Dominator Coloring and CD Coloring in Almost Cluster Graphs

Aritra Banik; Prahlad Narasimhan Kasthurirangan; Venkatesh Raman

WADS 2023

NISER Bhubaneswar; Stony Brook University; Institute of Mathematical Sciences

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Introduction

Problem of Interest

Consider a *social networking graph* G. What is the minimum number of *stranger groups* required to partition $V(G)$ such that each stranger group has a *common friend*?

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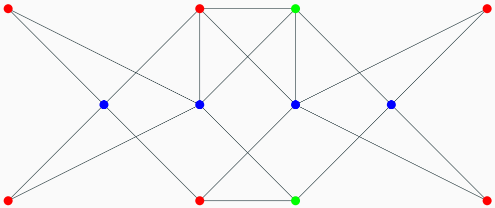

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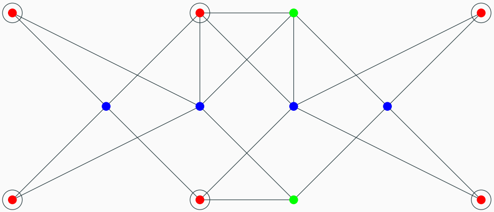

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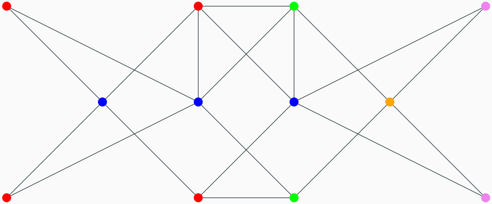

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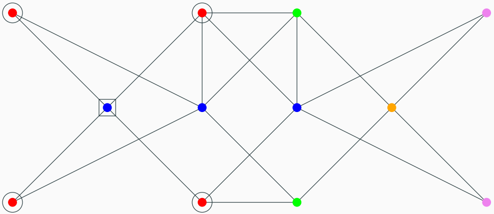

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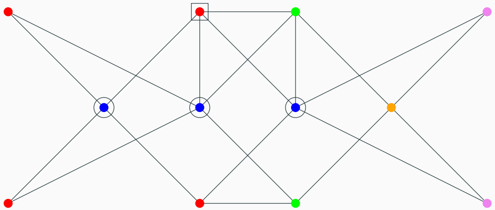

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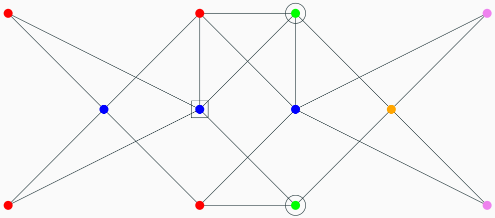

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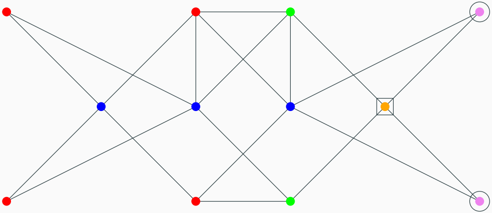

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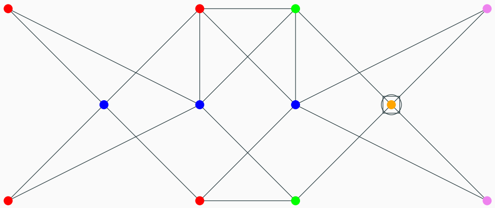

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CD Coloring is NP-hard for $l \geq 4$ [Merouane *et al.*, 2015].

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Fixed-Parameter Tractability

Fixed-Parameter Tractable Algorithm

An algorithm $\mathcal{A}$ is FPT with respect to the parameter k if there exists a computable function f and a constant c such that for every instance I of $\mathcal{P}$, $\mathcal{A}$ runs in $f(k) \cdot |I|^c$-time.

