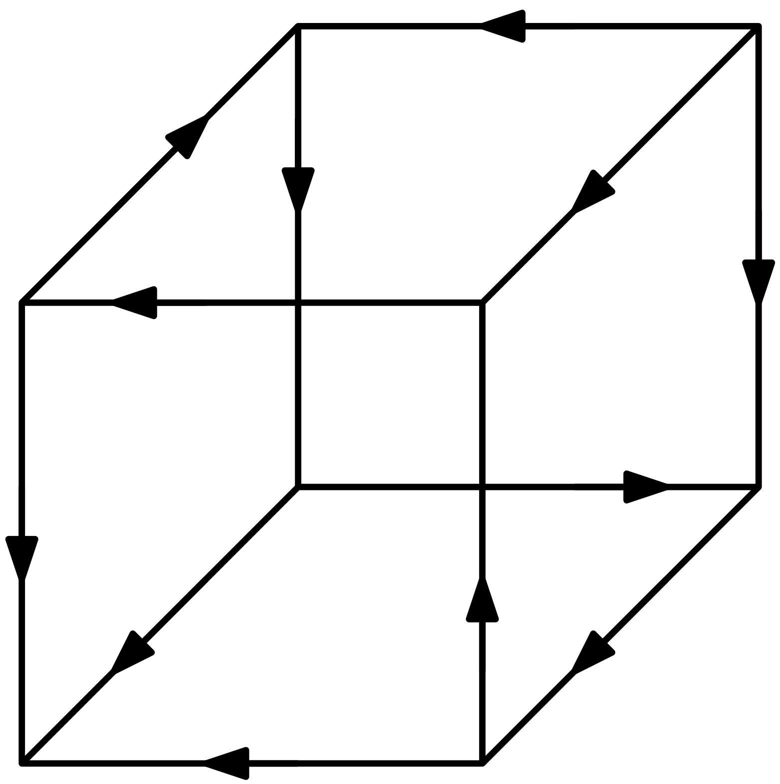
Realizability Makes a Difference: A Complexity Gap for Sink-Finding in USOs

Simon Weber and Joel Widmer

Unique Sink Orientations



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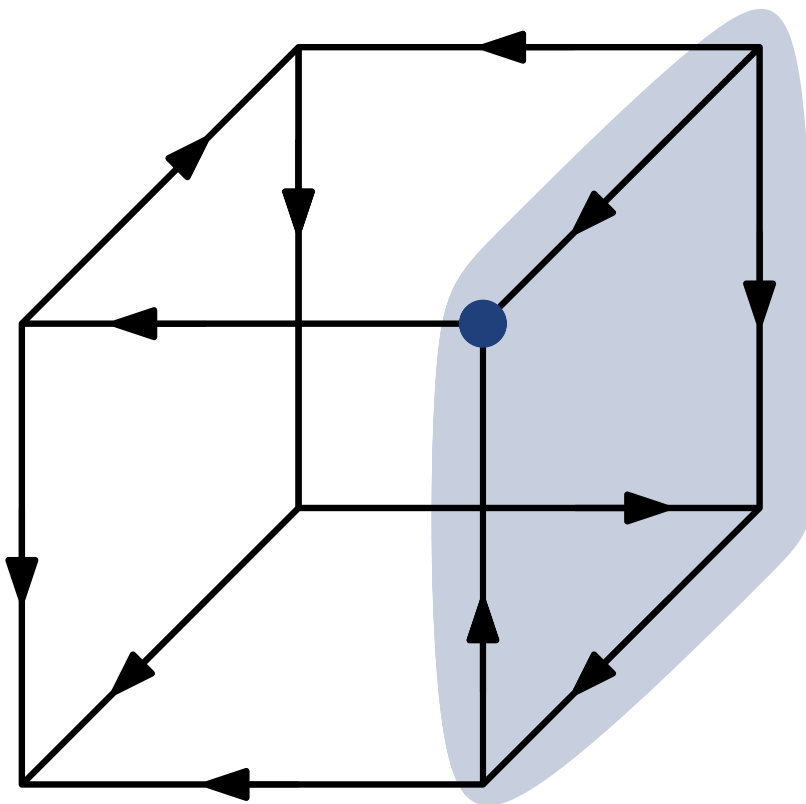

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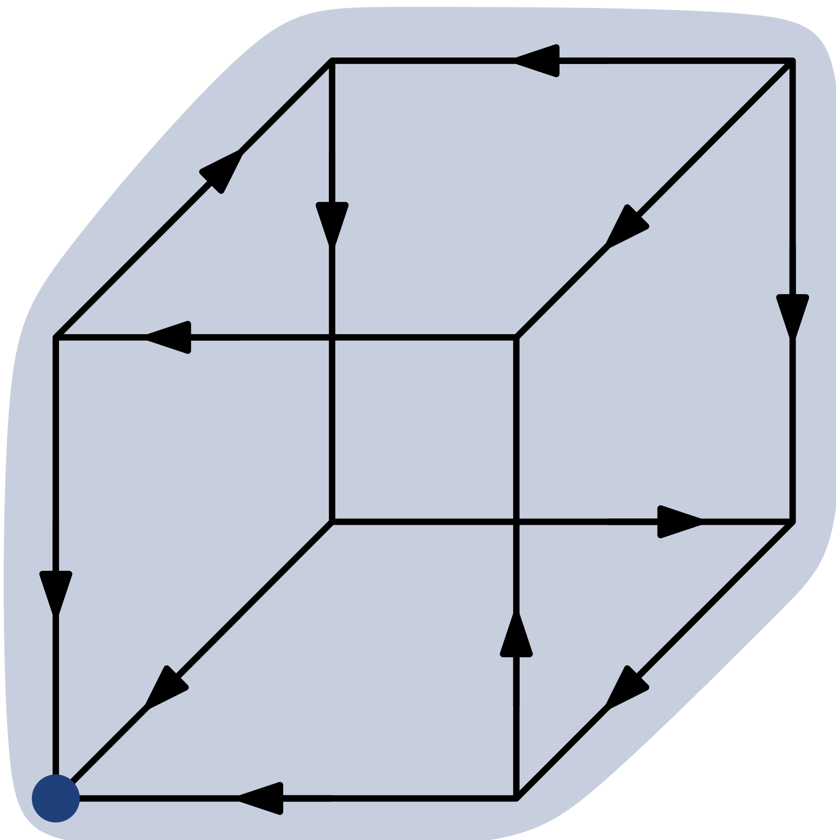

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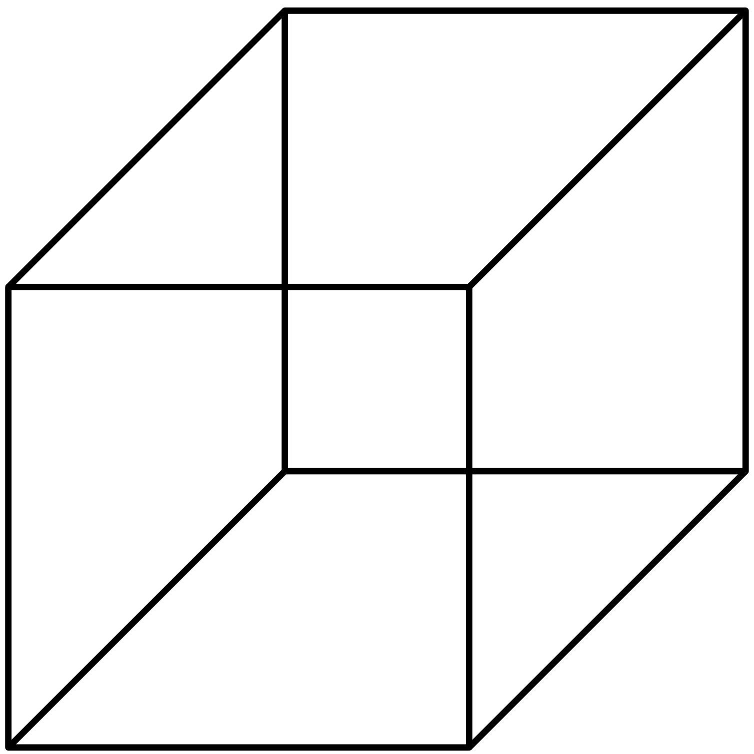

Sink-finding:

Find the global sink using *vertex evaluations*

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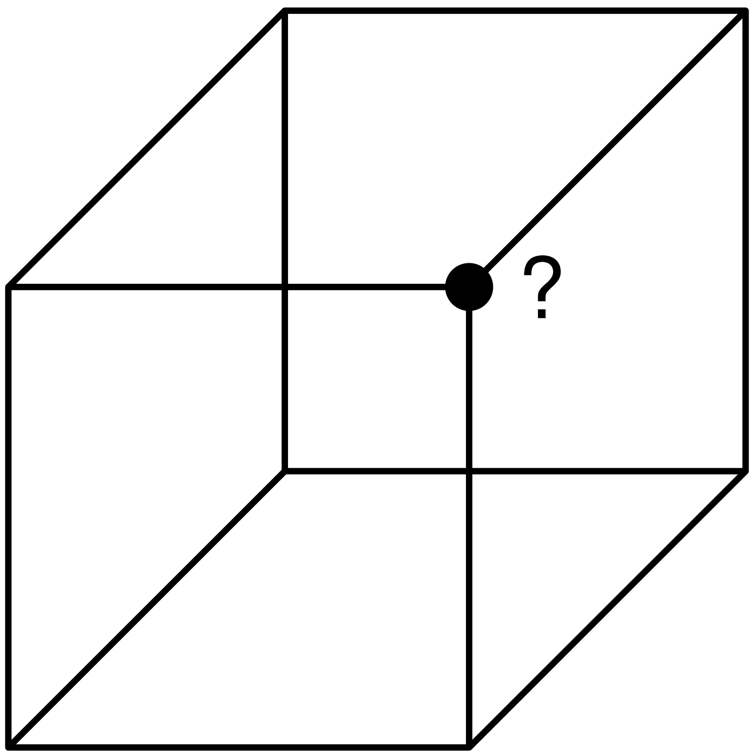

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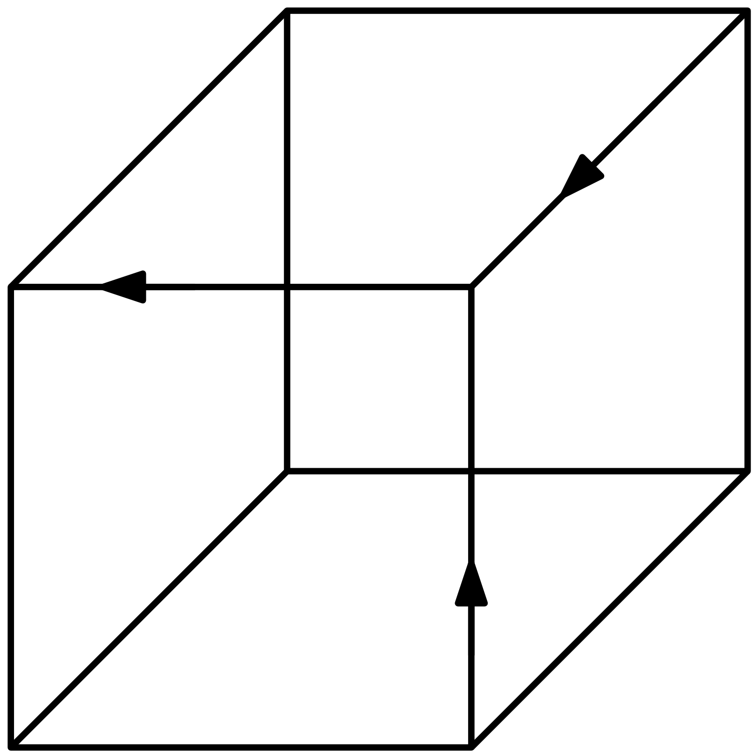

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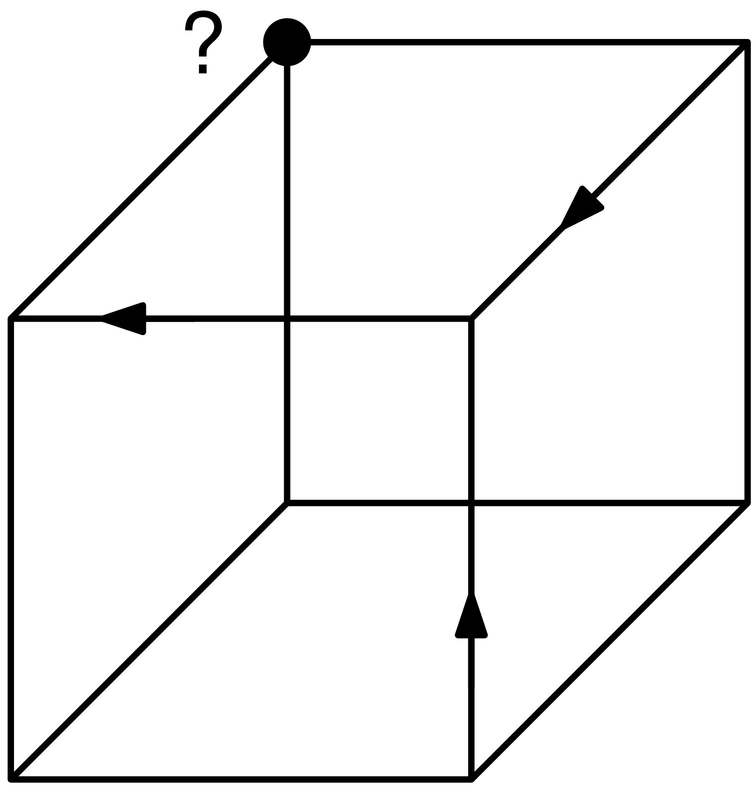

