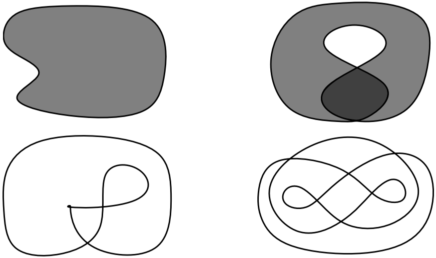

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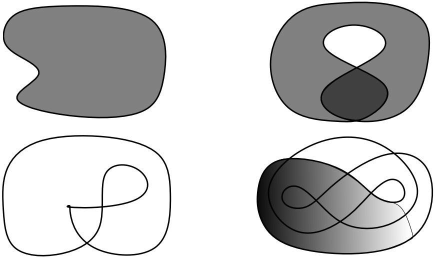
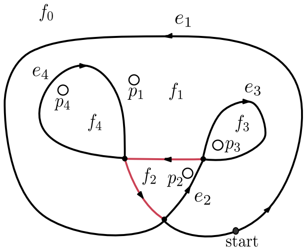


Nie's Algorithm

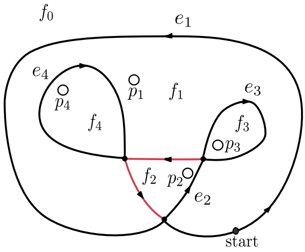
Words From Boundaries



- Choose a spanning tree

Nie's Algorithm

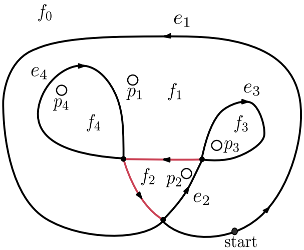
Words From Boundaries



- Choose a spanning tree
- Write each face as a boundary of edges
  - $\partial(f_1) = e_3 \bar{e}_2 e_1 e_4$
  - $\partial(f_2) = e_2$
  - $\partial(f_3) = \bar{e}_3$
  - $\partial(f_4) = \bar{e}_4$

Nie's Algorithm

Words From Boundaries



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Nie's Algorithm

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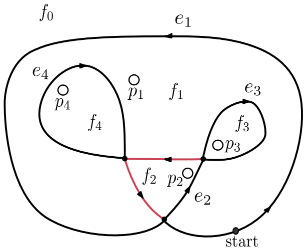

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- Write the curve as $\gamma = e_1 e_2 e_3 e_4$
- $\gamma = \partial(f_2)\partial(f_3)\partial(f_1)\partial(f_4)\partial(f_2)\partial(\overline{f_3})\partial(\overline{f_4})$

Nie's Algorithm

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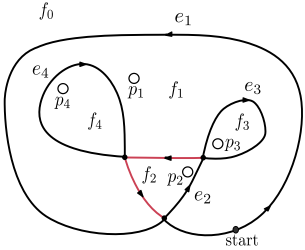
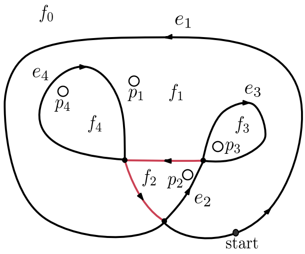


$w = [231423\bar{3}\bar{4}]$

- Folding - set of unlinked parings

Nie's Algorithm

Words From Boundaries

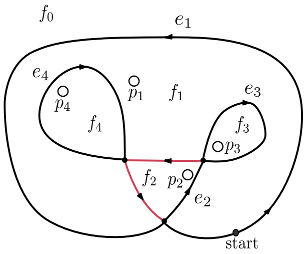


$w = [23142\bar{3}\bar{4}]$

- Folding - set of unlinked parings
- Weighted cancellation norm

Nie's Algorithm

Words From Boundaries



$w = [2314 2\bar{3}\bar{4}]$

- Folding - set of unlinked parings
- Weighted cancellation norm
- $||w|| := \min_{\mathcal{F}} \sum_i \text{Area}(f_i)$

Nie's Algorithm

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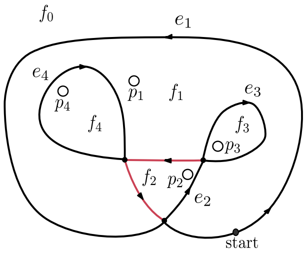

$w = [23142\bar{3}\bar{4}]$

- Folding - set of unlinked parings
- Weighted cancellation norm
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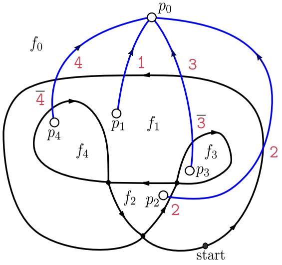
- Dynamic program - think matrix chain multiplication

Word Equivalence

Where Did We Make Choices?