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Social Networks

Social networking graphs have been empirically shown to have:

- High density of triangles.
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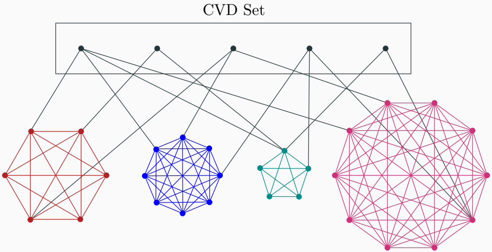
Social networking graphs are almost cluster graphs!

Parameter of Interest

For such graphs, a popular parameter is *Cluster Vertex Deletion set* size:



Overview of Results

	CD Coloring	Dominator Coloring
Exact	$\tilde{\mathcal{O}}(2^n)^\dagger$	$\tilde{\mathcal{O}}(4^n)$
CLQ	$\mathcal{O}^*(2^k)$	$\mathcal{O}^*(16^k)$
TC	$\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$	$\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$
CVD Set	$\mathcal{O}^*(2^{\mathcal{O}(2^k kq \log q)})$	$\mathcal{O}^*(2^{\mathcal{O}(2^k)})$

We also establish some lower bounds for CD Coloring and Dominator Coloring with respect to these parameters.

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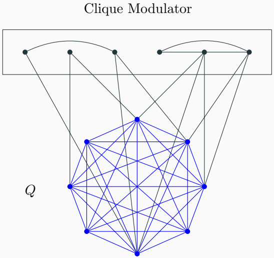
Questions?

- $M \subseteq V(G)$ is a *CVD set* if $G - M$ is a cluster graph.
- A proper coloring χ of G is a *CD coloring* of G if every color class is dominated by a vertex in $V(G)$.
- We now design a randomized algorithm which solves CD Coloring in $\mathcal{O}^*(2^k)$ time where k is the size of a *clique modulator*.

Clique Modulator

A $M \subseteq V(G)$ is a *clique modulator* if $G - M$ is a clique.



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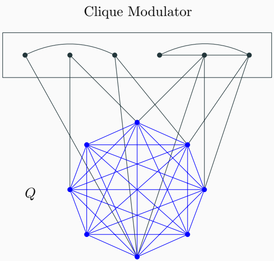

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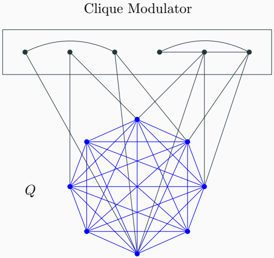

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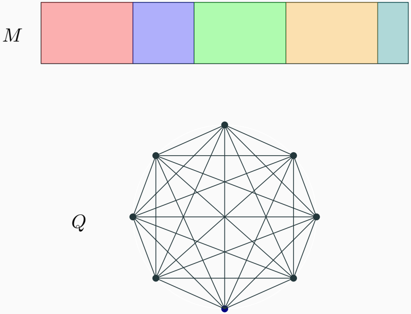
An optimal clique modulator can be found “quickly” [Gutin *et al.*, 2021].

Coloring the Clique



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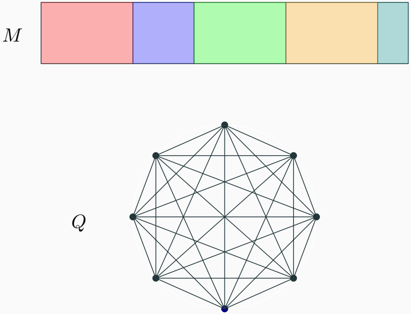

List Coloring can be solved in polynomial time on cliques [Arora *et al.*, 2020].

