Block Crossings in One-Sided Tanglegrams

Alexander Dobler, Martin Nöllenburg

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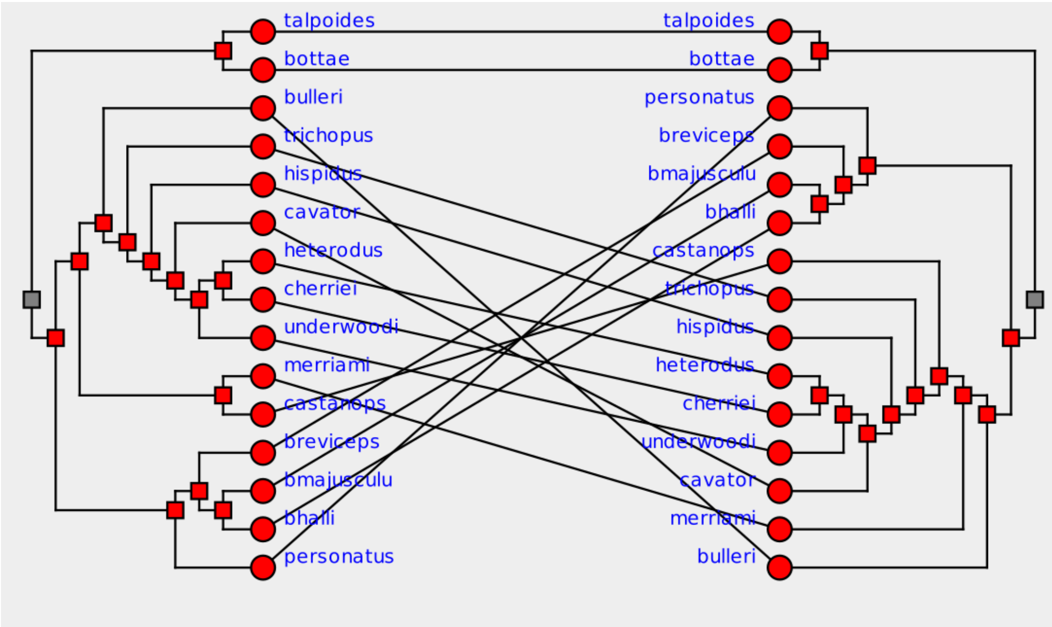
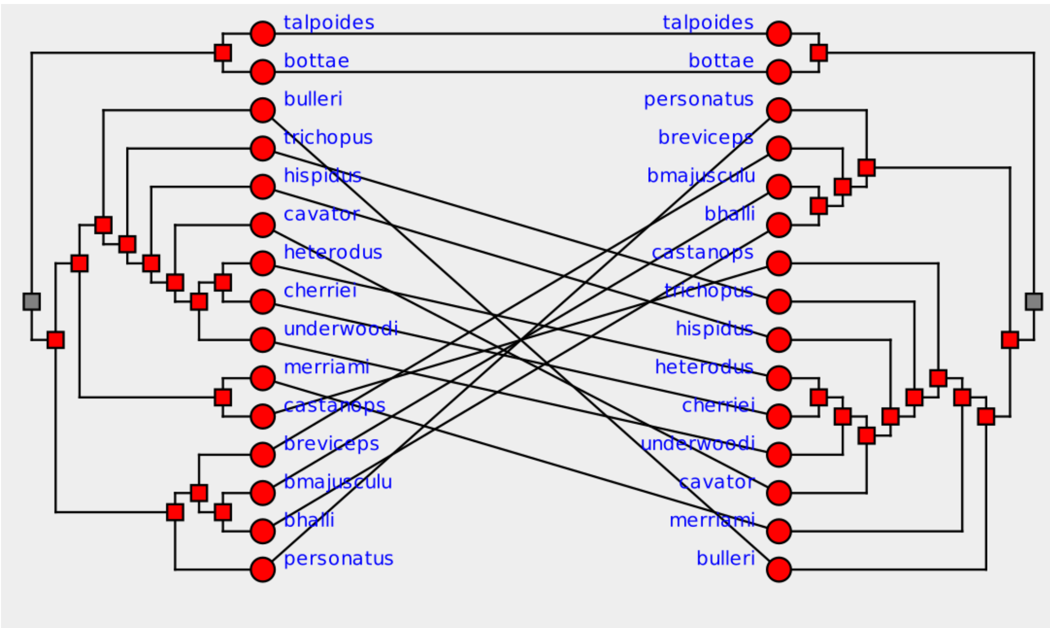
ALGORITHMS AND
COMPLEXITY GROUP

Tanglegrams



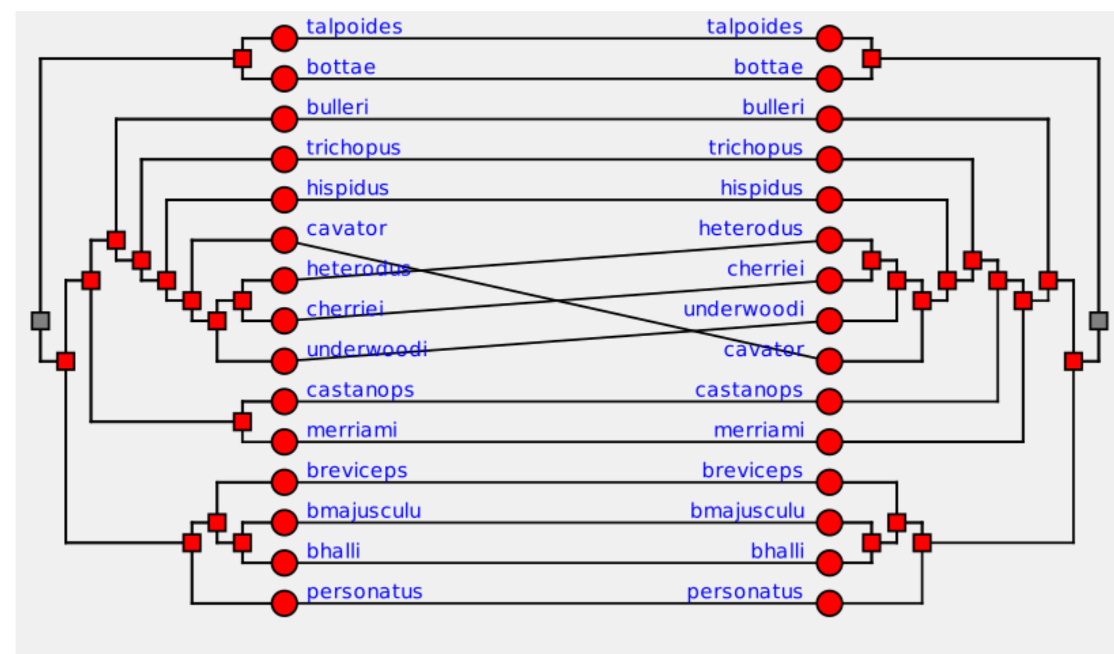
Comparison of species trees of **same leaf set**

Tanglegrams



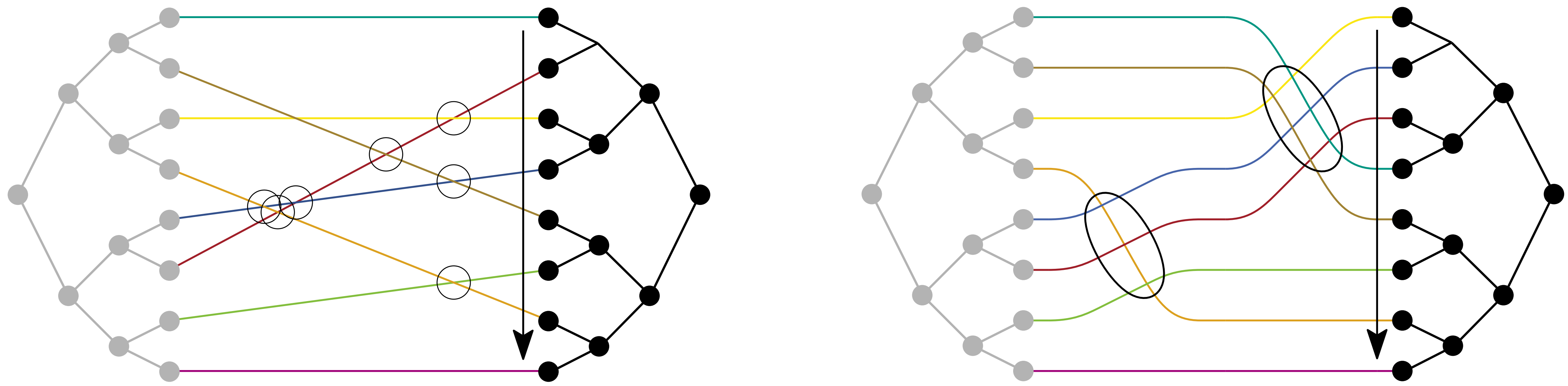
Comparison of species trees of **same leaf set**