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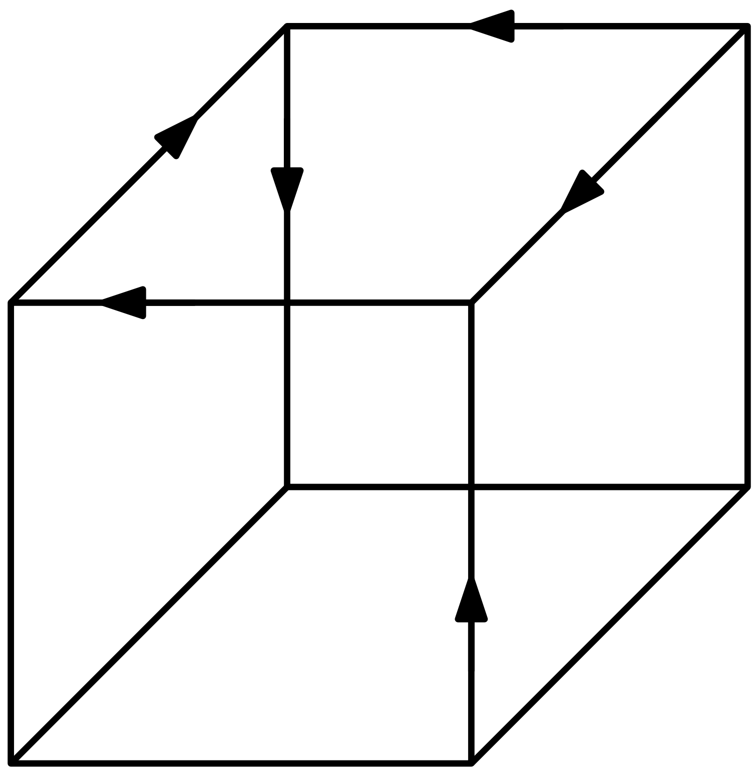

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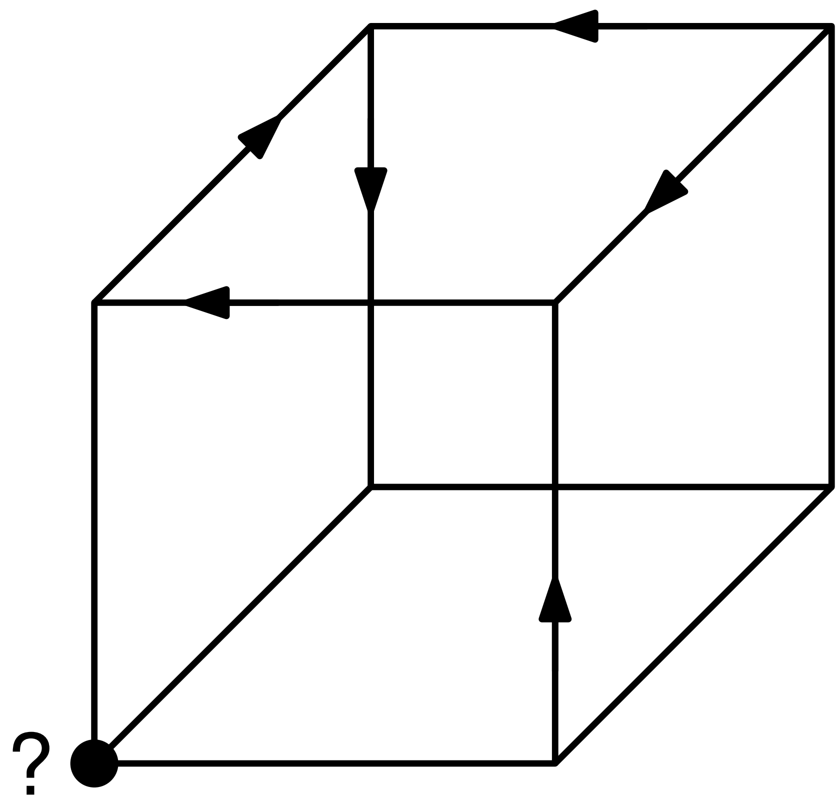

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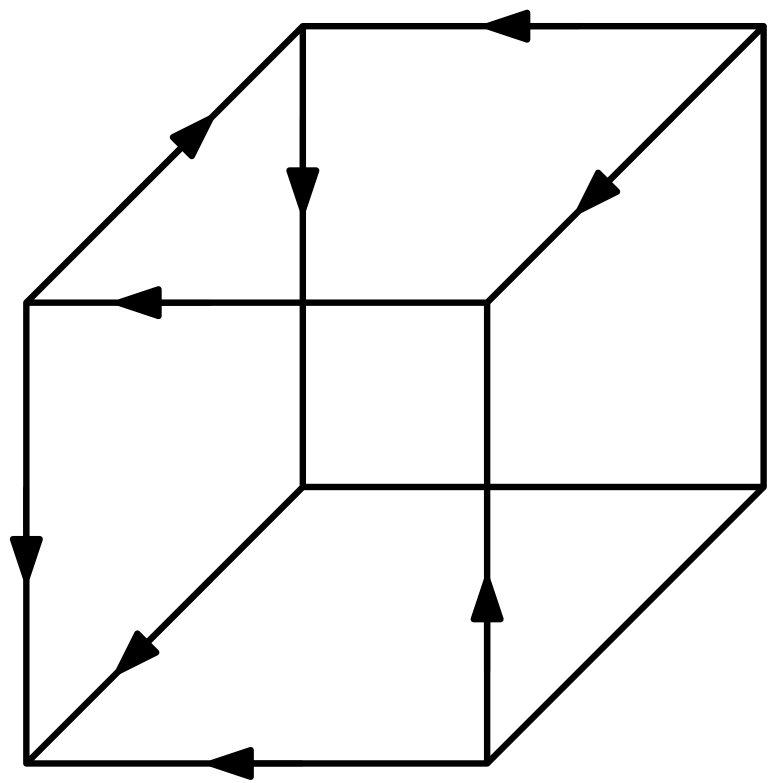

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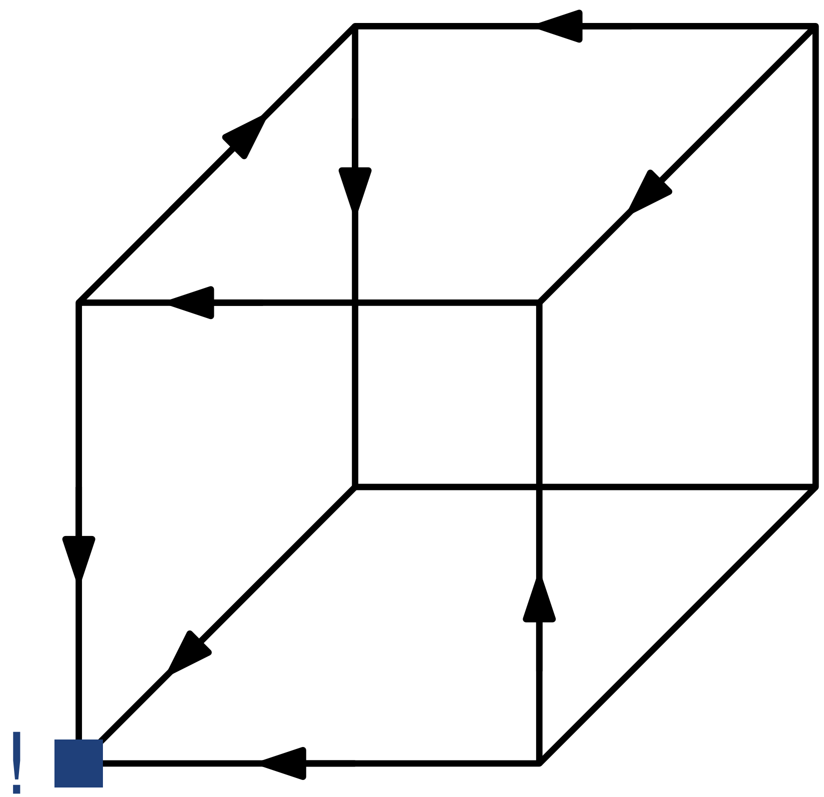

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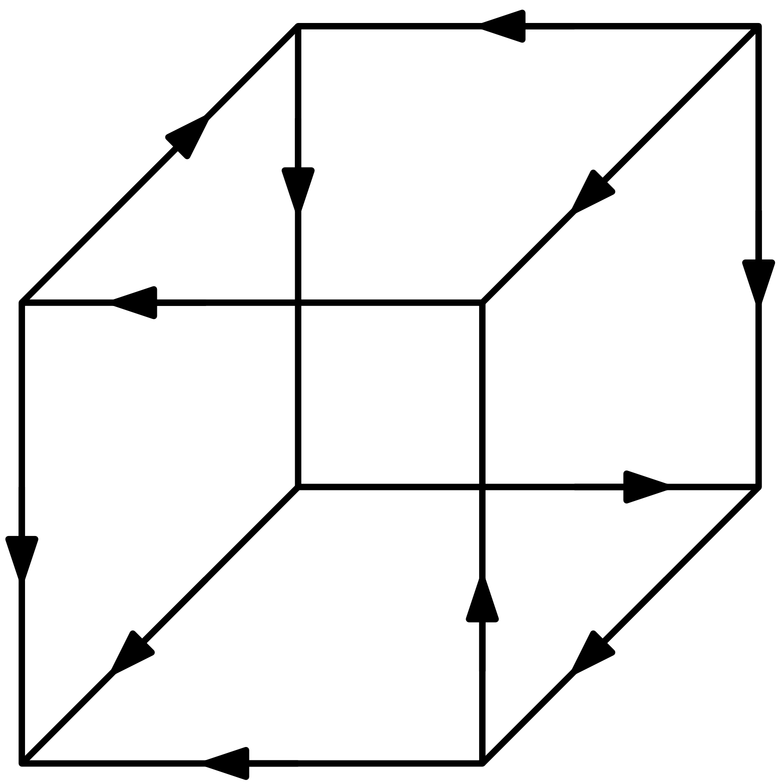

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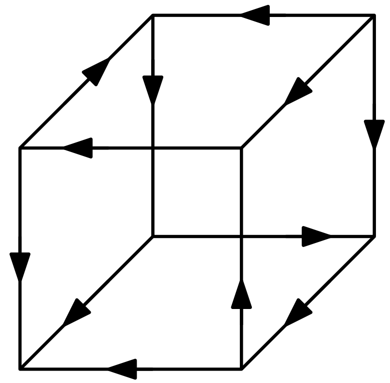
Find the global sink using *vertex evaluations*

Runtime = Number of vertex evaluations

Realizability



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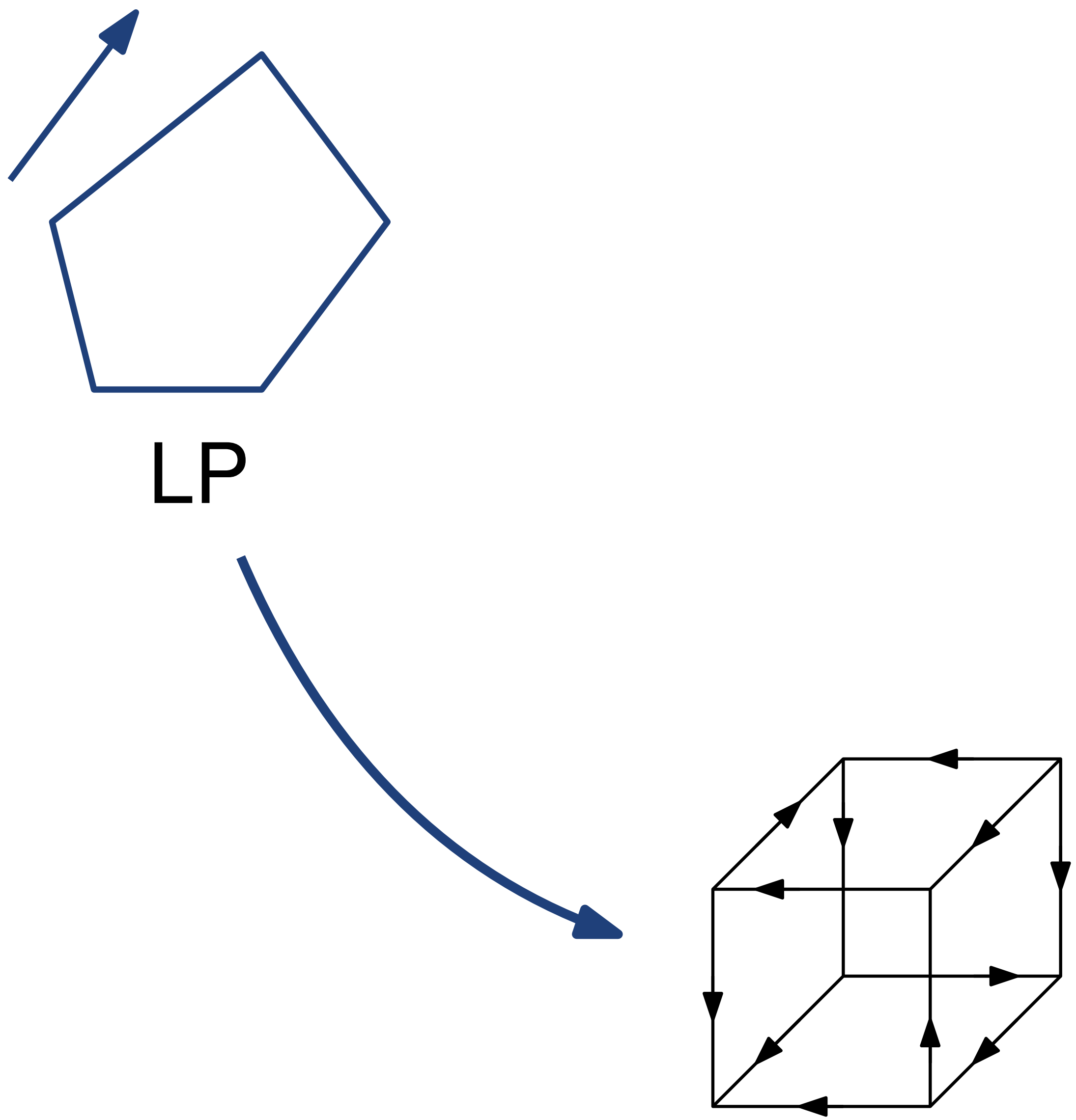