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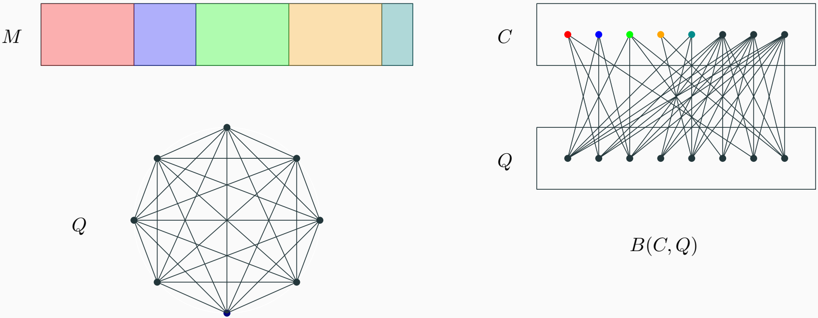

Lemma

(G, l) is a YES instance of CD COLORING with this coloring of M, if, and only if, $B(C, Q)$ has a matching saturating Q.

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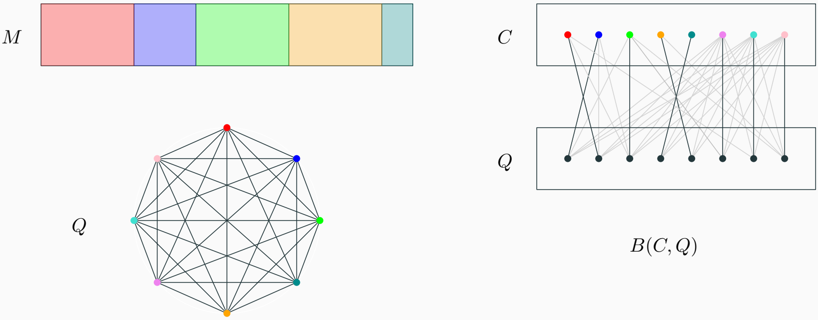

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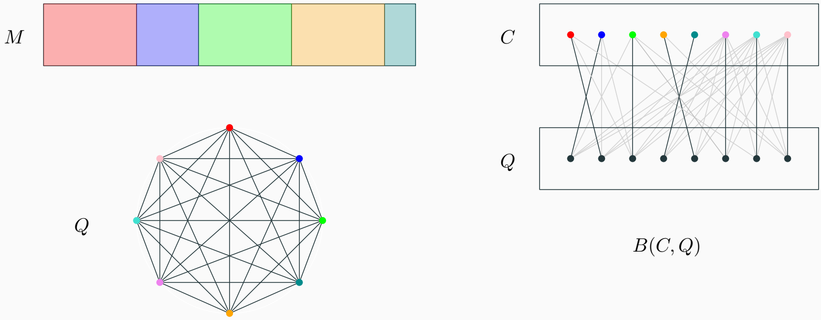

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Takes $\mathcal{O}^*(k^k)$-time!

Goal

Parameterized by Clique Modulator Size

There exists a randomized algorithm that solves CD Coloring in $\mathcal{O}^*(2^k)$ time where k is the size of an optimal clique modulator of the input graph.

Polynomials

Divisibility Determination

Given a polynomial $p(x_1, x_2 \ldots x_n)$ over $\mathbb{R}$, and a $J \subseteq \{1, 2, \ldots n\}$, determine if p contains a monomial m such that $\prod_{j \in J} x_j \mid m$.

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Addressed in [Wahlström, 2013].

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q can be constructed and evaluated in $\mathcal{O}^*(2^{|J|})$-time.

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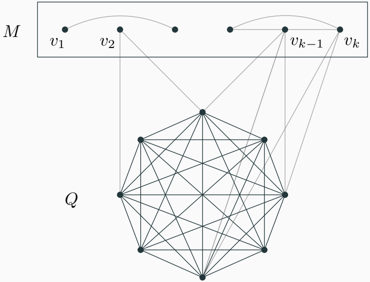