Known **combinatorial problem**: reorder leaves of one/both trees to **minimize crossings**



Edge Crossings vs. Block Crossings

Same instance, left leaf order fixed



Optimal: 8 edge crossings

Optimal: 2 block crossings
(but 9 edge crossings)

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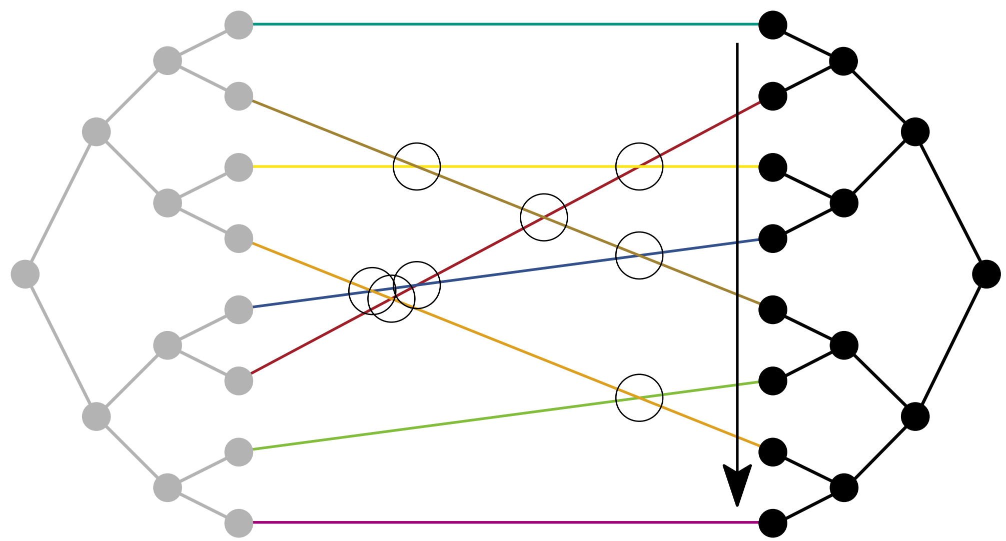

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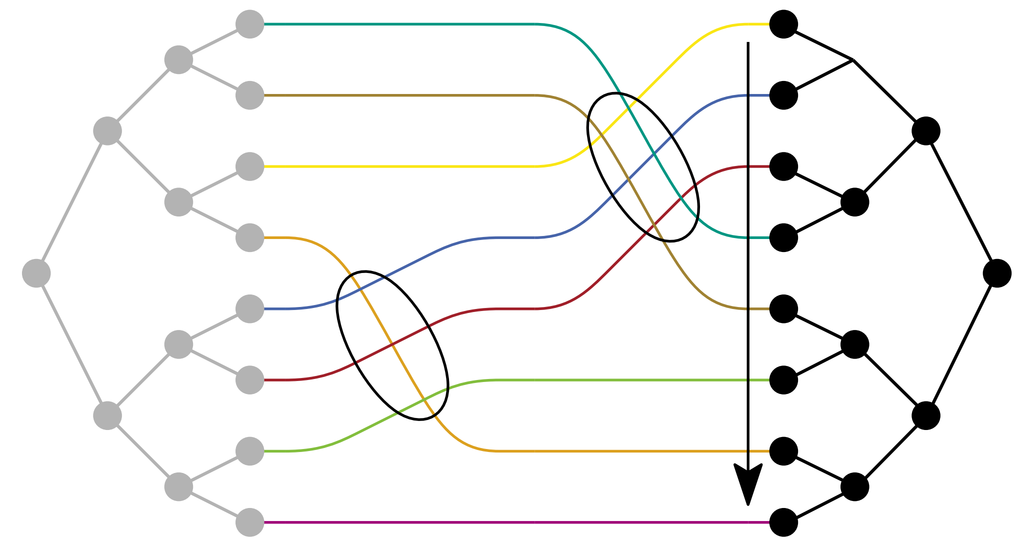
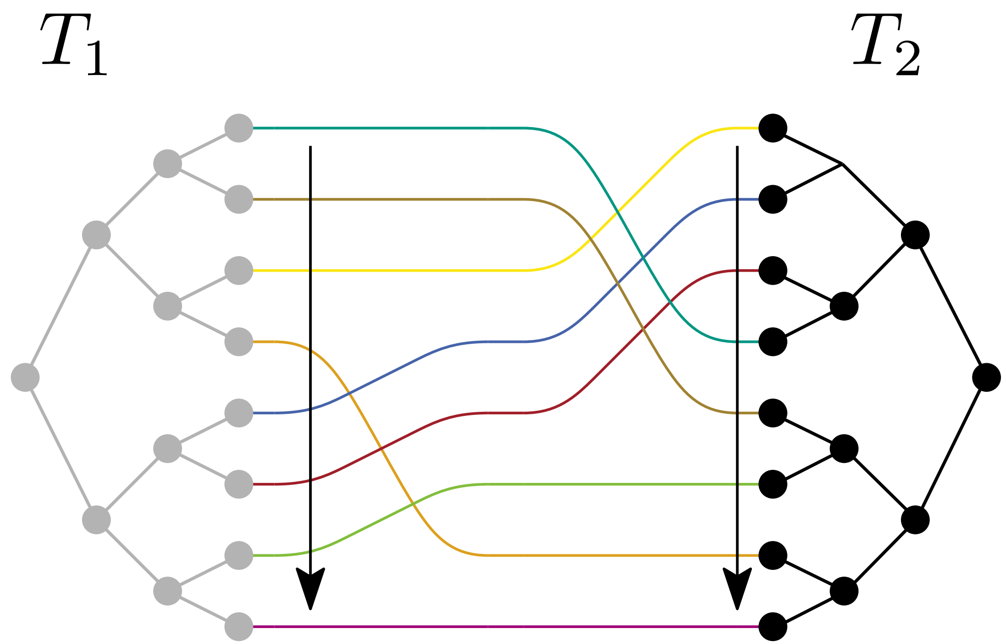


Optimal: 2 block crossings (but 9 edge crossings)

Reduces **visual clutter**, but edges **non-straight-line**

One-Sided Block Crossing Minimization Problem



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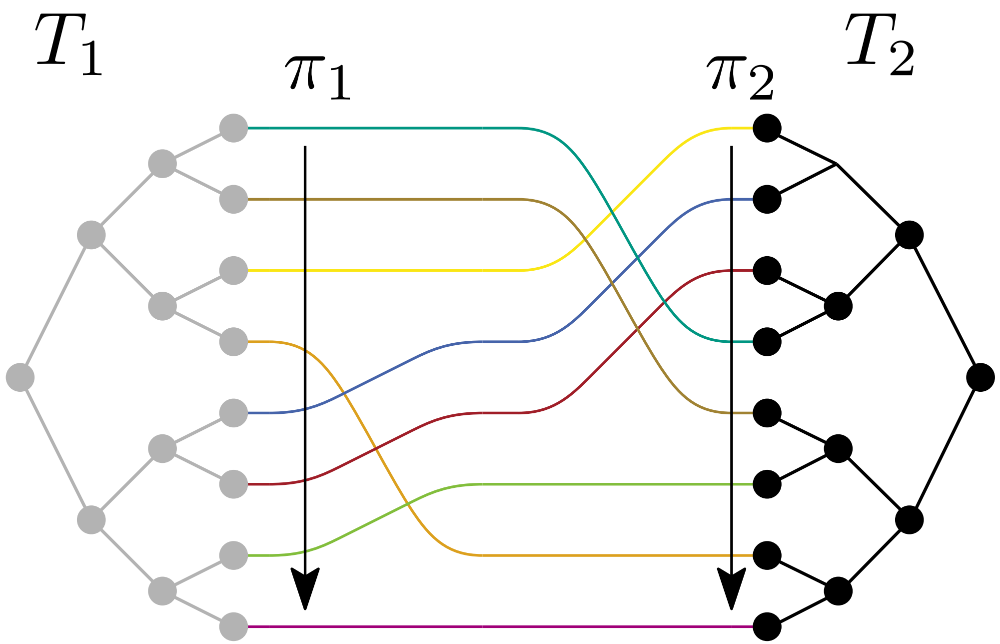