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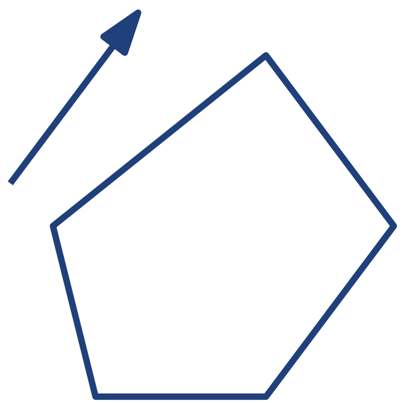
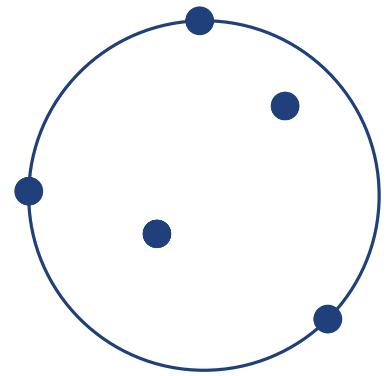
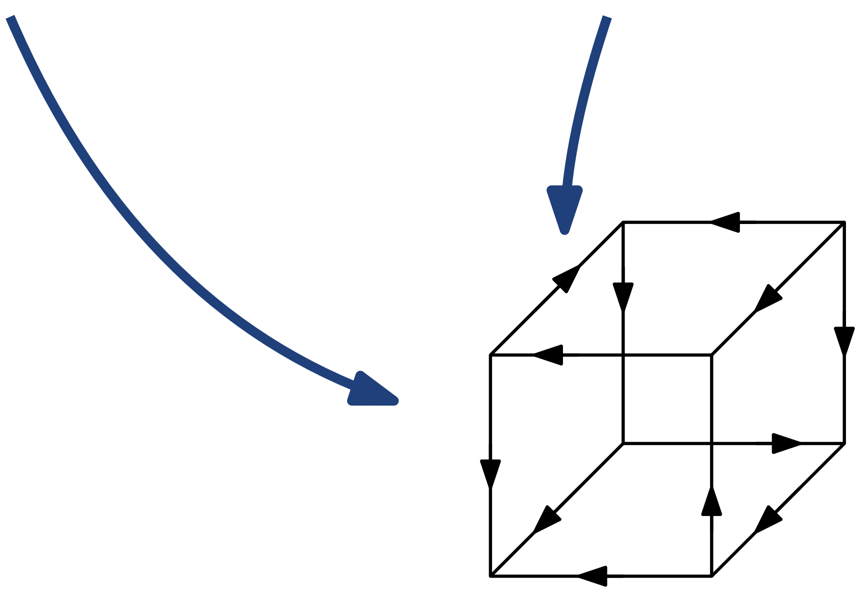




LP

Smallest Enclosing Ball



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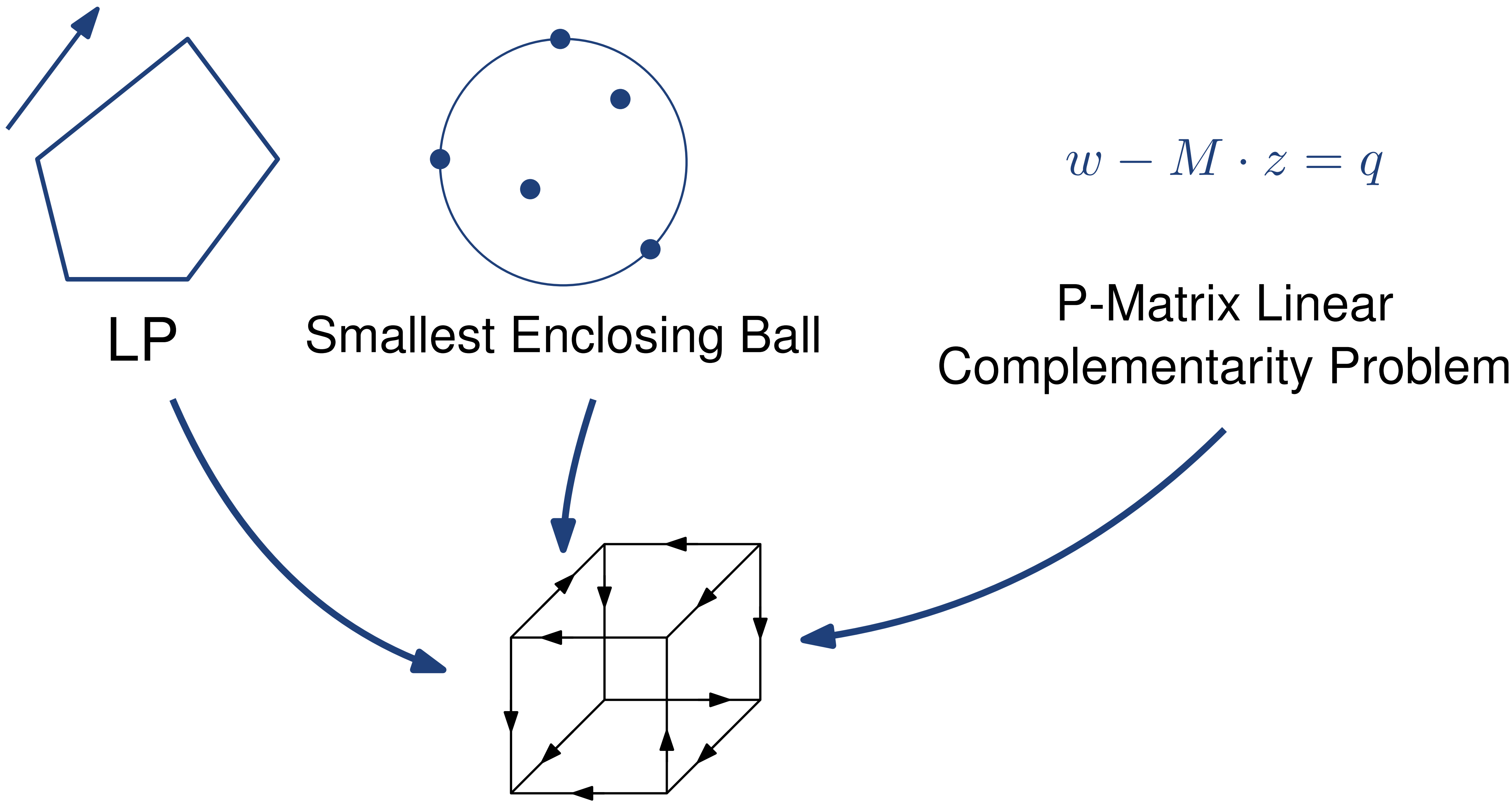

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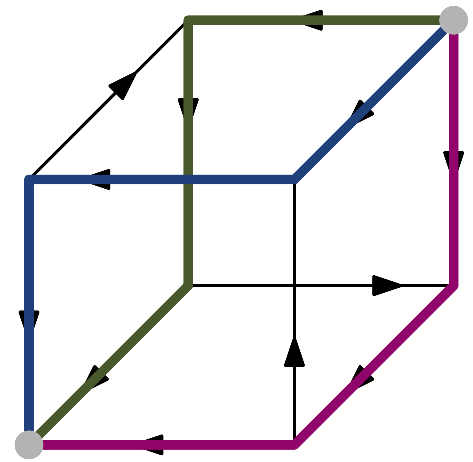

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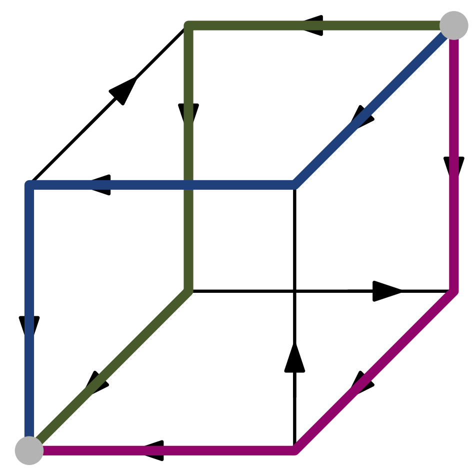

Necessary, but not sufficient!
Fulfilled by $2^{\Omega(2^n/\sqrt{n})}$ USOs

Query Complexity Lower Bounds

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The best algorithms are exponential!

Query Complexity of Random Facet

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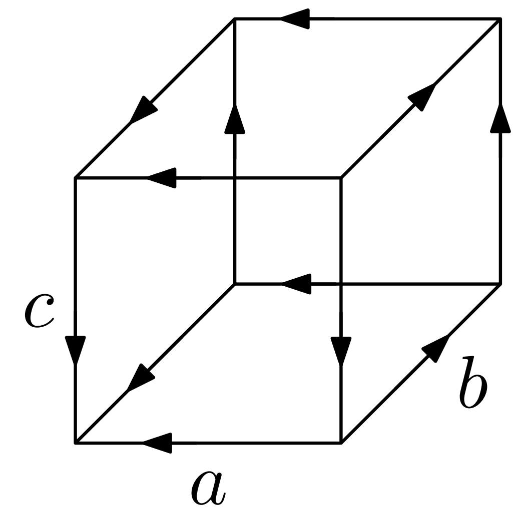

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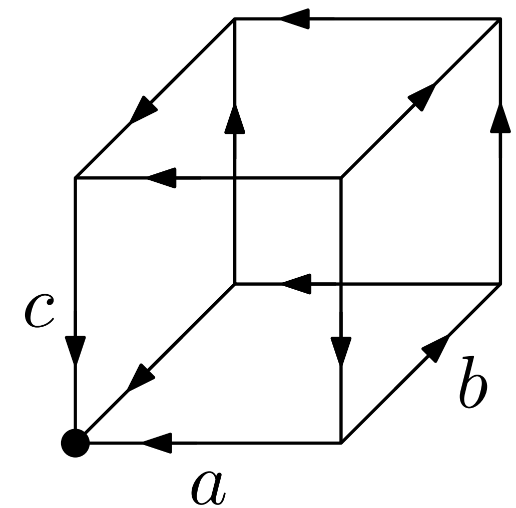

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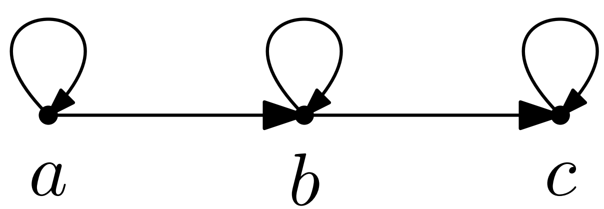
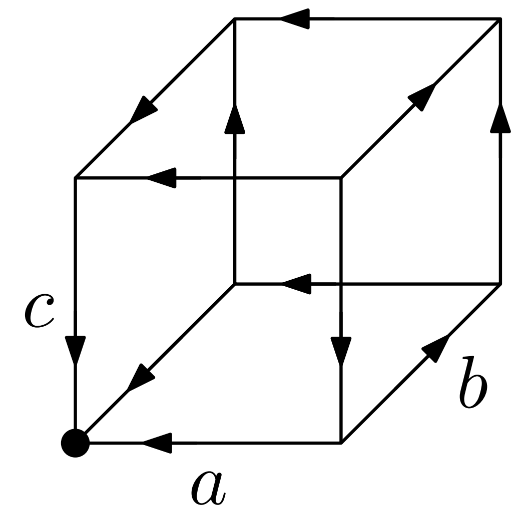


dimension influence graph



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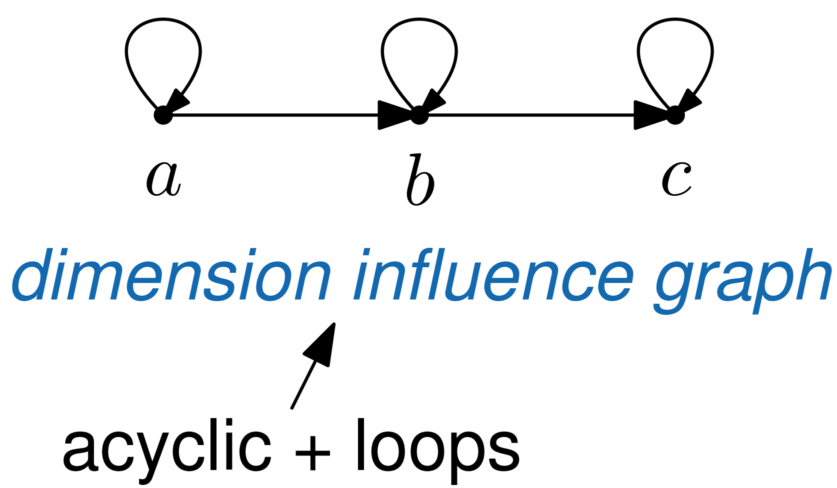