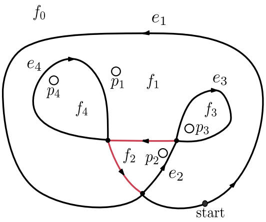
Blank's Word



$w = [123423\bar{3}\bar{4}]$

Nie's Word

$\partial(f_1) = e_3 \bar{e}_2 e_1 e_4$



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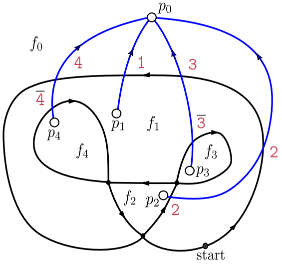

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How to draw the cables.

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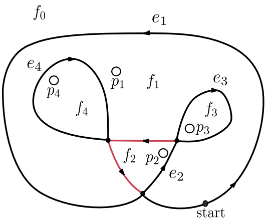

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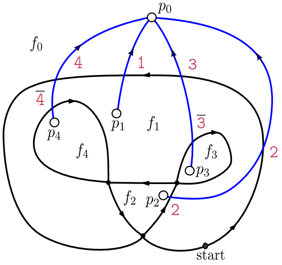

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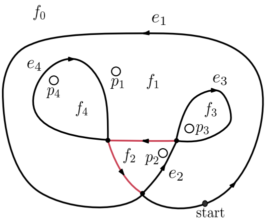

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Spanning tree.

Word Equivalence

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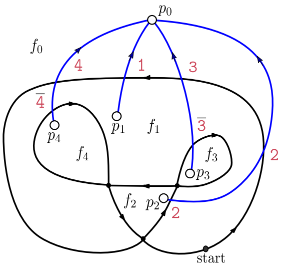

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How to draw the cables.

Order around p_0.

Nie's Word

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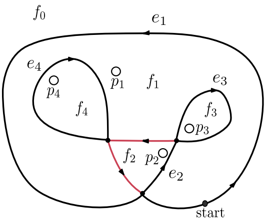

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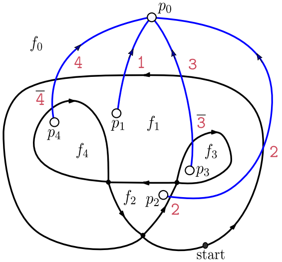

$w = [12342\bar{3}\bar{4}]$

How to draw the cables.

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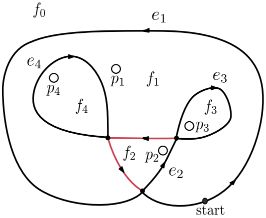

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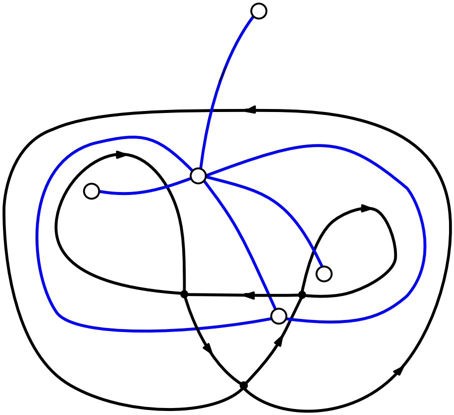
Spanning tree.

Cyclic order of boundary.

Word Equivalence

Tree Co-Tree Decomposition



Word Equivalence

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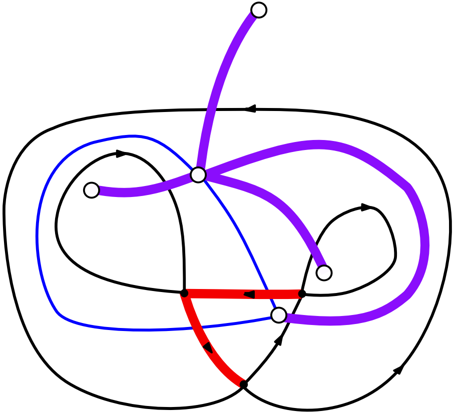

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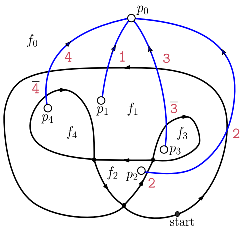

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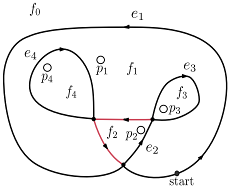

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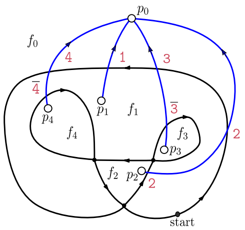
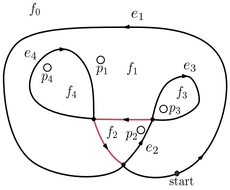


Where to place cable one.