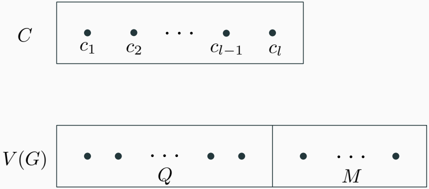

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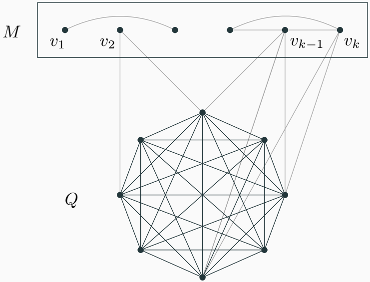

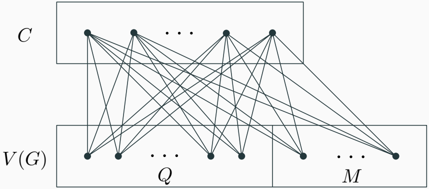

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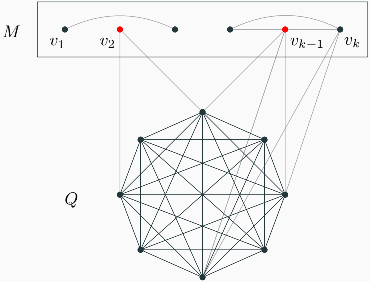

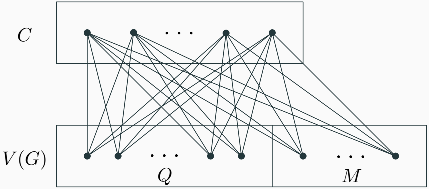

How do we assign two vertices the same color?

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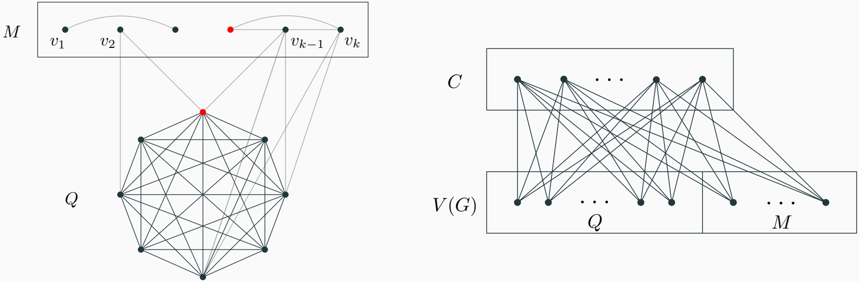

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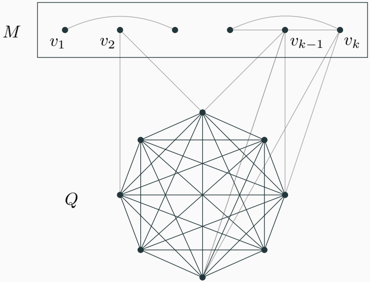

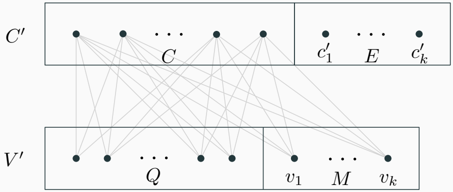

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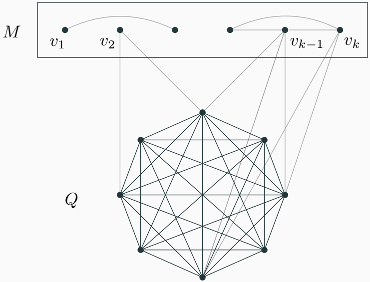

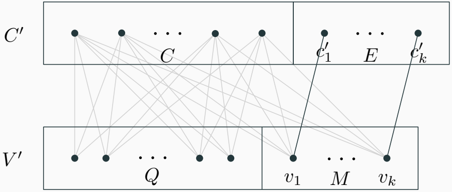

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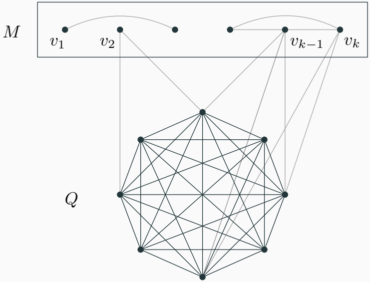

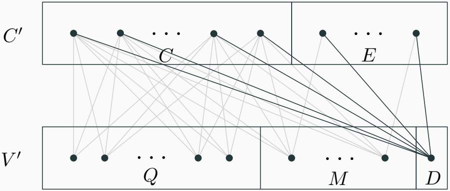

Add a set of “dummy” vertices for balance.