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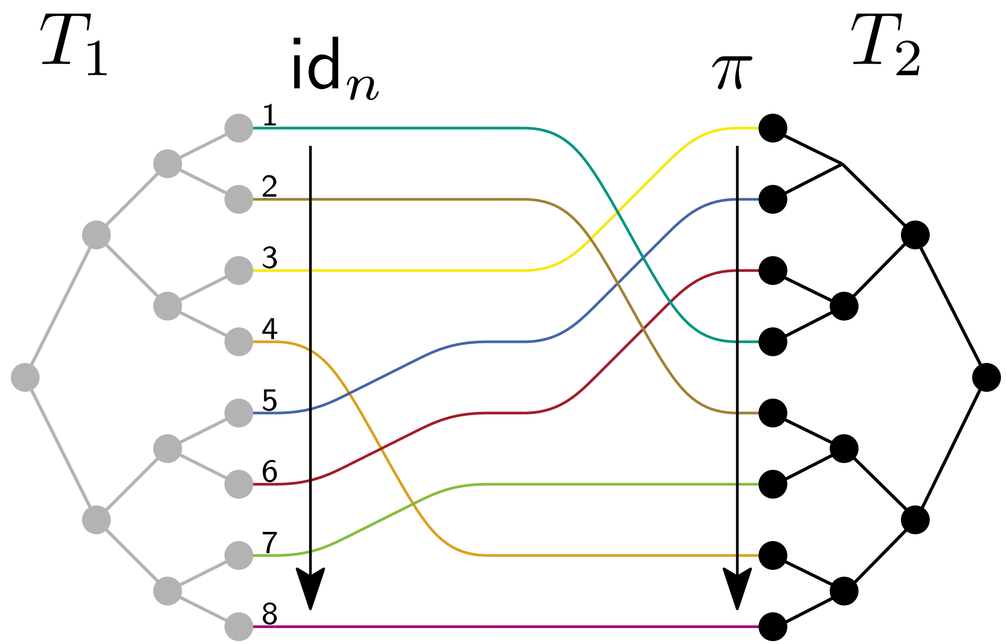

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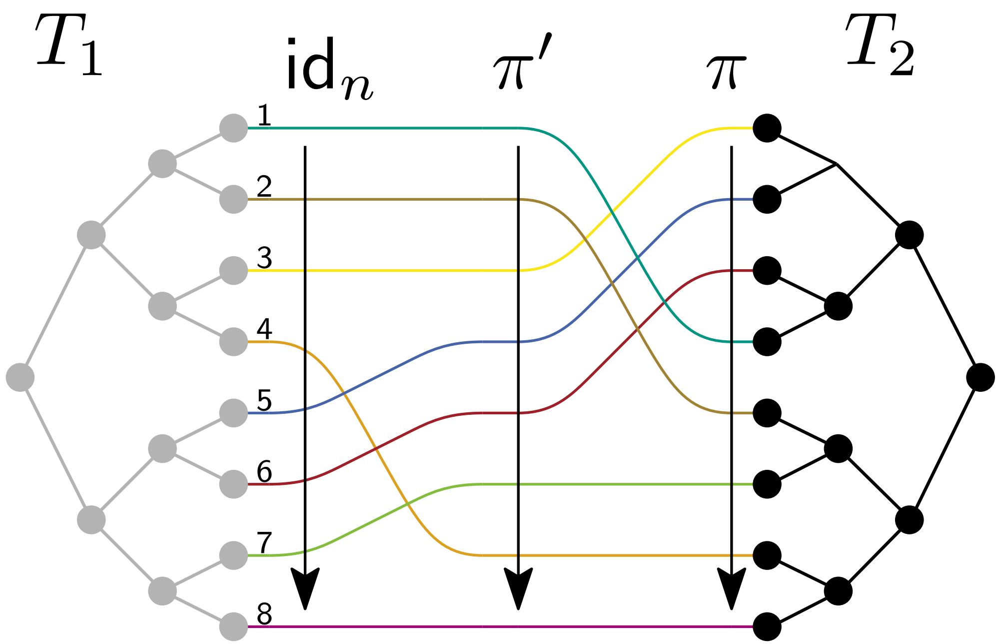

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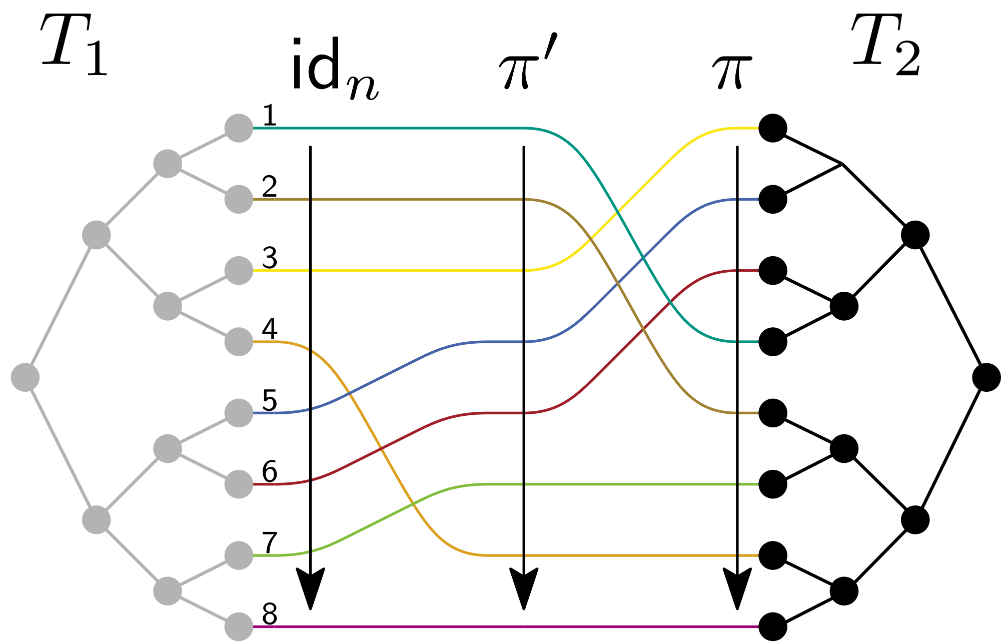

Between π and π' two bundles of edges exchange positions

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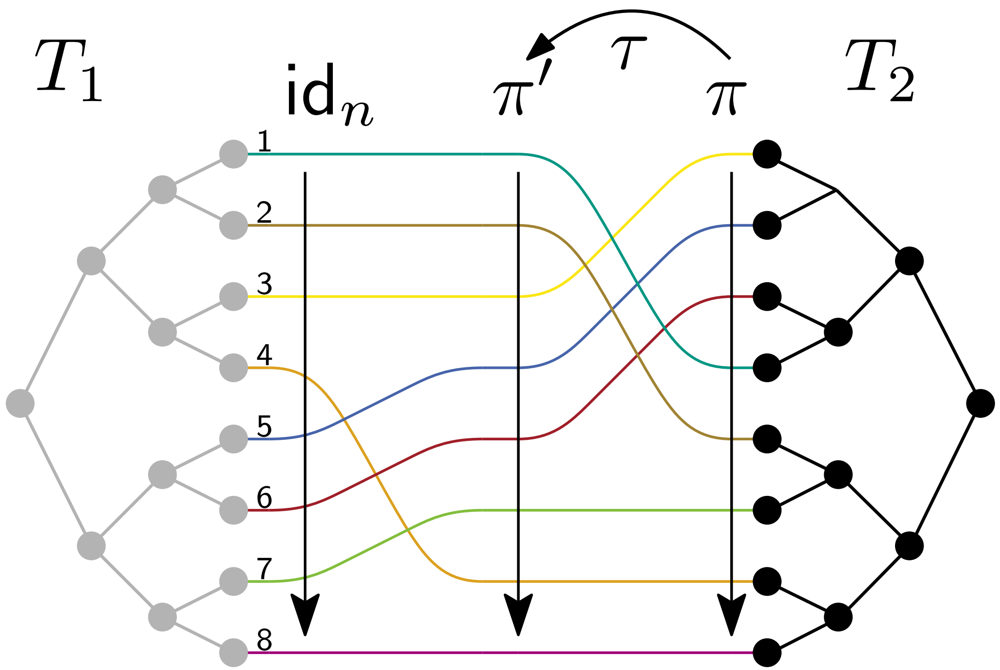

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$\pi' = \pi \circ \tau$ for **transposition** τ