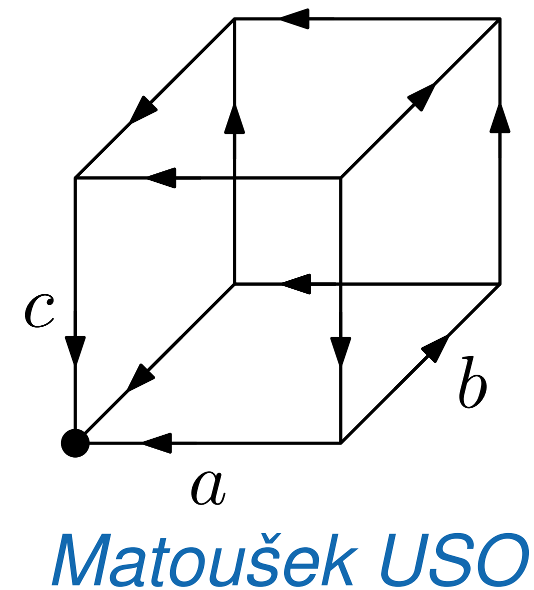

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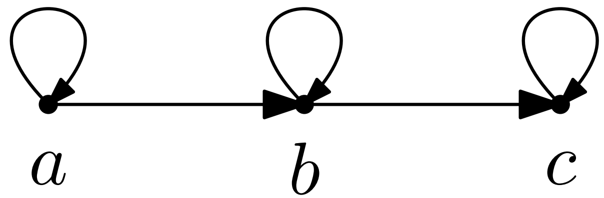
Is this specific to Random Facet or *inherent to the problem*?

Matoušek-type USOs



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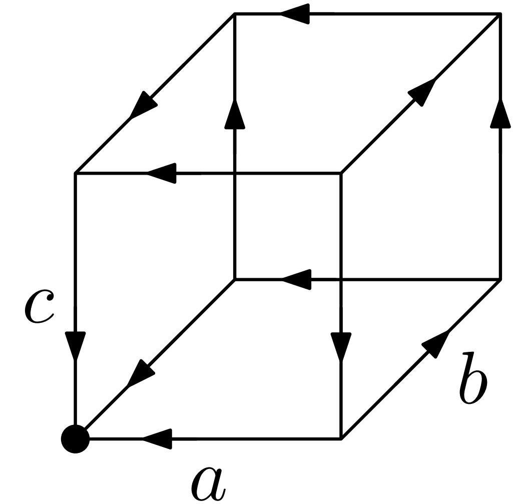

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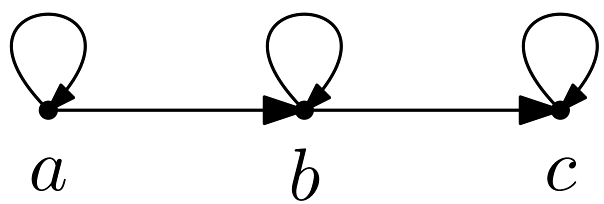

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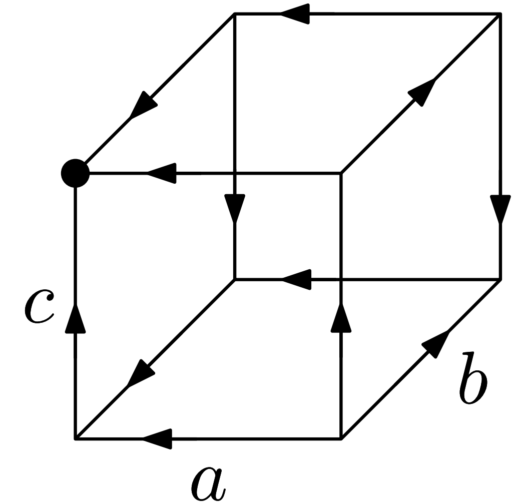

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The Realizable Matoušek-type USOs

Theorem [W., Gärtner, 2021]: A Matoušek-type USO is realizable if and only if its dimension influence graph is the *reflexive transitive closure of an arborescence*.

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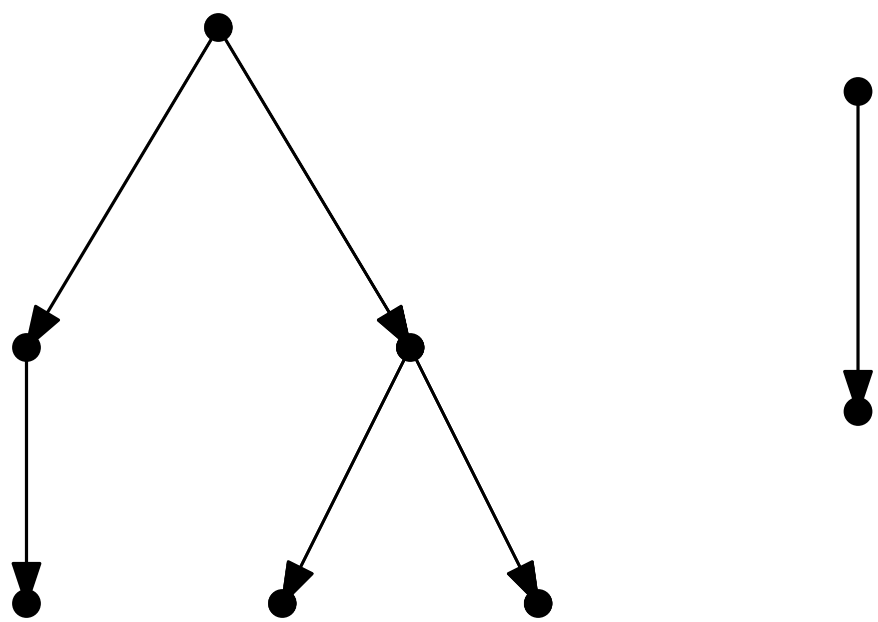

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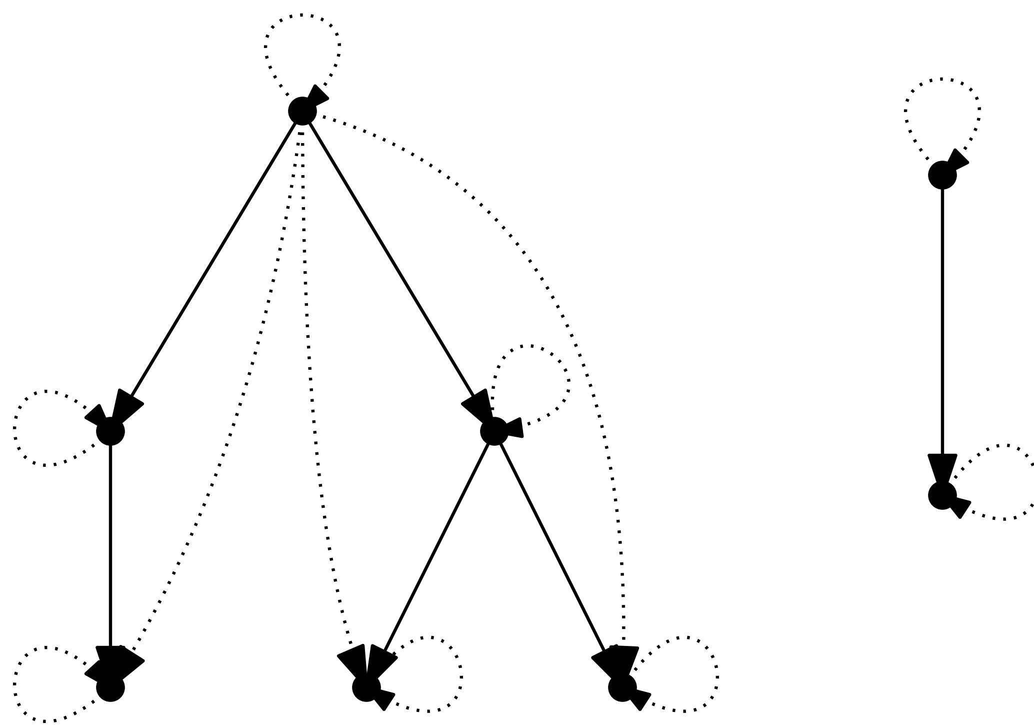
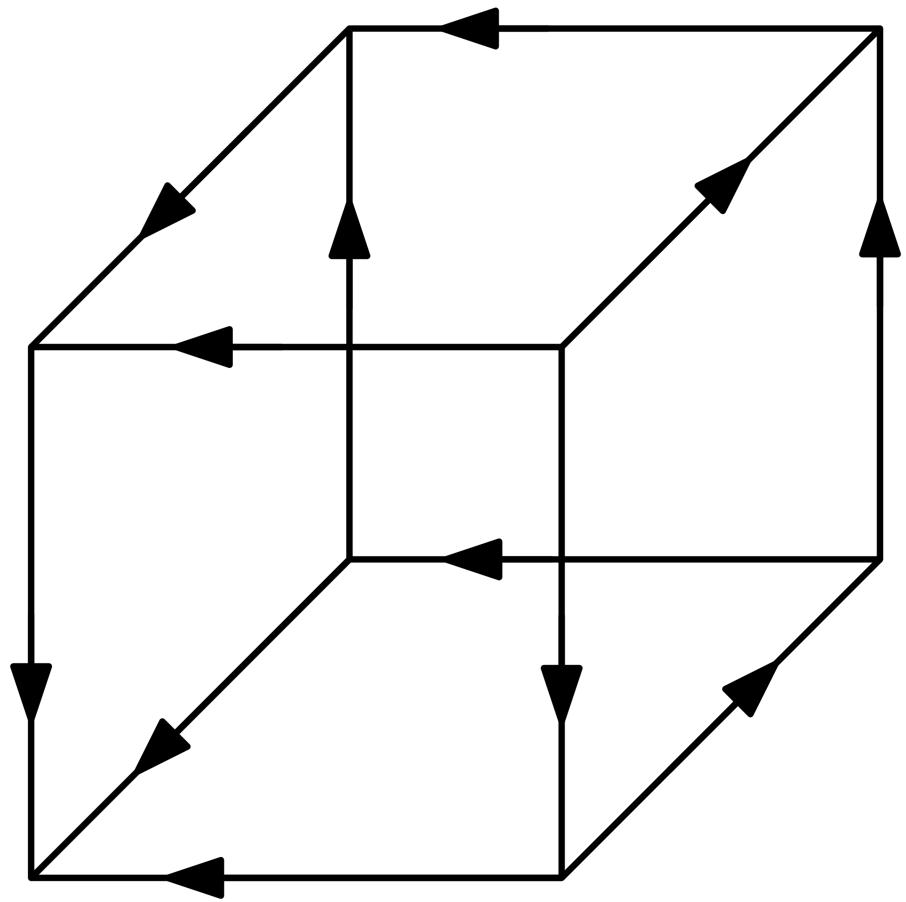


General Matoušek-type USOs: Upper Bound



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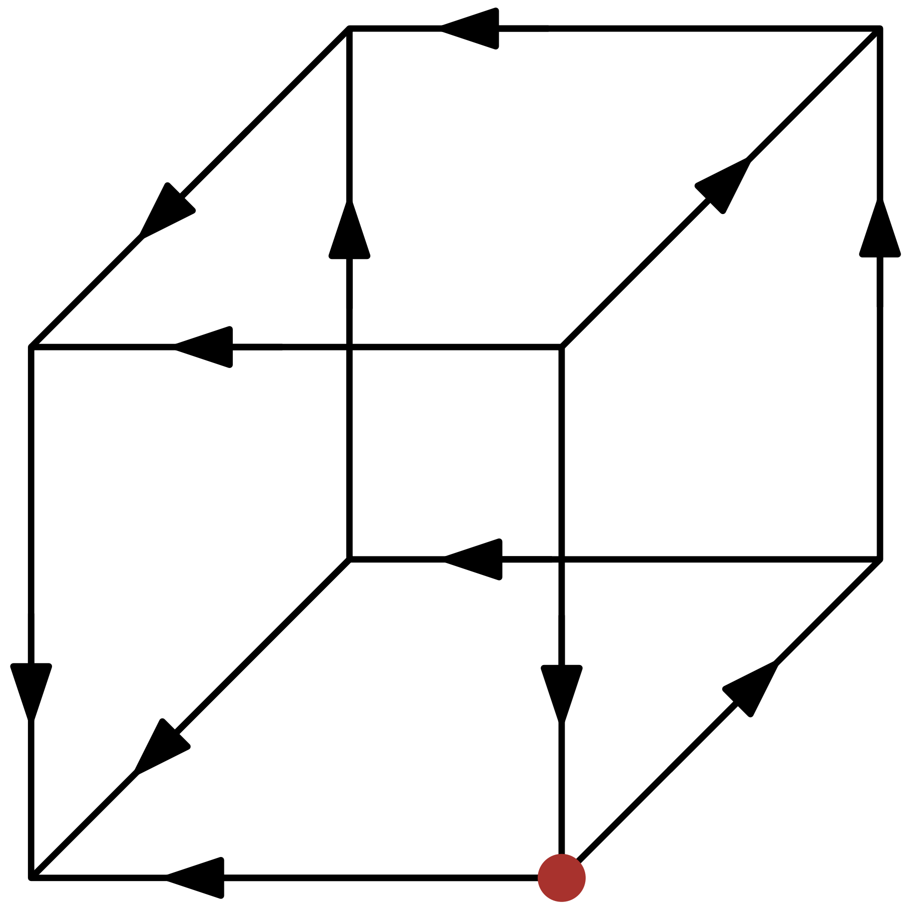