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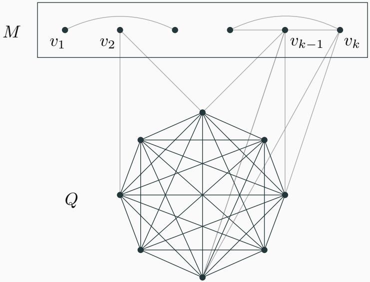
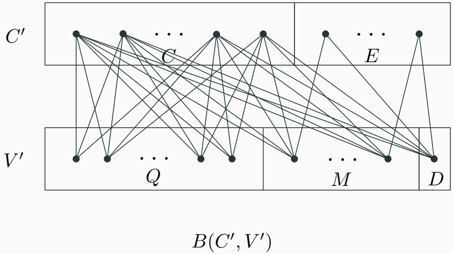
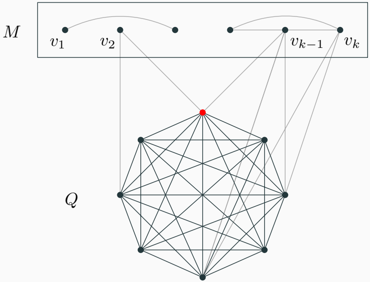
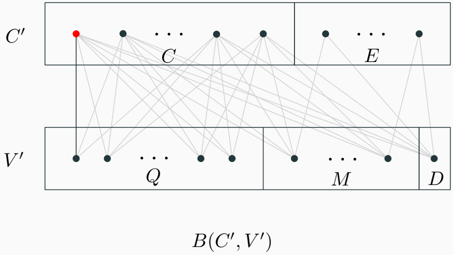




Step 2: Constructing its Edmond's Matrix





Which vertices in M can also be colored red?

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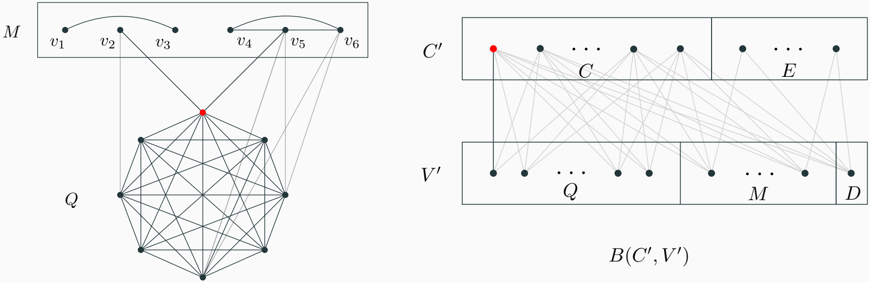

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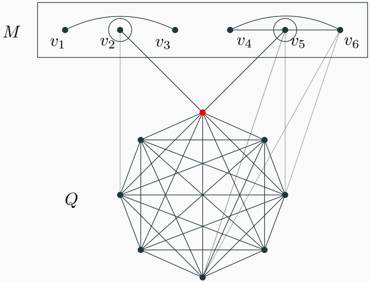