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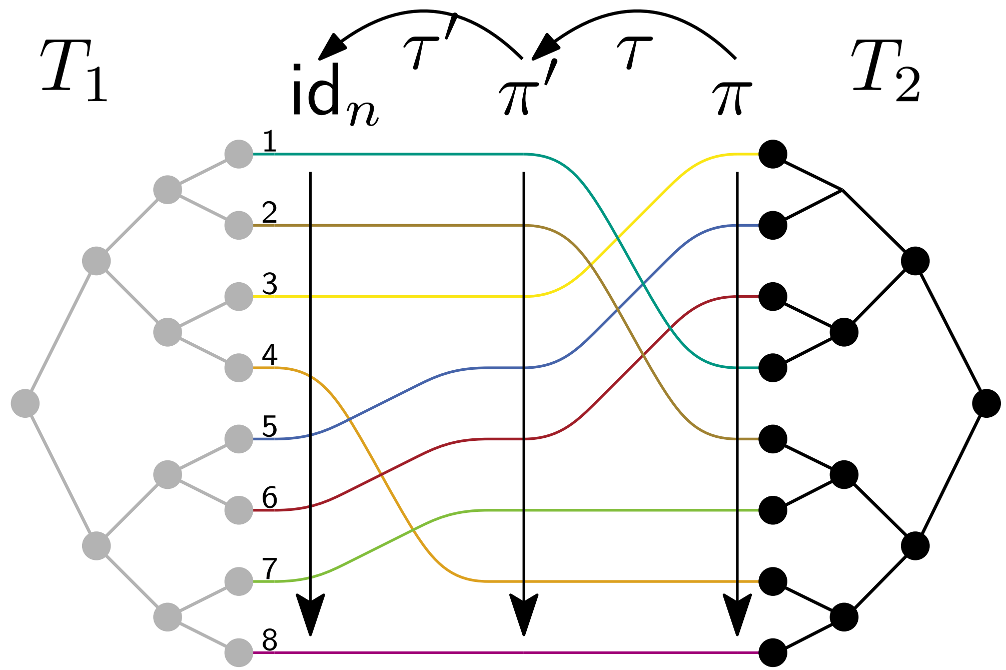

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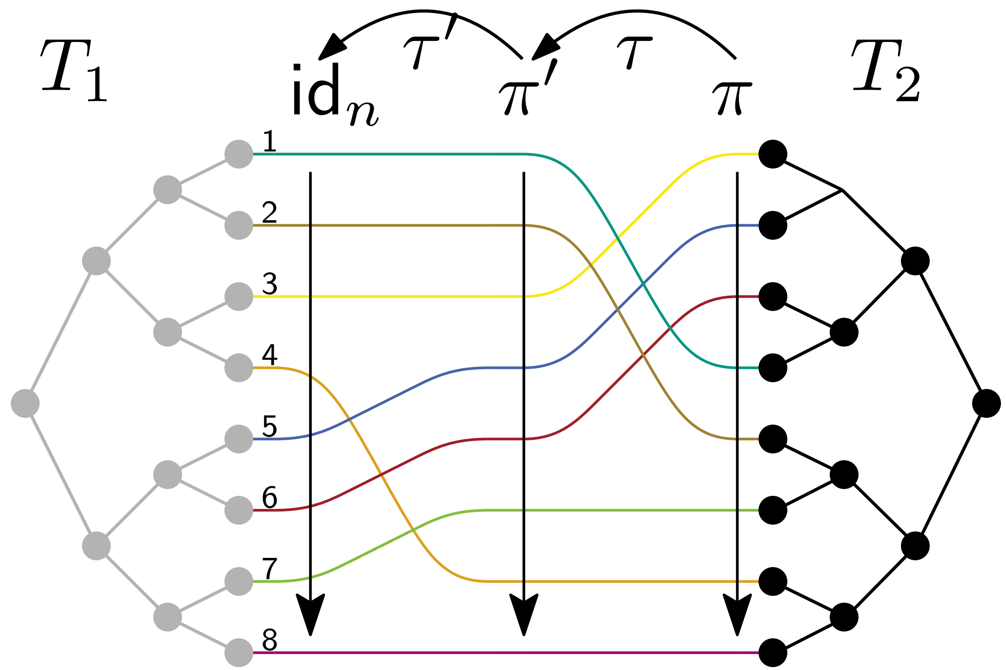

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A **transposition** $\tau = \tau(i, j, k) \in \Pi_n$ with $1 \leq i < j < k \leq n + 1$ is the permutation

$$(1, \ldots, i-1, j, \ldots, k-1, i, \ldots, j-1, k, \ldots, n).$$

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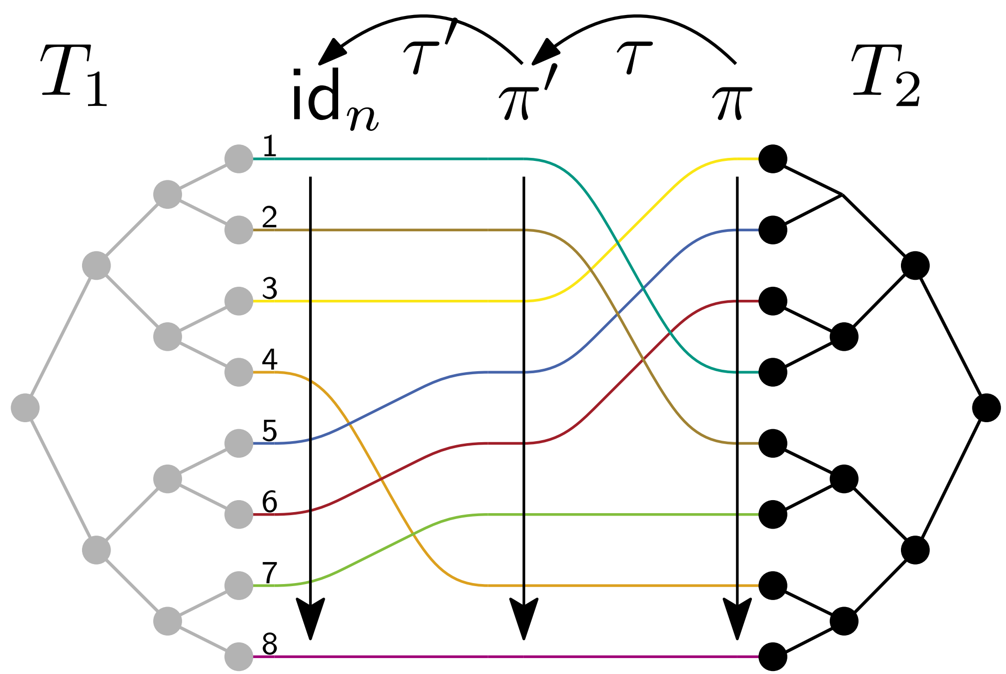

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The **transposition distance** $d_t(\pi)$ of $\pi \in \Pi_n$ is the min. number $k \in \mathbb{N}$ s.t. there exist transpositions $\tau_1, \ldots, \tau_k$ with $\pi \circ \tau_1 \circ \tau_2, \cdots \circ \tau_k = \text{id}_n$

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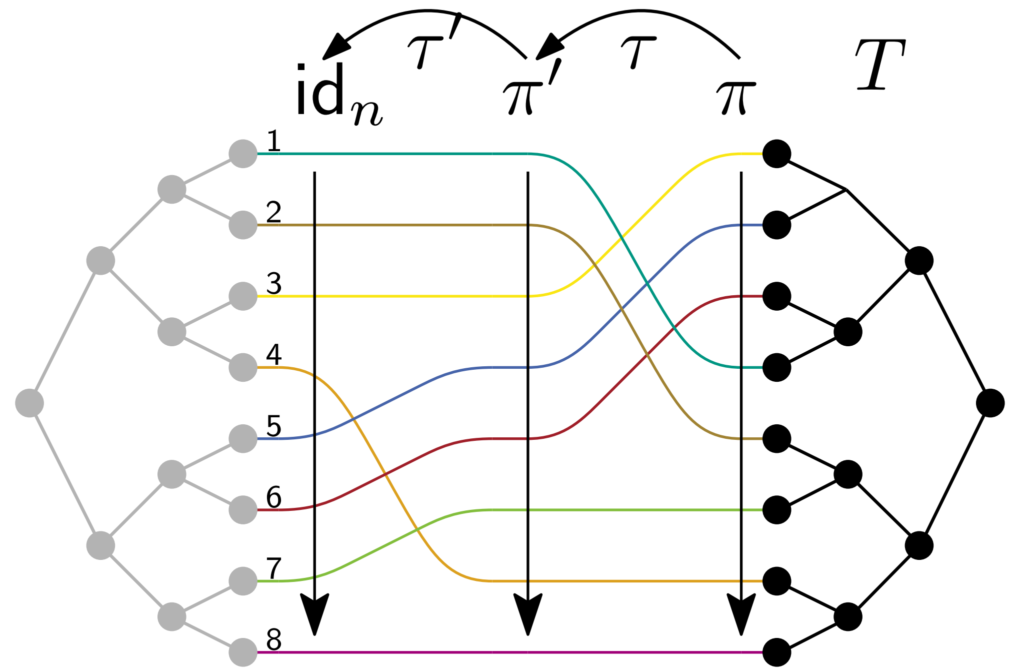

One-Tree Block Crossing Minimization (OTBCM)

Instance: A rooted tree T with leaves$(T) = [n]$ and a positive integer k.