Nie's Word



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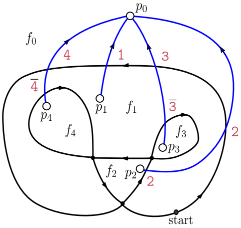

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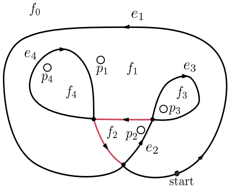
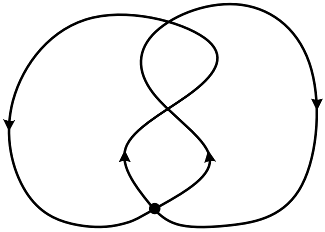


$$\partial(f_1) = e_4 e_3 \bar{e}_2 e_1 \sim e_1 e_4 e_3 \bar{e}_2$$

Foldings $\iff$ Self-Overlapping Decompositions

Theorem (Fasy, Karakoç, Wenk - 2017)

Any curve has a self-overlapping decomposition whose area is minumum over all null-homotopies



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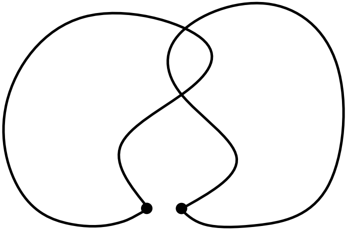

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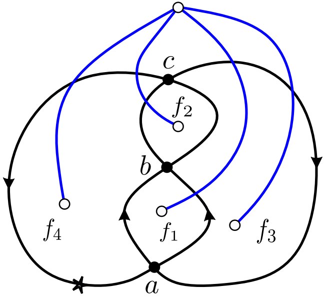
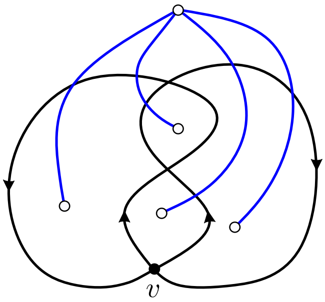


$$w = [a1b\overline{2}c\overline{13}abc24].$$

Possible Issue



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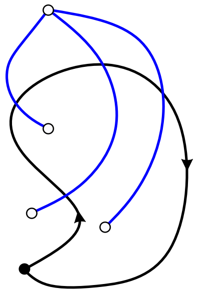
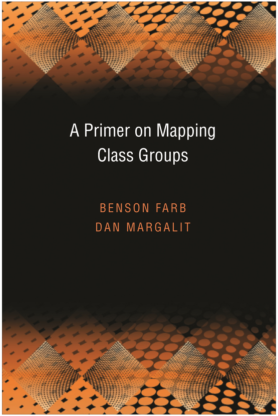


Cable Independence

Mapping Class Groups

- surface with genus g and n punctures $S_{g,n}$



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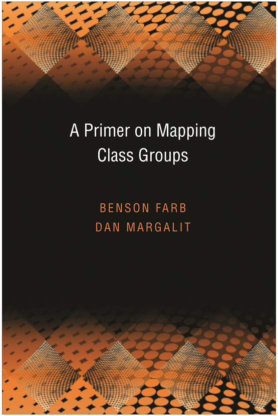

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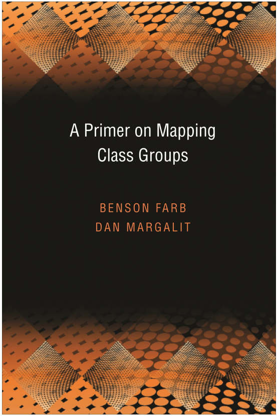
  $\phi \sim \psi$ if ϕ and ψ isotopic



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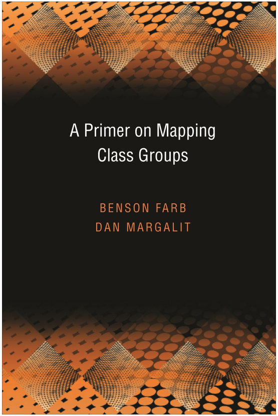

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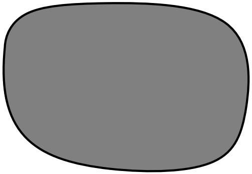

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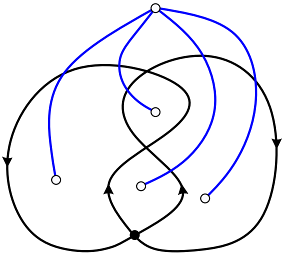

- our surface - punctured disk

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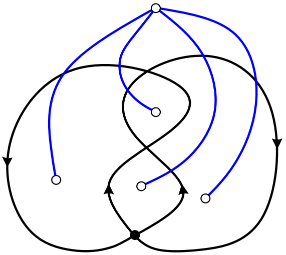
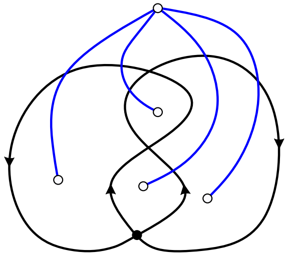