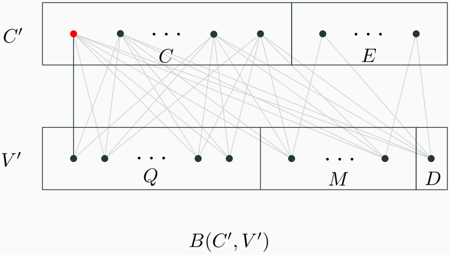

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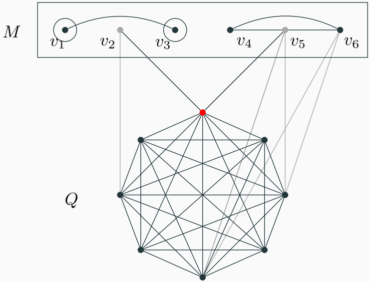

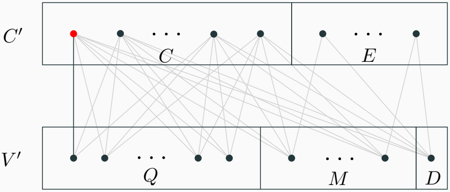

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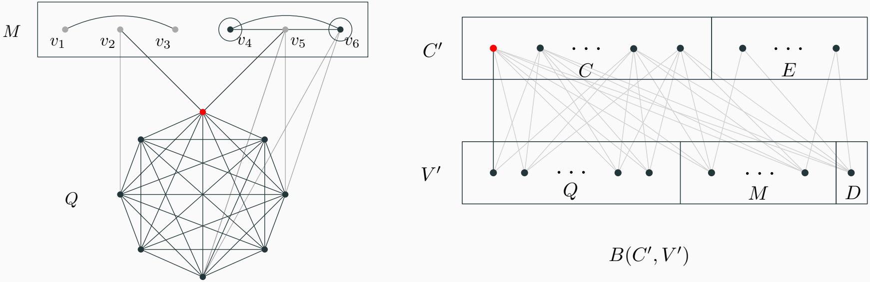

Vertices in *M* that can be colored red:

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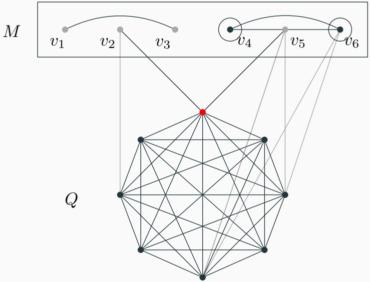

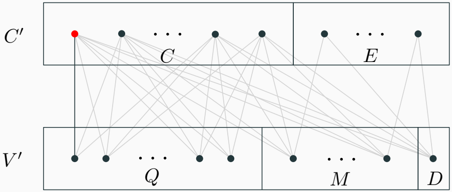

$B(C', V')$

Vertices in M that can be colored red: $\mathcal{S}_{(v,c)} = \{\emptyset, \{v_4\}, \{v_6\}\}$.

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- For an edge $(v, c) \in V(G) \times C$, $\mathcal{S}_{(v,c)}$ is the collection of subsets of M which can also be colored c if v is colored c.

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Edmond's Matrix

$$A(v, c) = \begin{cases} z_{(v,c)} p_{(v,c)} \text{ if } (v, c) \in E(B) \\ 0 \text{ otherwise} \end{cases}$$

Step 3: Profit?

Theorem

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By [Wahlström, 2013], this takes $\mathcal{O}^*(2^{|M|}) = \mathcal{O}^*(2^k)$ time.

Questions?

	CD COLORING	DOMINATOR COLORING
Exact	$\tilde{\mathcal{O}}(2^n)^\dagger$	$\tilde{\mathcal{O}}(4^n)$
CLQ	$\mathcal{O}^*(2^k)$	$\mathcal{O}^*(16^k)$
TC	$\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$	$\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$
CVD Set	$\mathcal{O}^*(2^{\mathcal{O}(2^k kq \log q)})$	$\mathcal{O}^*(2^{\mathcal{O}(2^k)})$

We also establish some lower bounds for CD COLORING and DOMINATOR COLORING with respect to these parameters.

†Proved by Krithika *et al.* in 2021.