Question: Is there a permutation $\pi \in \Pi_n$ consistent with T such that there exist transpositions $\tau_1, \ldots, \tau_k$ with $\pi \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k = \mathsf{id}_n$?

Results

Complexity results for One-Tree Block Crossing Minimization:

Restr. on T	Block Crossing Min.	Crossing Min.
Complete Binary	**NP-complete** $\mathcal{O}(n^2)$ **2.25-approximation** FPT-algorithm in k	P [Dwyer and Schreiber 2004]
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NP-hardness:

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NP-hardness:
Index $i \in \{0, 1, \ldots, n\}$ is a **breakpoint** in $\pi \in \Pi_n$ if

- $i = 0$ and $\pi_1 \neq 1$,
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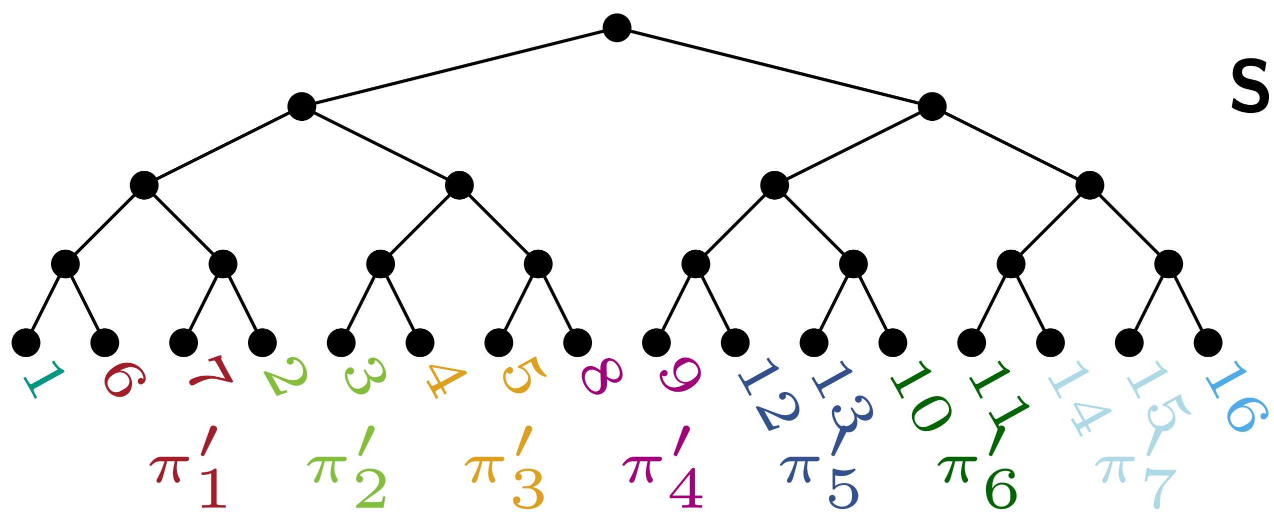

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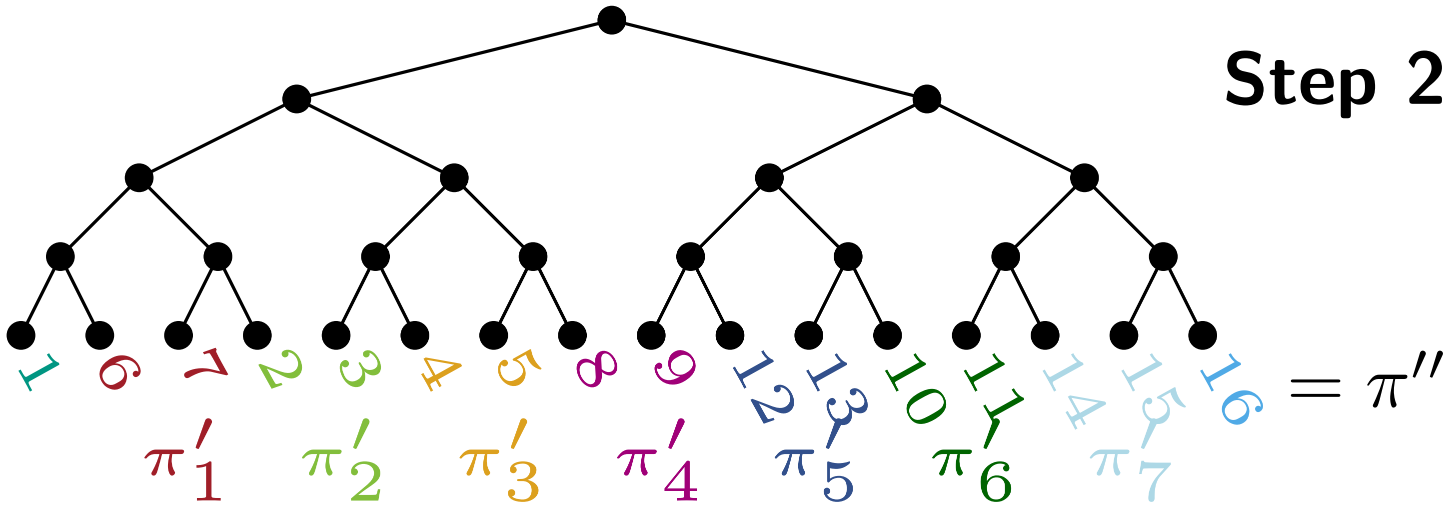

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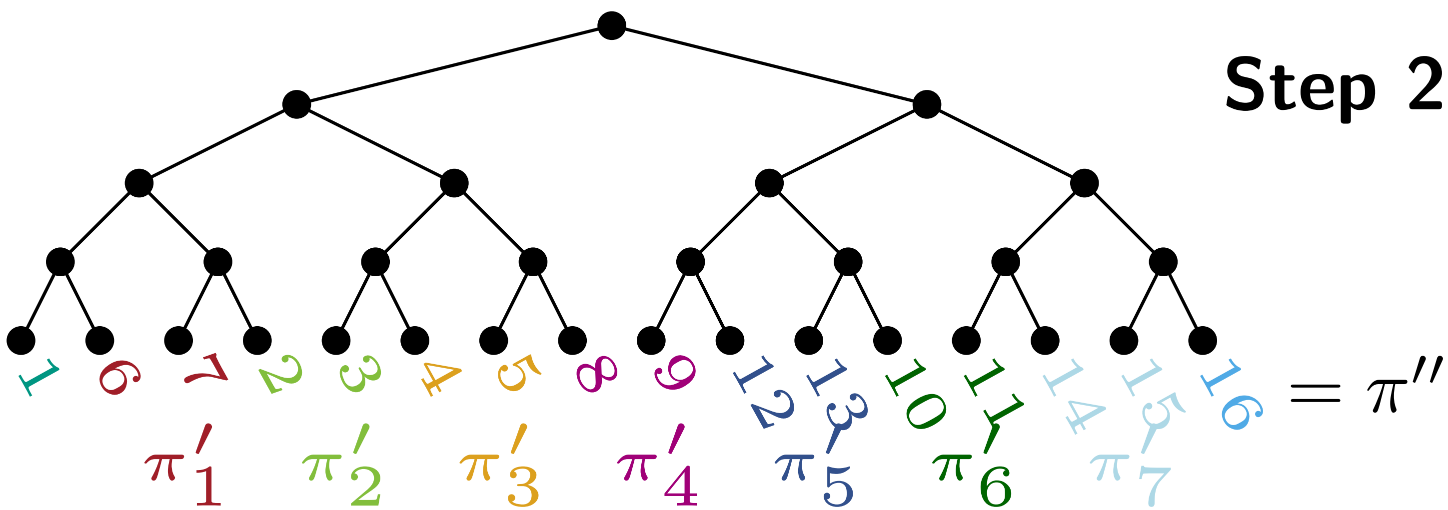