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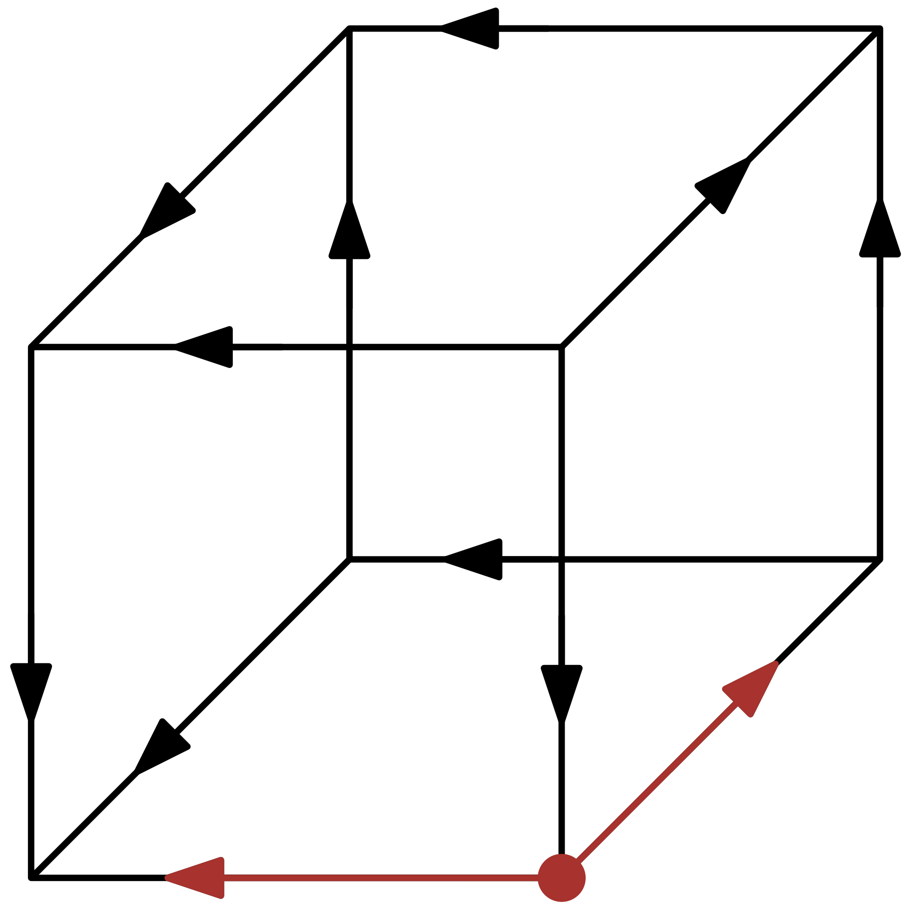

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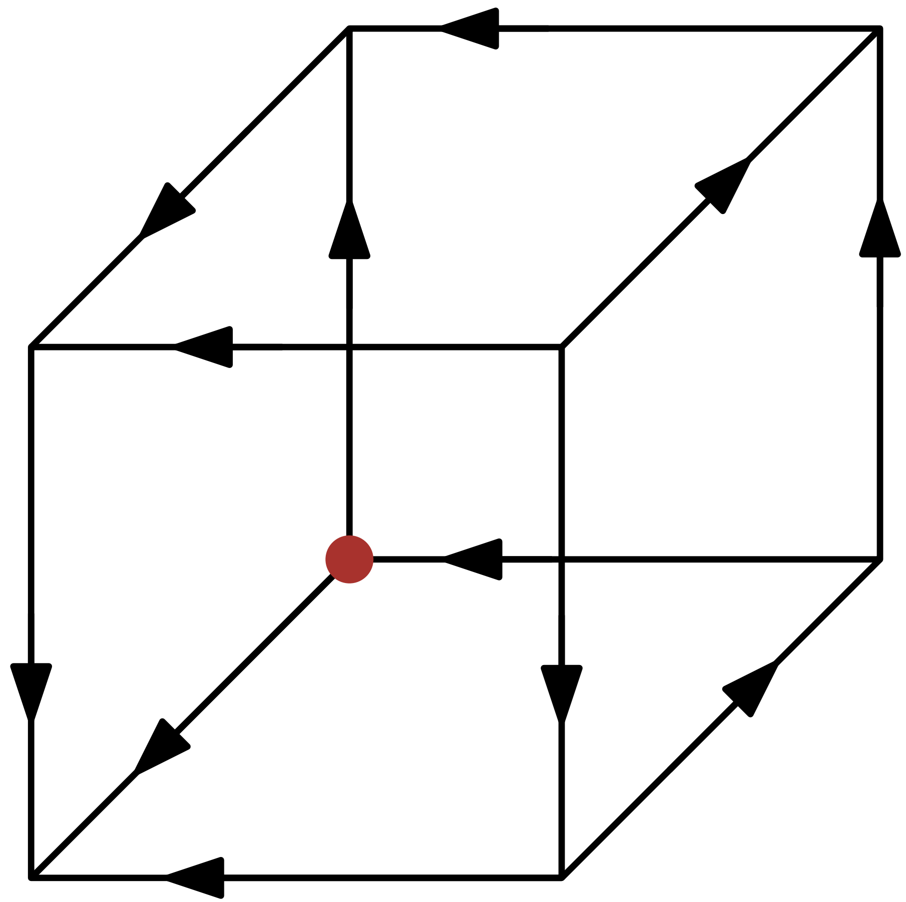

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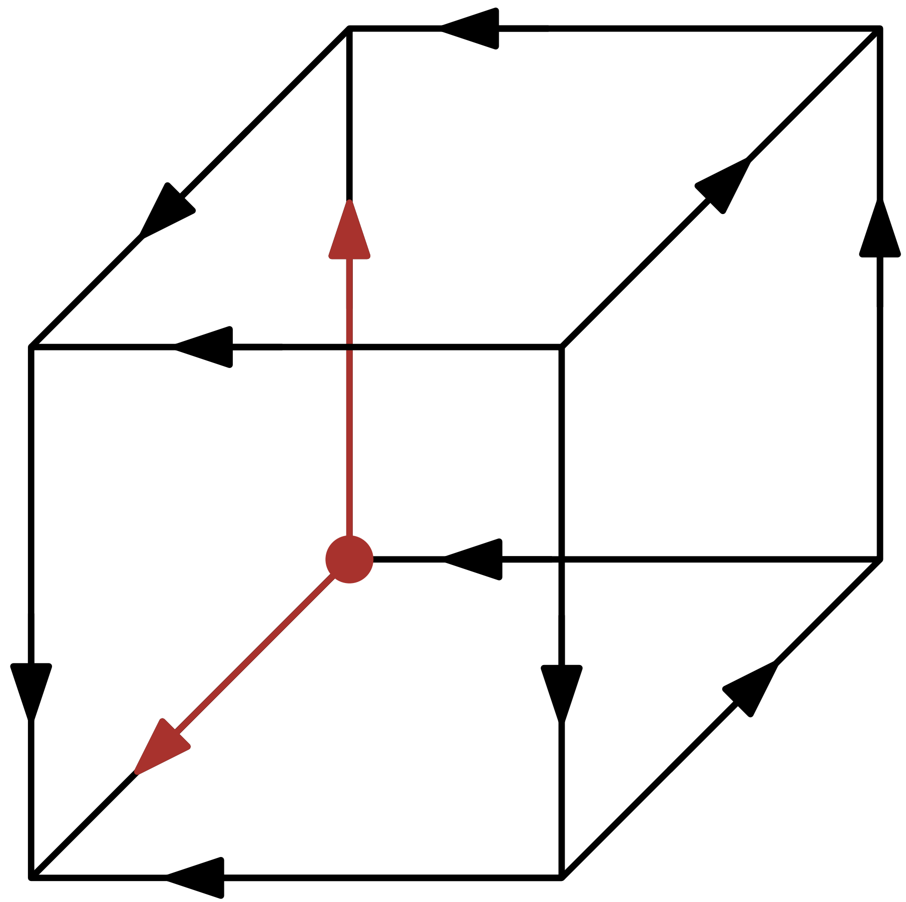

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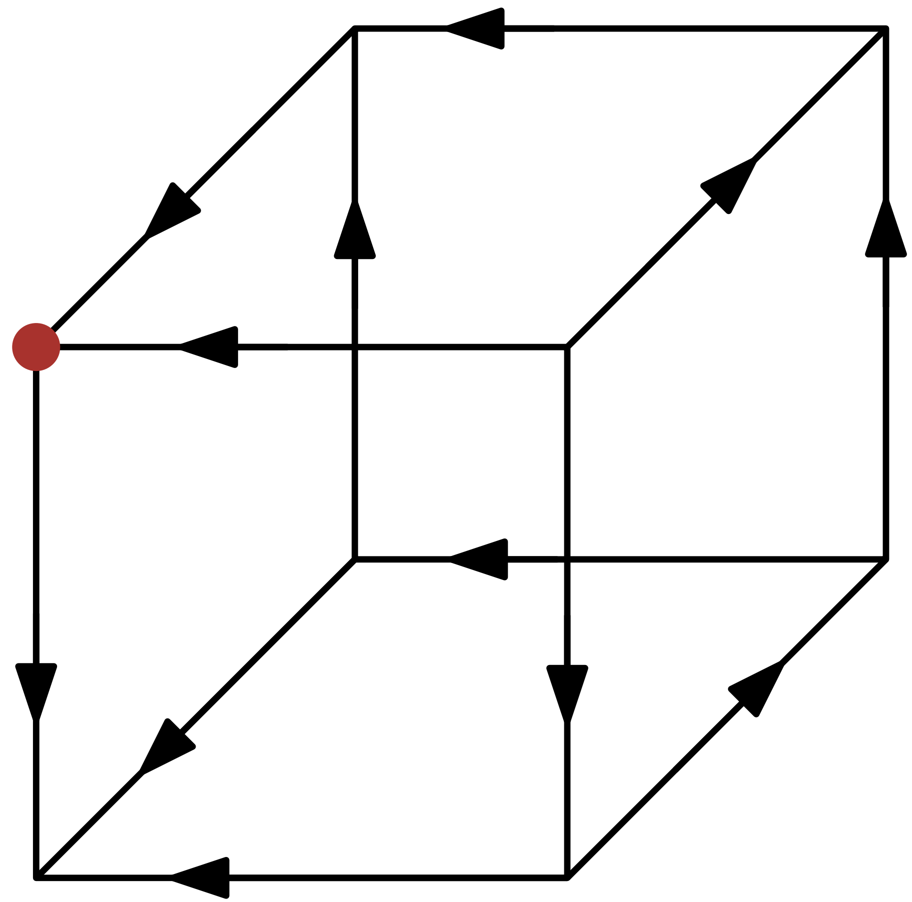

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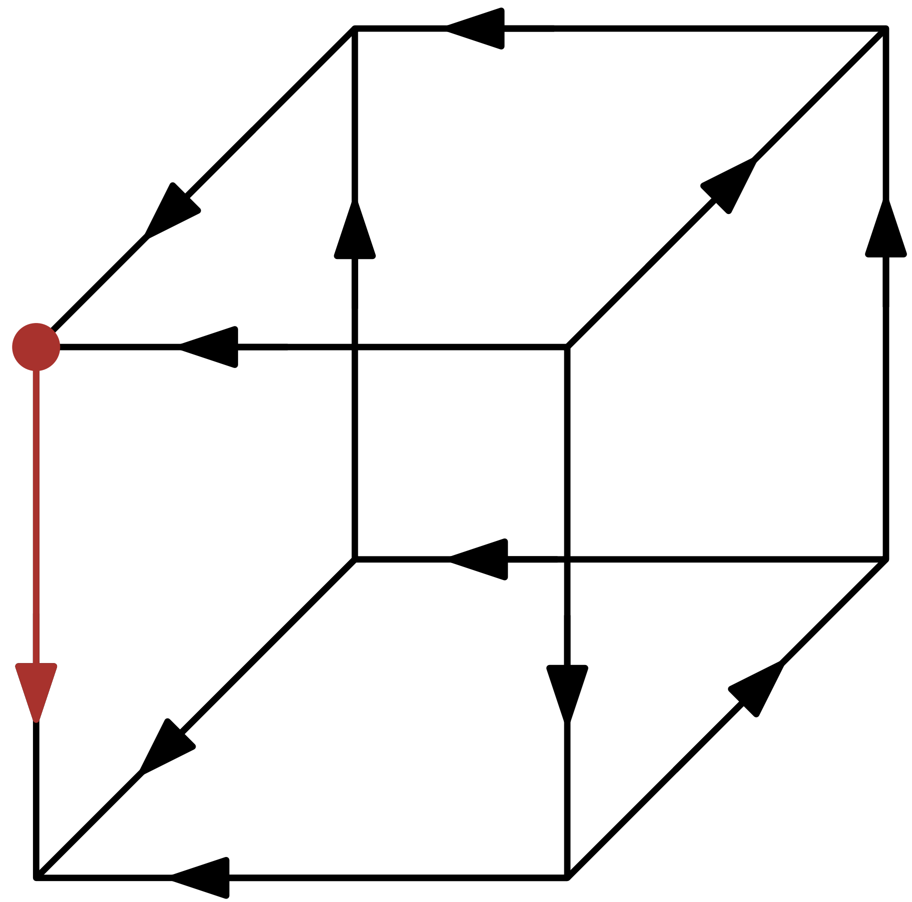