- our surface - punctured disk
- cable - nonseparating puncture-to-puncture arc

Cable Independence



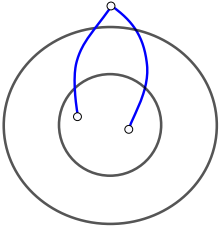
- our surface - punctured disk
- cable - nonseparating puncture-to-puncture arc
- pure mapping class group - punctures fixed

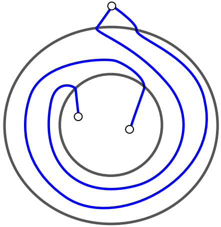
Cable Independence

Dehn Twists

Theorem (9.3 Farb, Margalit)

The pure mapping class group of D_n is generated by Dehn twists about the set of simple closed curves that surround exactly two punctures.





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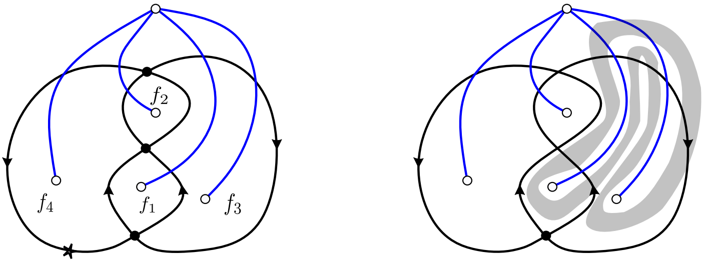

$$[1\overline{213}24] \to [311\overline{132}31\overline{113}31\overline{313}24]$$

Cable Independence

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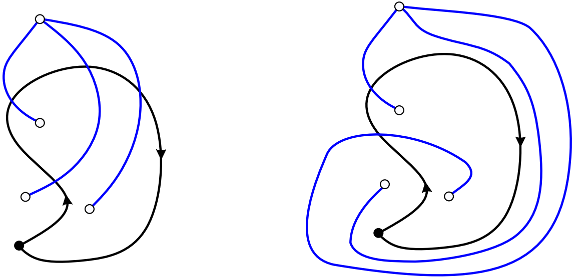

The Results

Lemma

Let γ *be a curve. For each folding* F *there exists a null-homotopy of* γ *with area equal to the area of* F.

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Corollary

New proof of correctness.

The Results

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Given a maximal folding F of w, there is a self-overlapping decomposition of γ whose area is equal to the area induced by the folding F.

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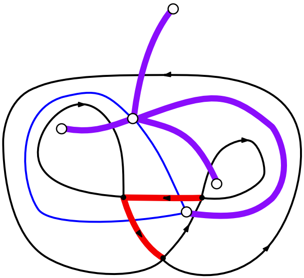
Corollary

Polynomial time minimum area self-overlapping decomposition.

Highlight Cool Ideas

Cut Cycle Duality



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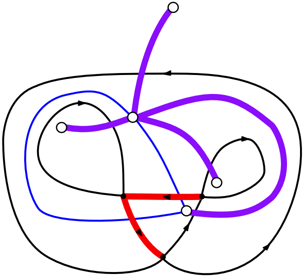

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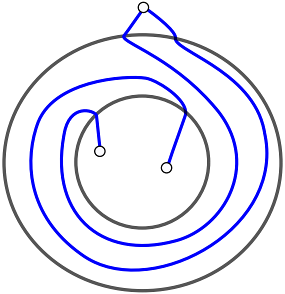

An Open Problem

Which Words are Curves?

Given a word, construct a curve with word equal to the given word or say no curve exists.

$w_1 = [12]$ $w_2 = [122]$

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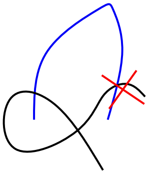

$w_2 = [122]$

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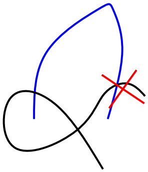

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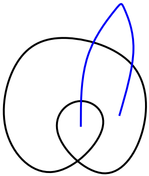
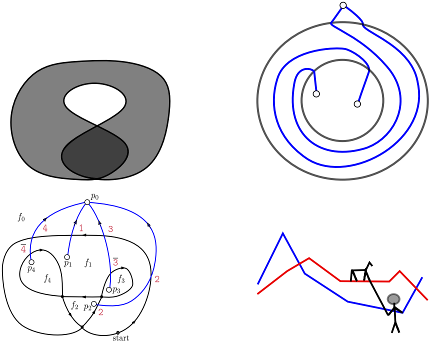


Questions?

Thank You



bradleymccoy@montana.edu