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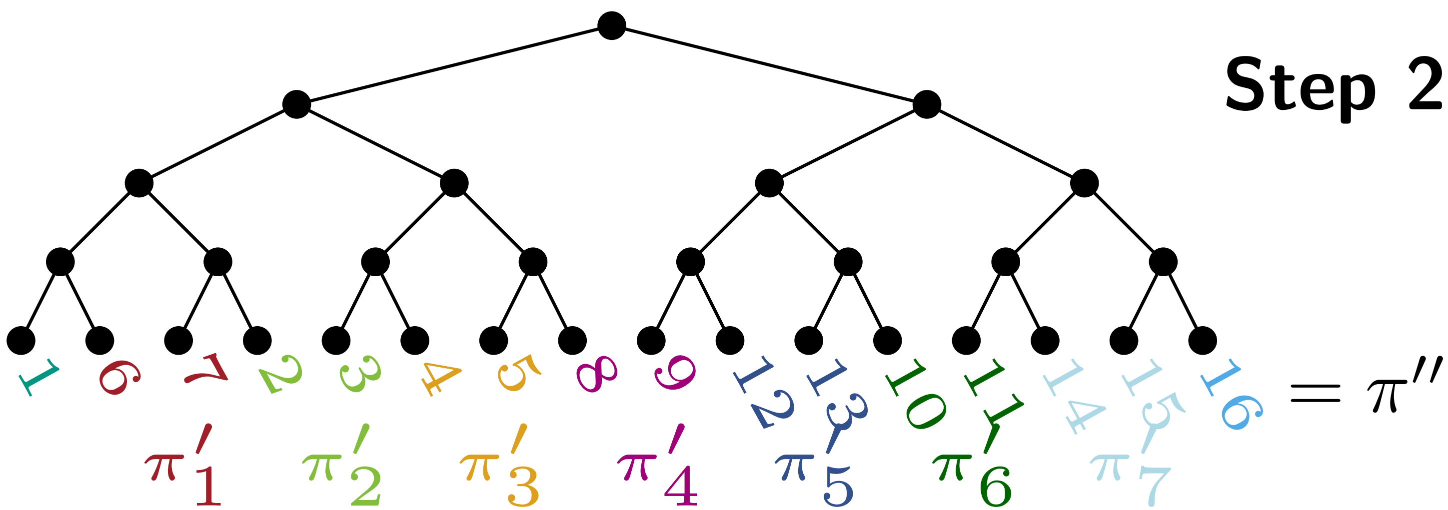

Lemma. $d_t(\pi'') = d_t(\pi)$, $\text{bp}(\pi'') = \text{bp}(\pi)$.

Lemma. π'' is the only permutation consistent with T that has $\text{bp}(\pi'') \leq \text{bp}(\pi)$.

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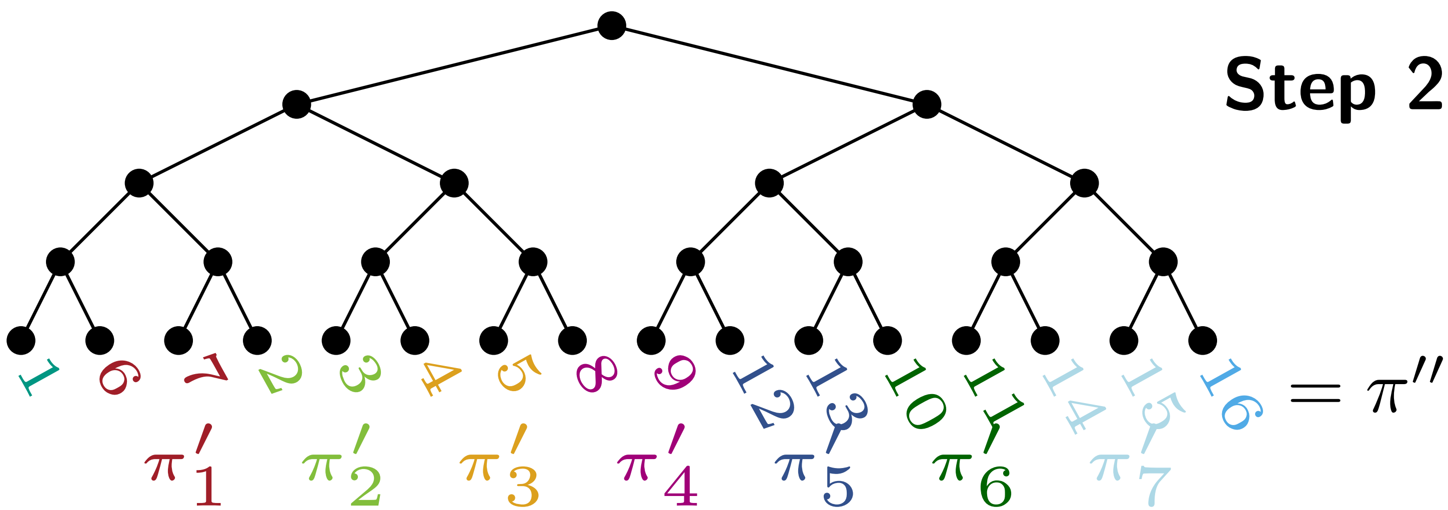

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$\Rightarrow d_t(\pi) = \lfloor \frac{\mathsf{bp}(\pi)}{3} \rfloor$ iff. (T, k) is a yes instance for OTBCM with $k = \lfloor \frac{\mathsf{bp}(\pi)}{3} \rfloor$. $\square$

FPT-Algorithm

Instance: A rooted tree T with leaves$(T) = [n]$ and a positive integer k.

Question: Is there a permutation $\pi \in \Pi_n$ consistent T such that there exist transpositions $\tau_1, \ldots, \tau_k$ with $\pi \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k = \mathrm{id}_n$?

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Theorem. For binary T, we can find π and $\tau_1, \ldots, \tau_k$ in time $f(k) \cdot n^c$ if they exist, and report NO otherwise in the same time.

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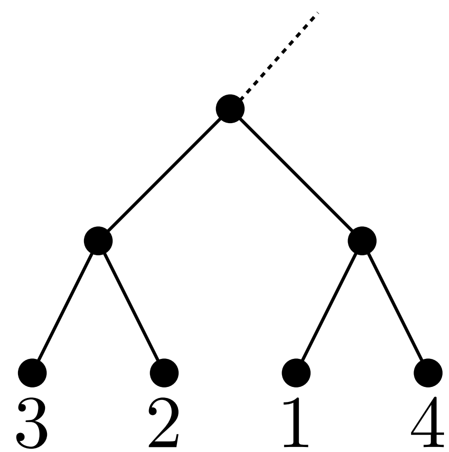
Ingredients:

- Two reduction rules
- Search tree algorithm fixing order of children of an inner node
- Leaf ordering π fixed by search tree $\rightarrow$ FPT-algorithm for finding $\tau_1, \ldots, \tau_k$

FPT-Algorithm

Reduction Rule 1. (Preprocessing)



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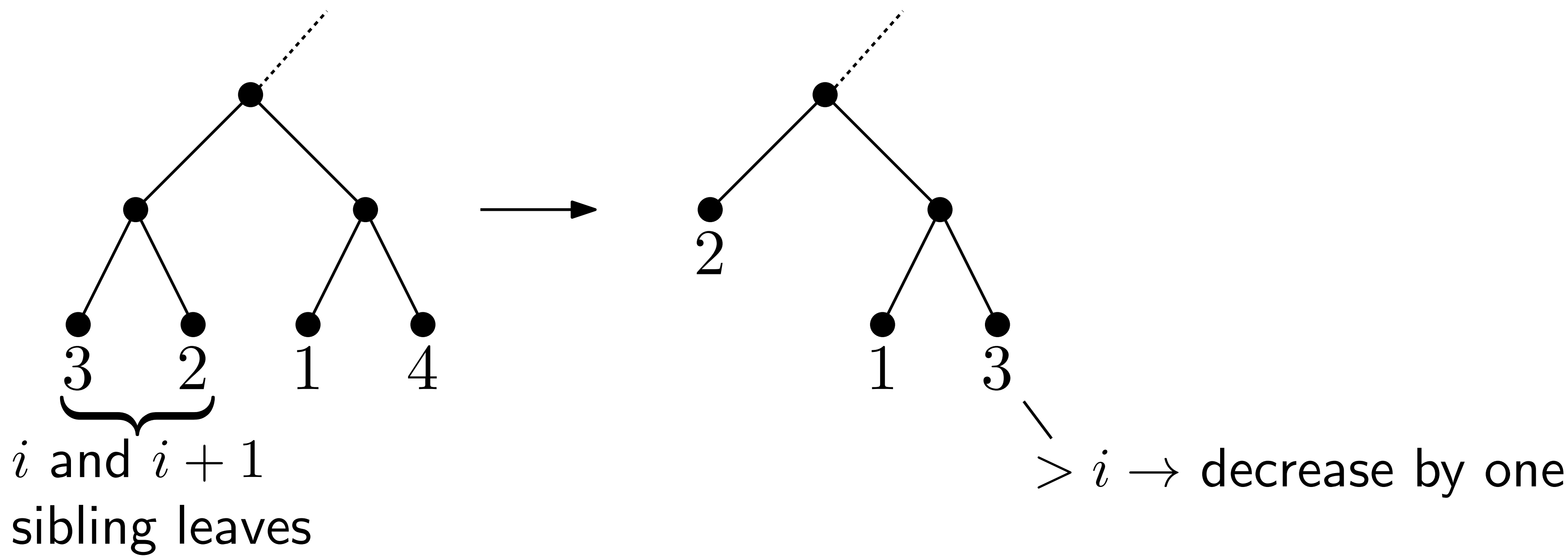
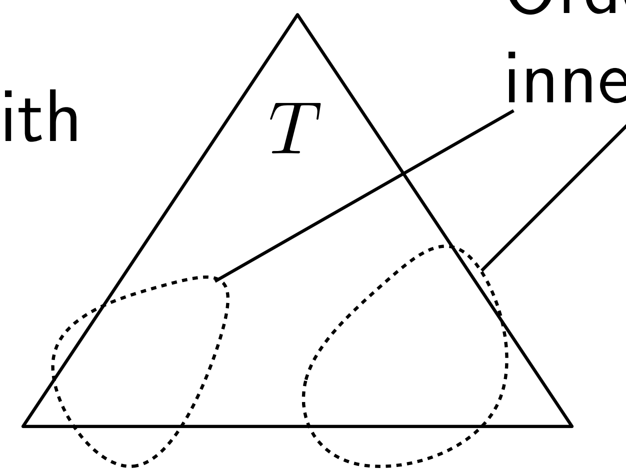


FPT-Algorithm

Search Tree.