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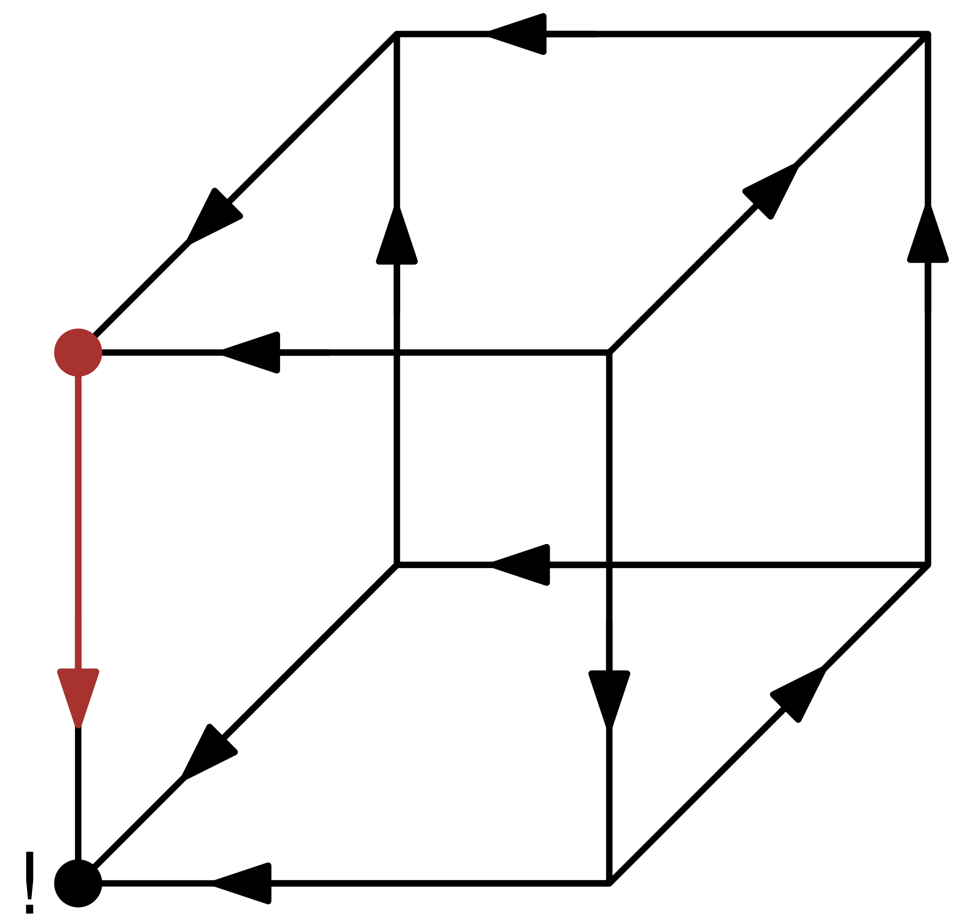
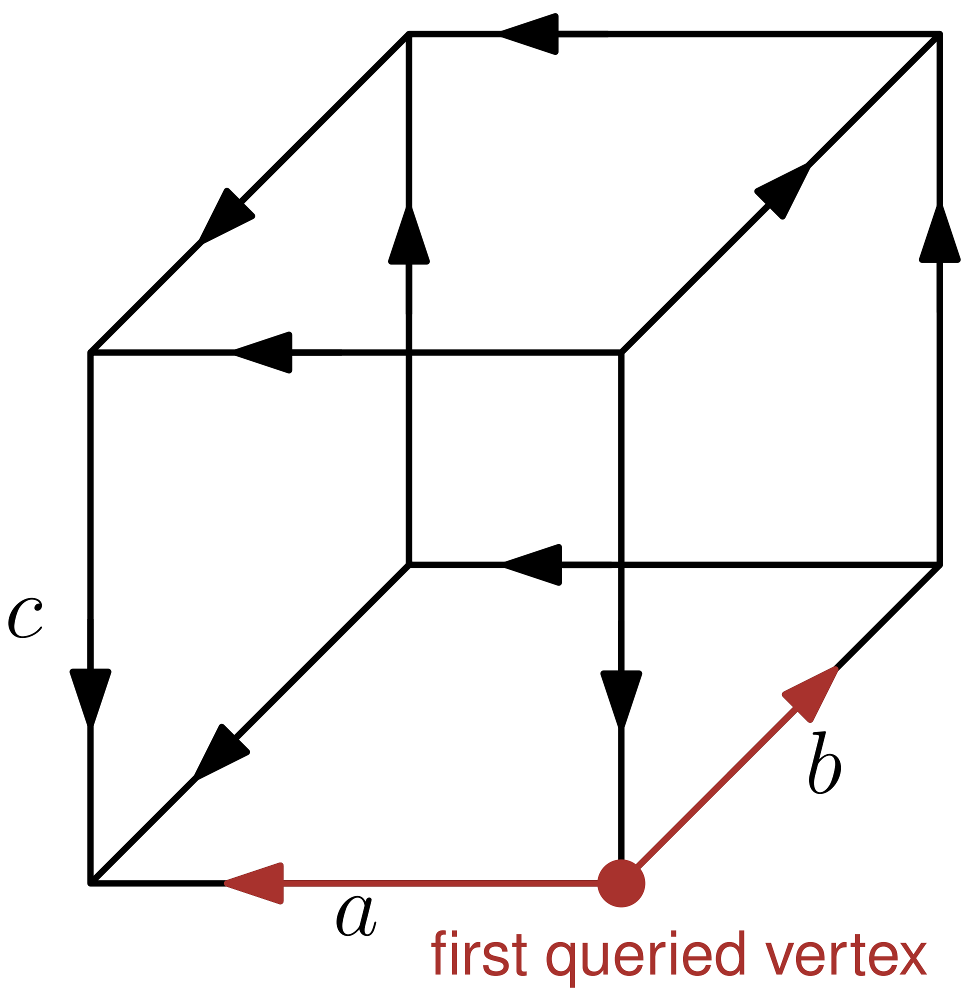


Alternative Views



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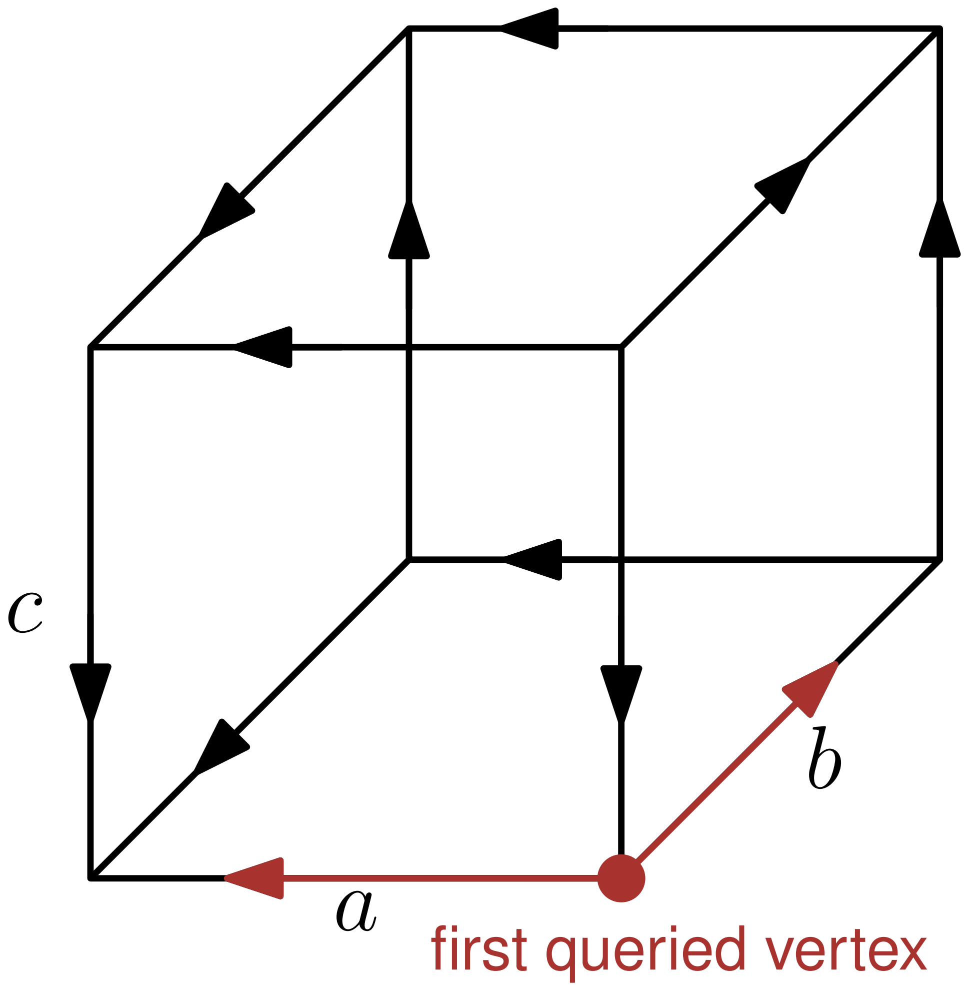
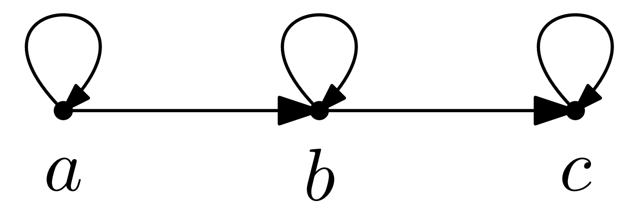


Find set of dimensions S such that a and b have odd numbers of in-neighbors in S, and c has an even number



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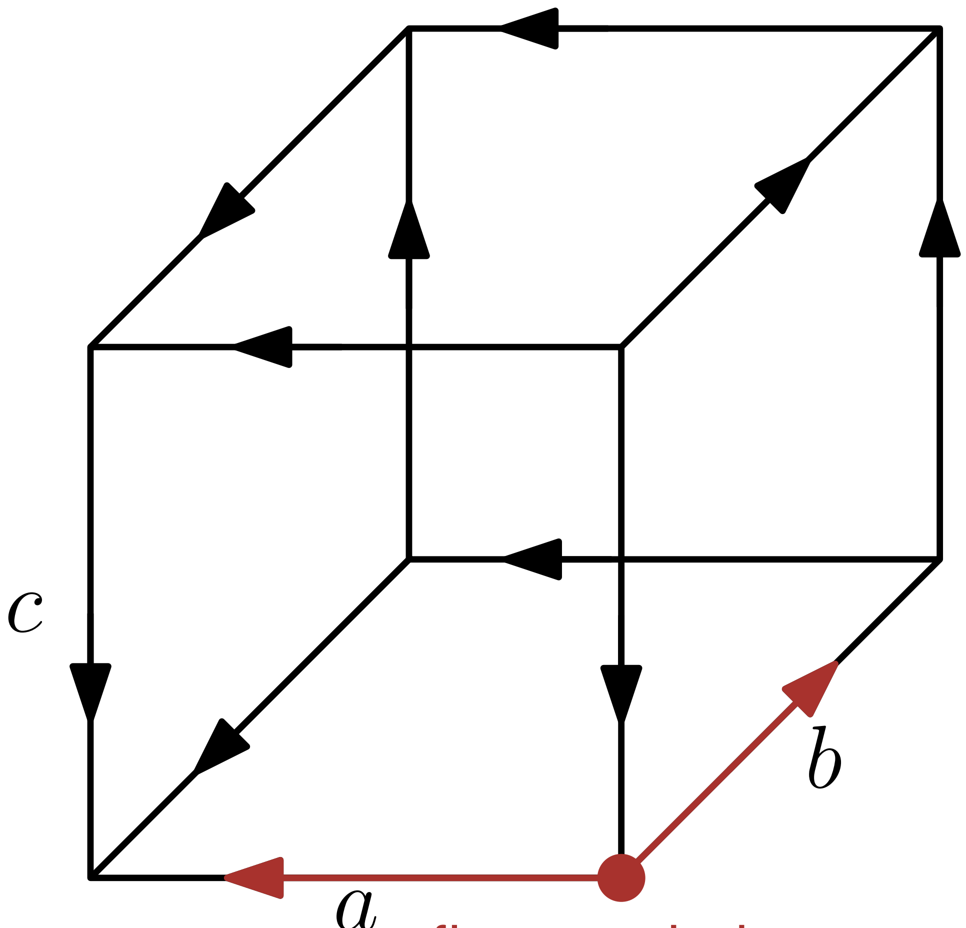

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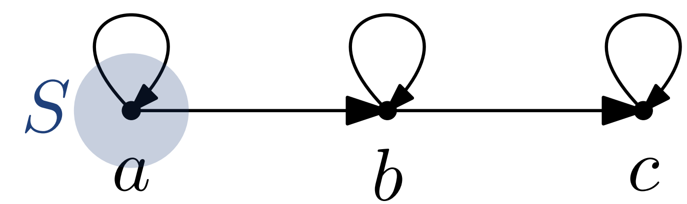

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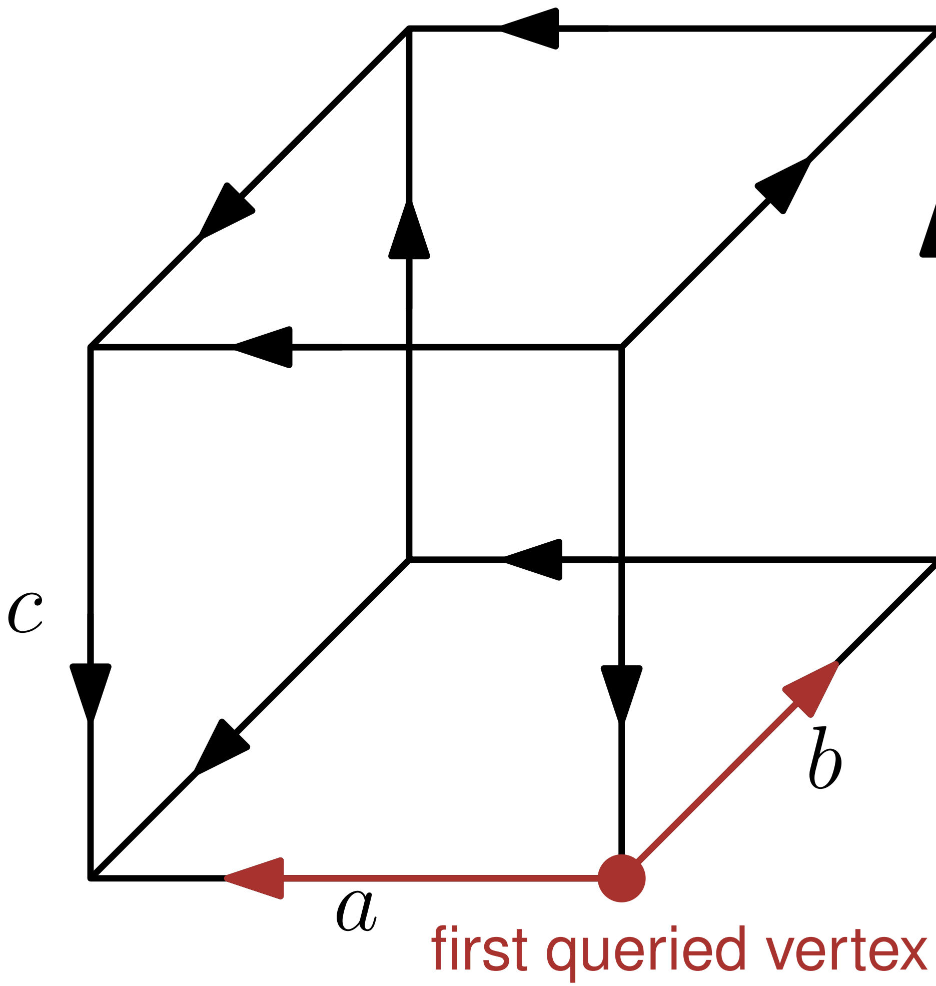

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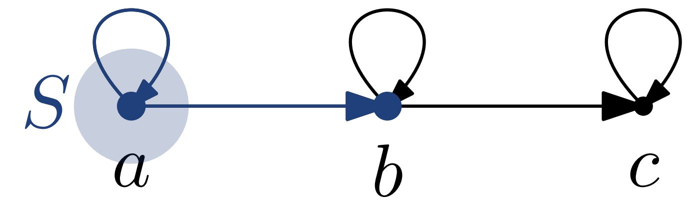

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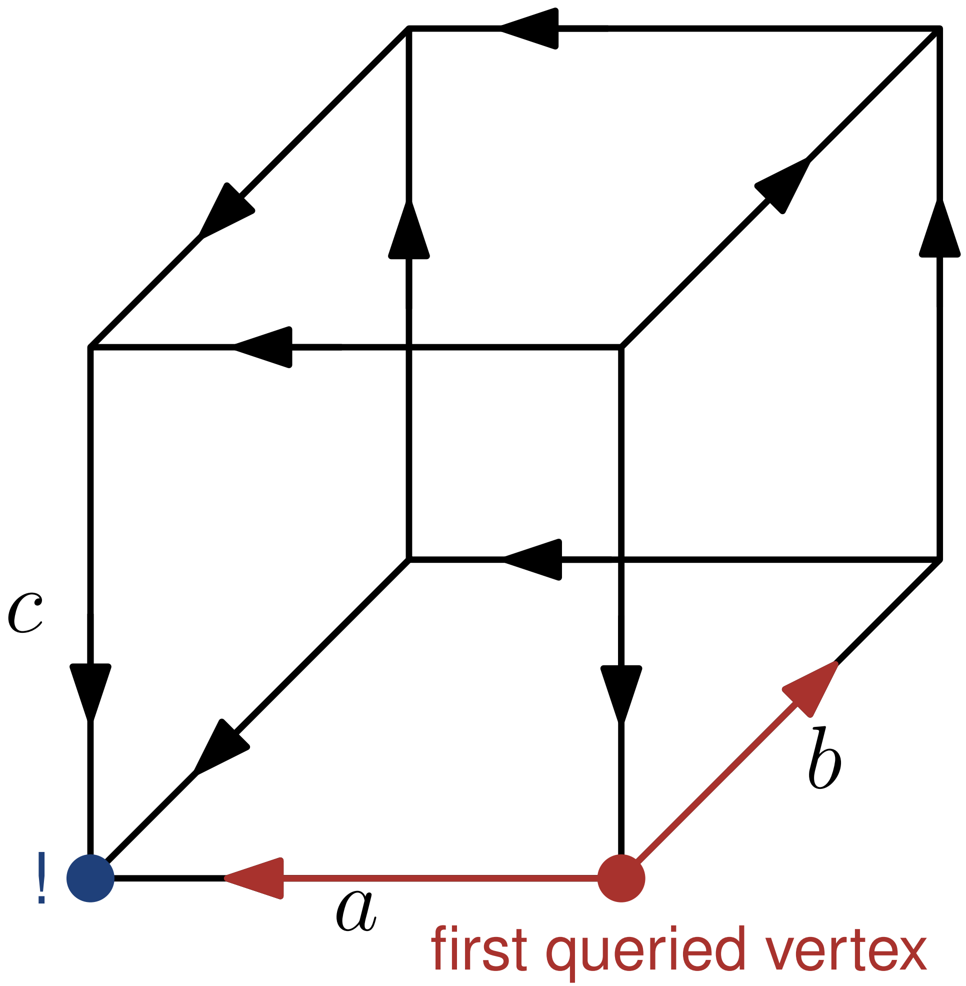

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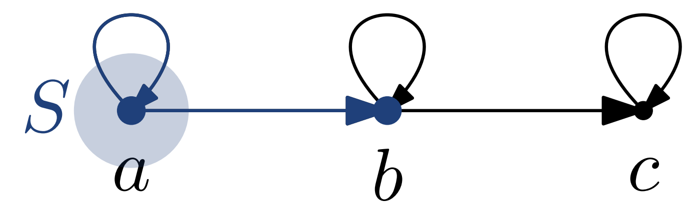

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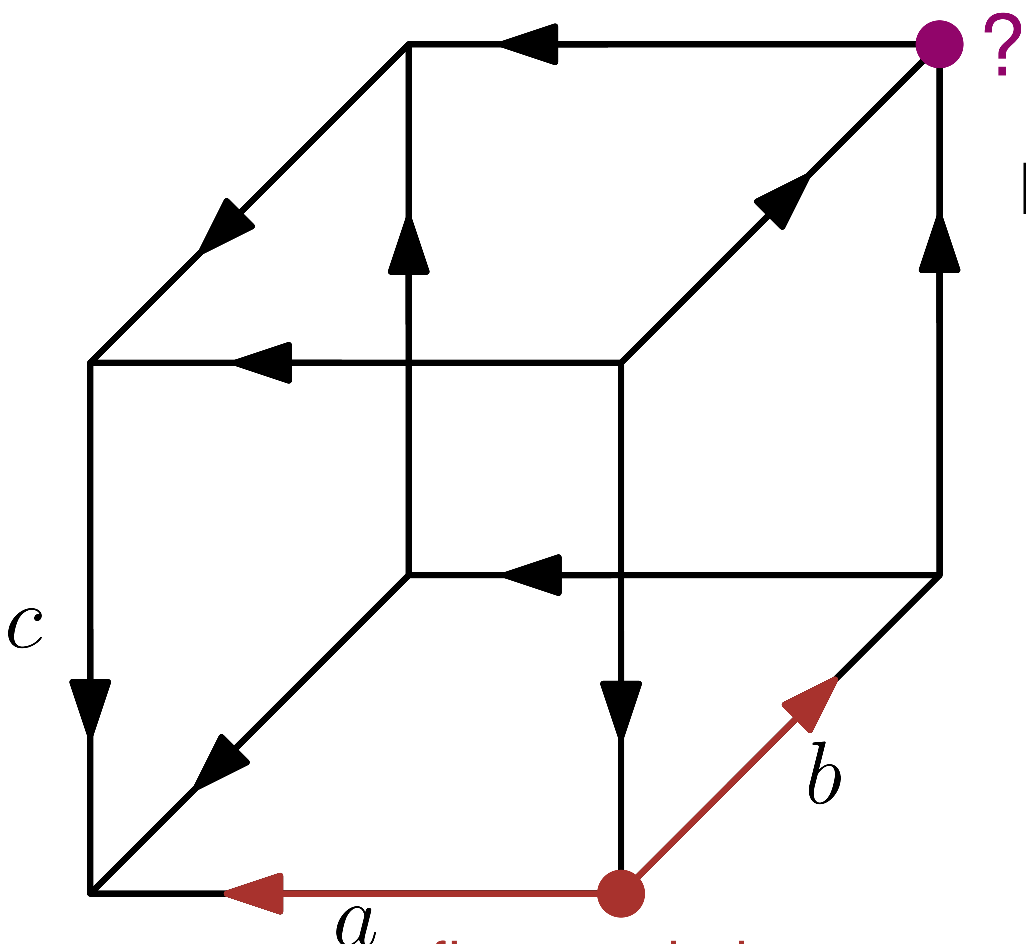

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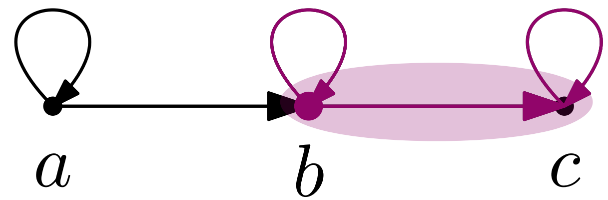

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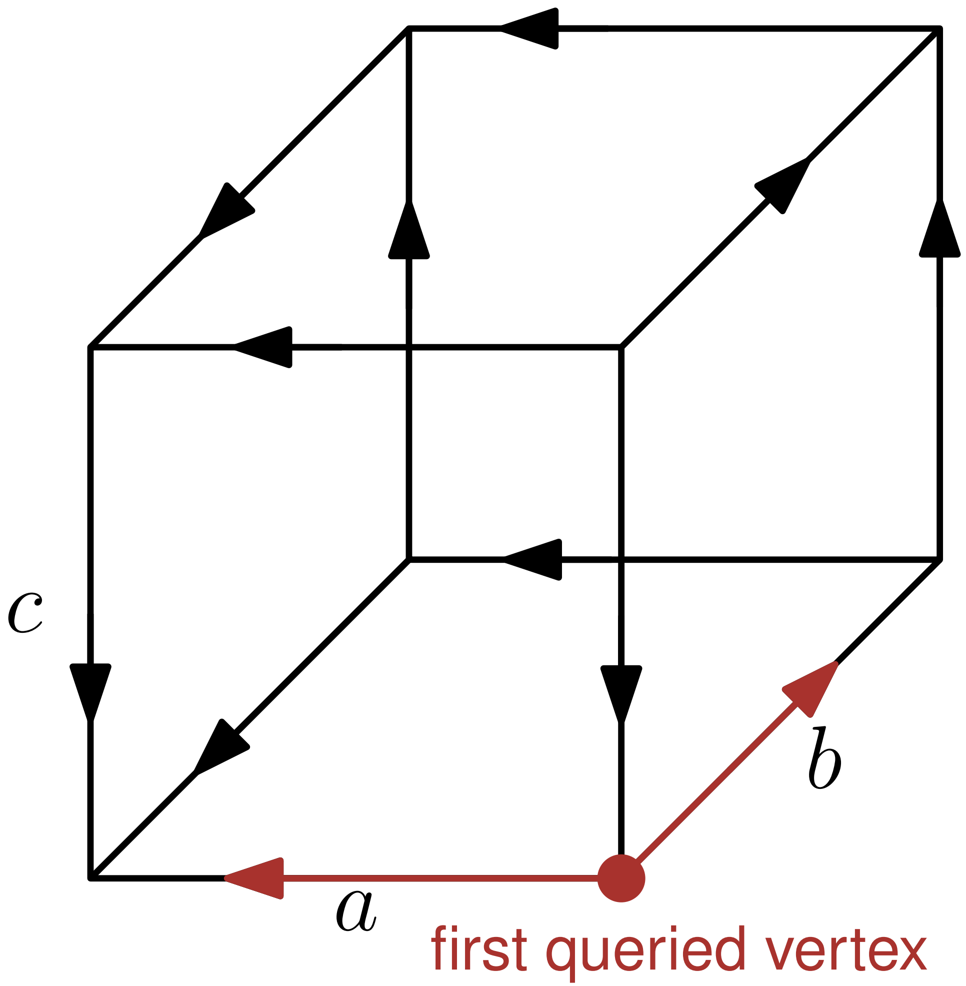

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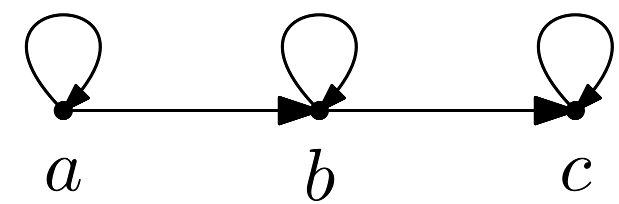

$$Mx = y$$

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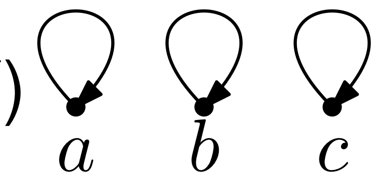

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Adversarial, adaptive oracle:

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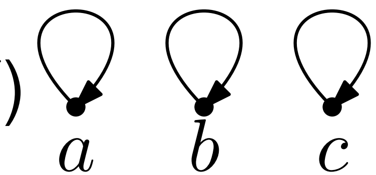
- On every query:
  Would replying according to the current dimension influence graph let the algorithm figure out the sink?