Problem assoc. with search tree node:



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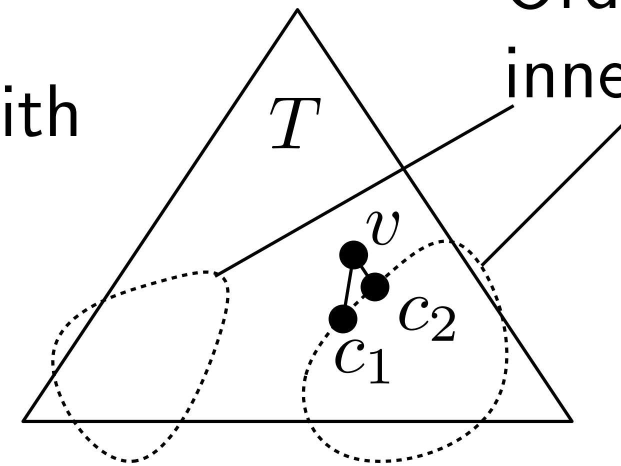

Ordering of children of inner nodes fixed

FPT-Algorithm

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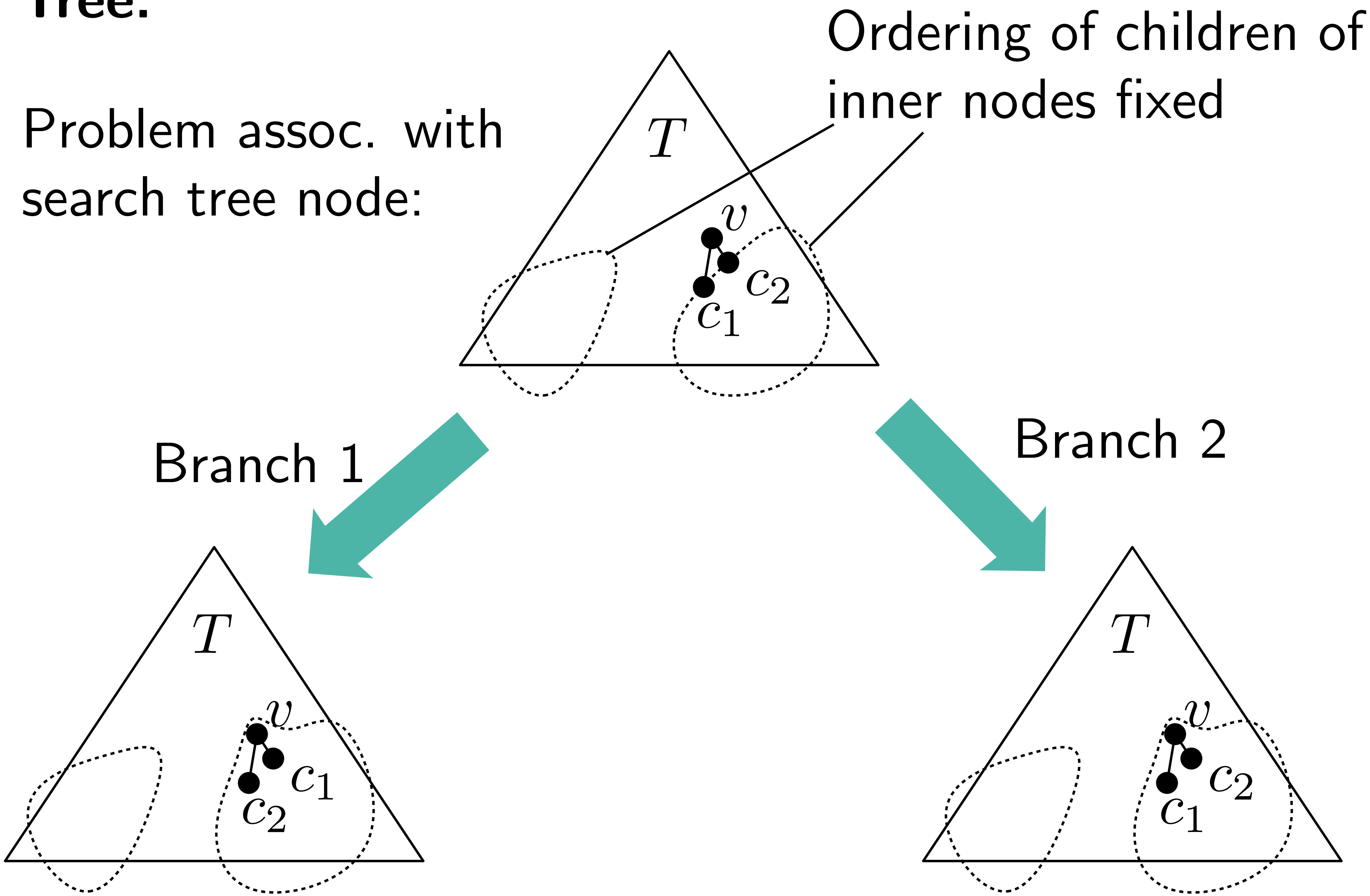

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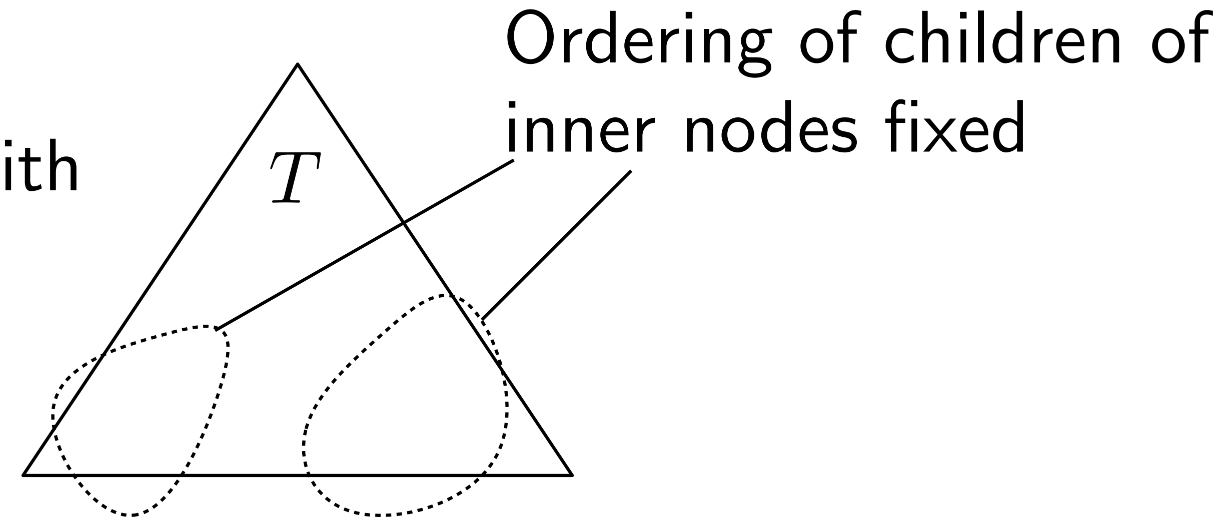

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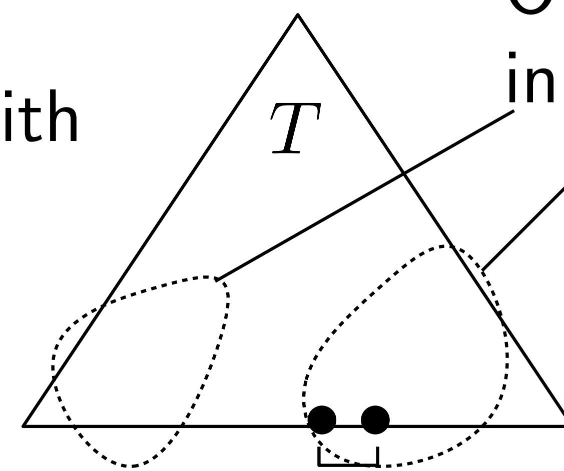
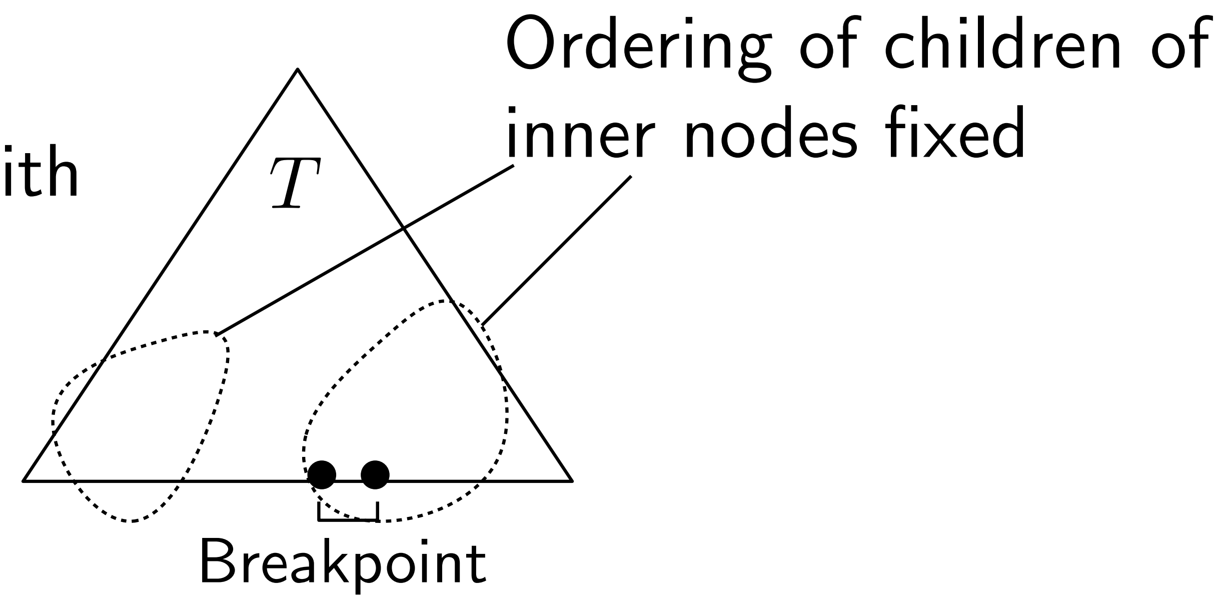


Breakpoint

FPT-Algorithm

Problem assoc. with search tree node:



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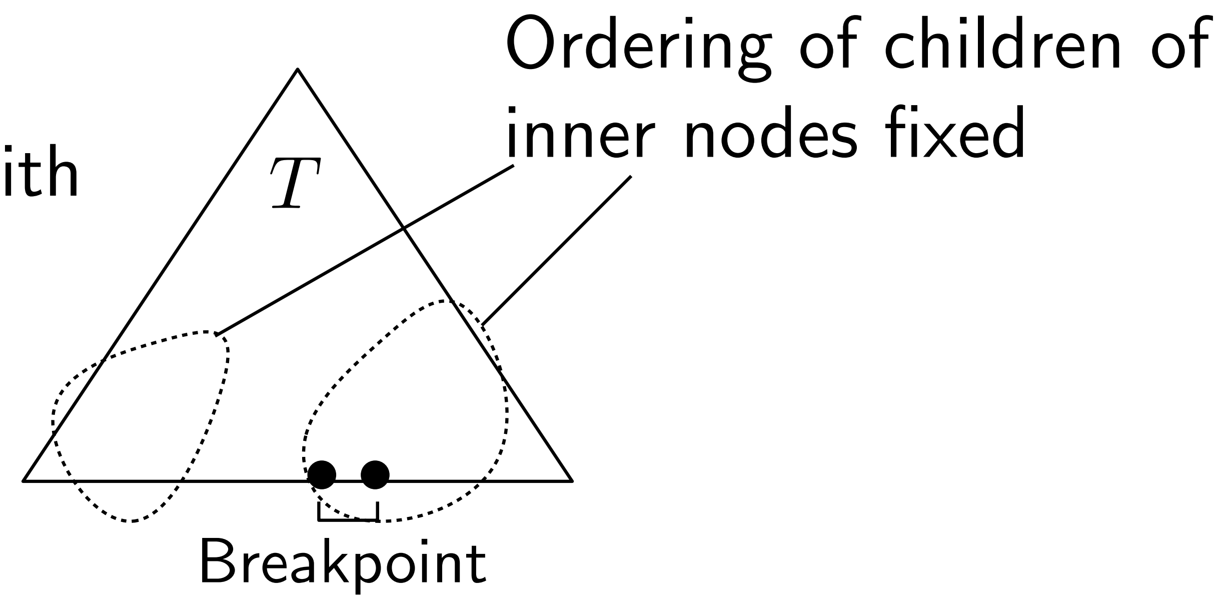

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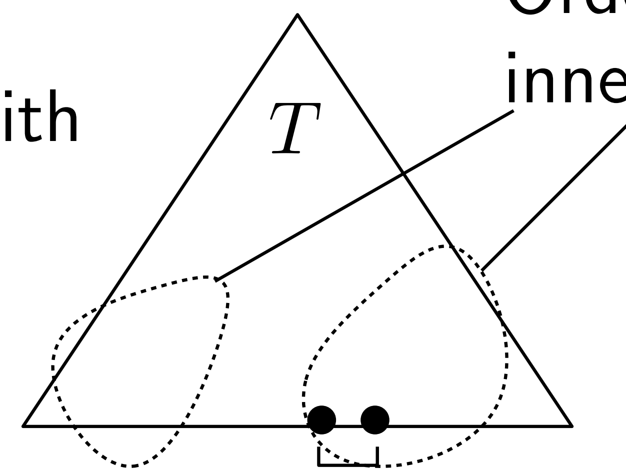

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Not enough for bounding runtime $\rightarrow$ **reduction rule 2**
Similar to reduction rule 1, fixes order of children of an inner node

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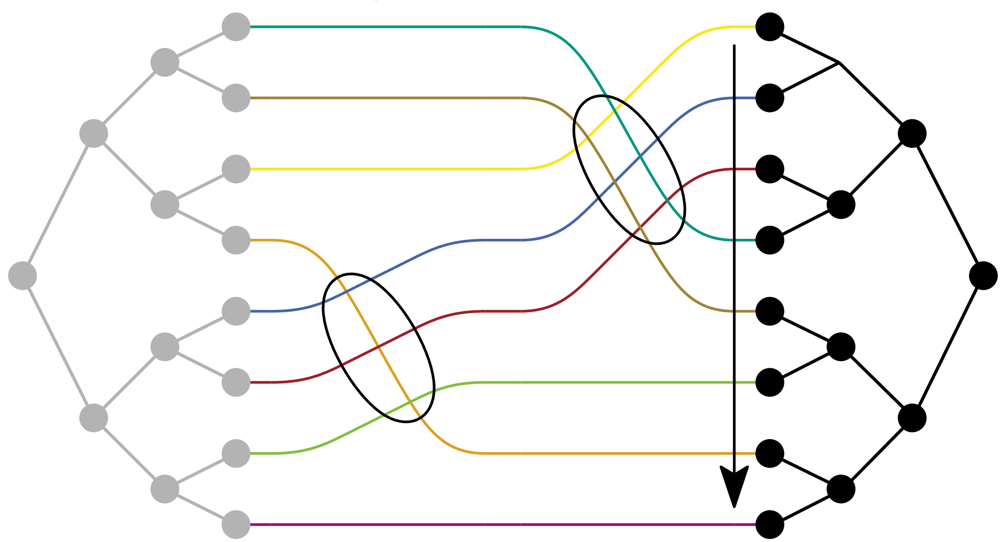
Restr. on T	Block Crossing Min.	Crossing Min.
Complete Binary	**NP-complete** $\mathcal{O}(n^2)$ **2.25-approximation** FPT-algorithm in k	P [Dwyer and Schreiber 2004]
Binary	NP-complete $\mathcal{O}(n^3)$ **2.25-approximation** **FPT-algorithm in k**	P [Dwyer and Schreiber 2004]
Non-binary	NP-complete **Approx. does not extend**	NP-complete [Bulteau et al. 2022]



Open problems:

- Permute leaves of both trees?
- Better approximations?
- Edges can cross multiple times: disallow this?
- Experiments