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Adversarial, adaptive oracle:

- The first queried vertex is never a sink ($y \neq 0$)
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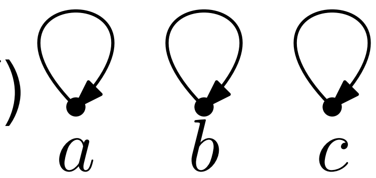
- On every query:
  Would replying according to the current dimension influence graph let the algorithm figure out the sink?
  - No? Answer according to G.



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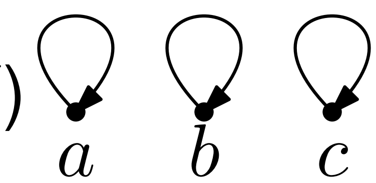

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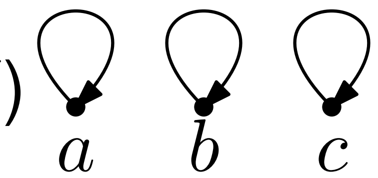
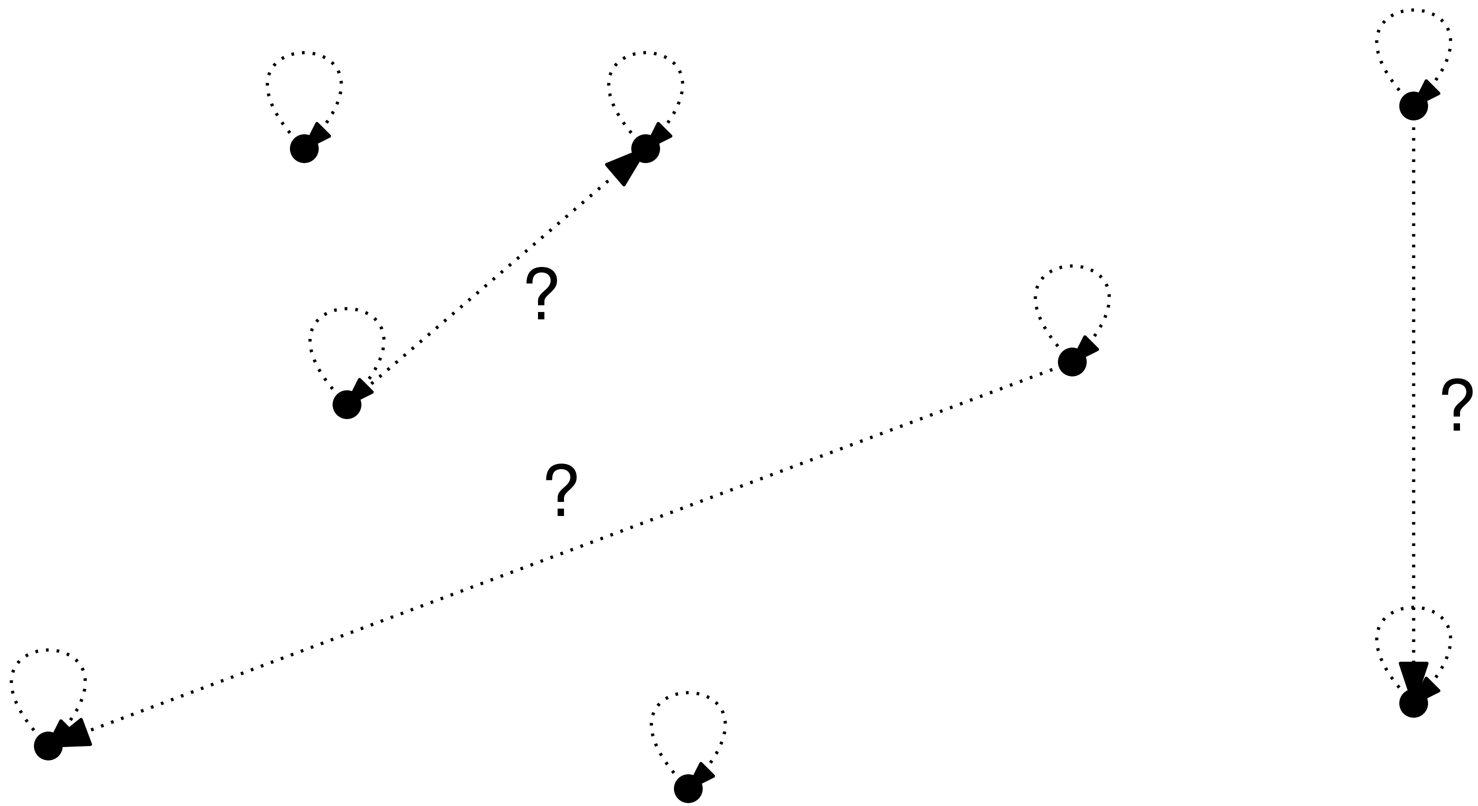


Need *consistency*, *legality*, and *uncertainty*.

Realizable Matoušek-type USOs: Upper Bound

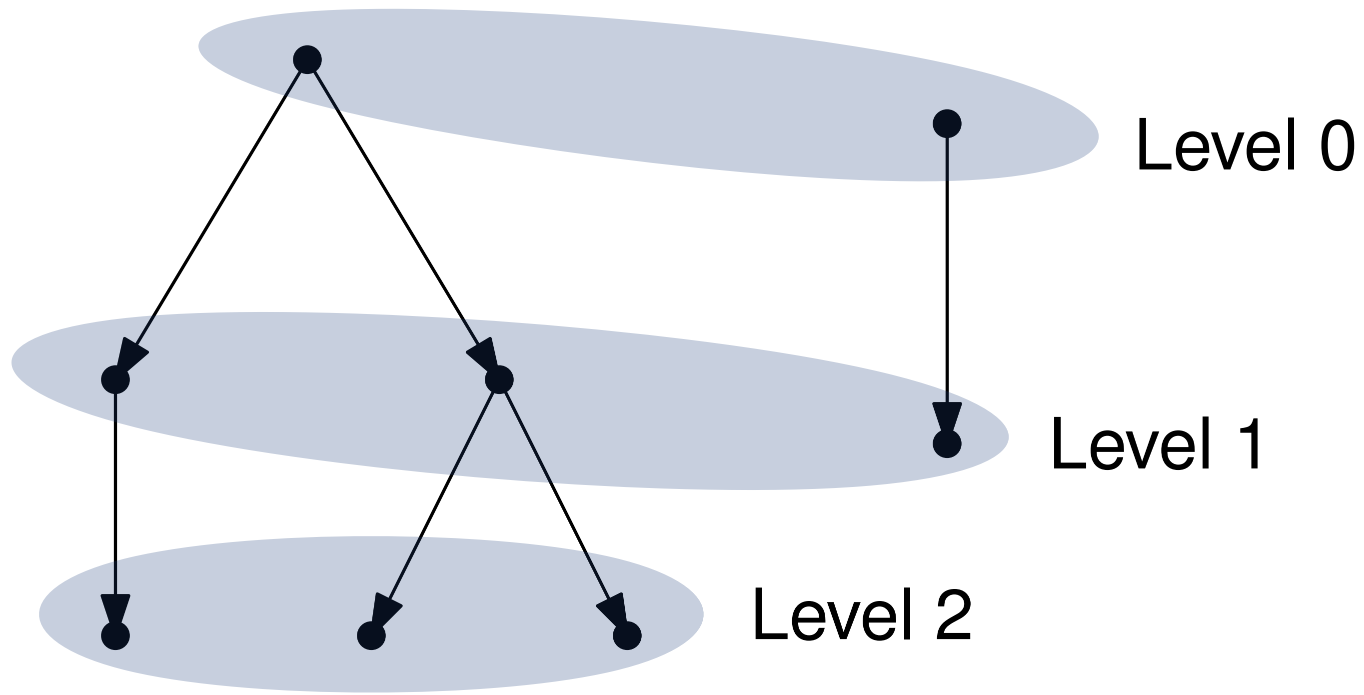
We recover the whole dimension influence graph!

Realizable Matoušek-type USOs: Upper Bound

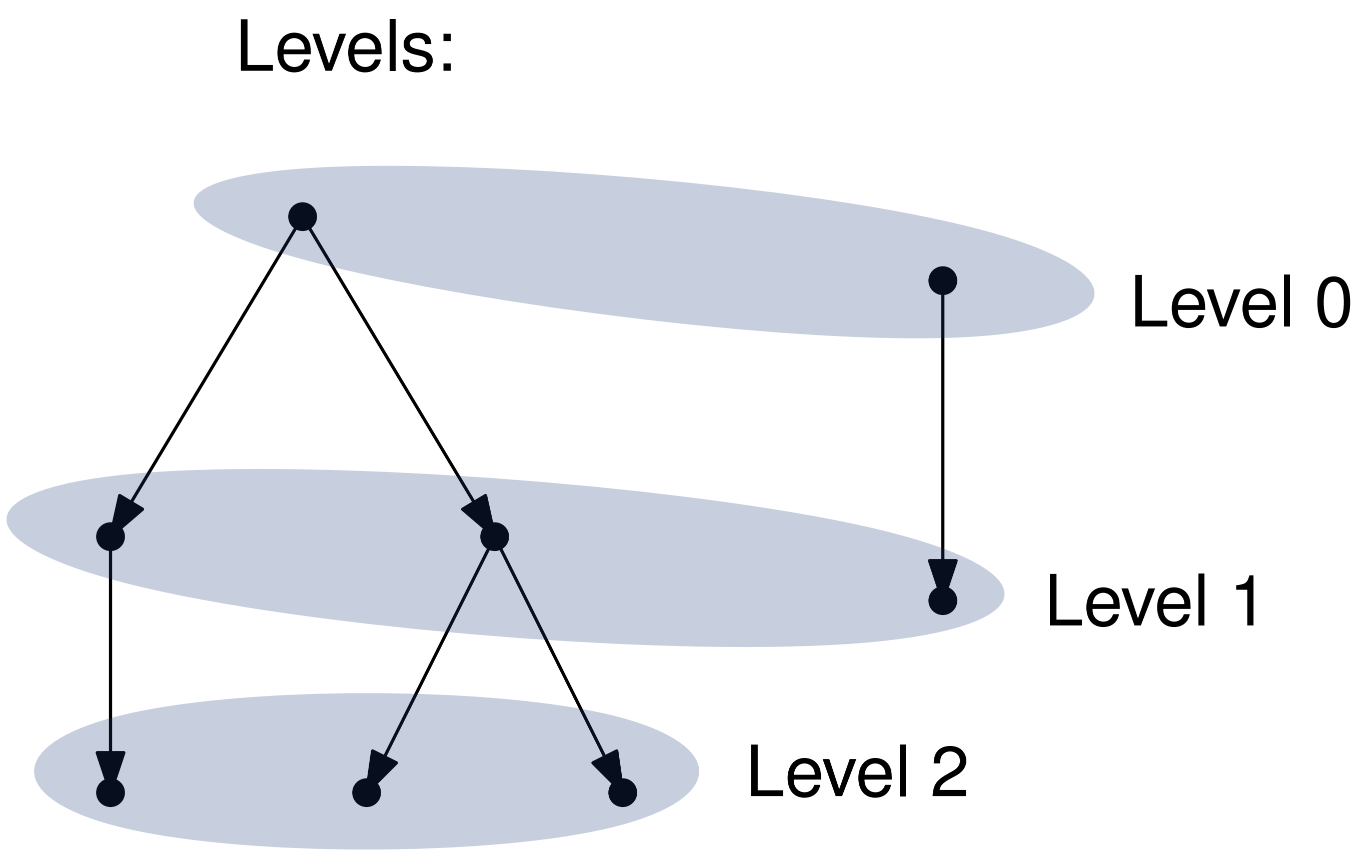


Realizable Matoušek-type USOs: Upper Bound

Levels:



Realizable Matoušek-type USOs: Upper Bound



Level $=$ in-degree -1

Realizable Matoušek-type USOs: Upper Bound

Levelling: $O(\log n)$

- First query $\{1, \ldots, n\}$: $\Rightarrow$ set of dimensions with odd in-degree

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- First query $\{1, \ldots, n\}$: $\Rightarrow$ set of dimensions with odd in-degree
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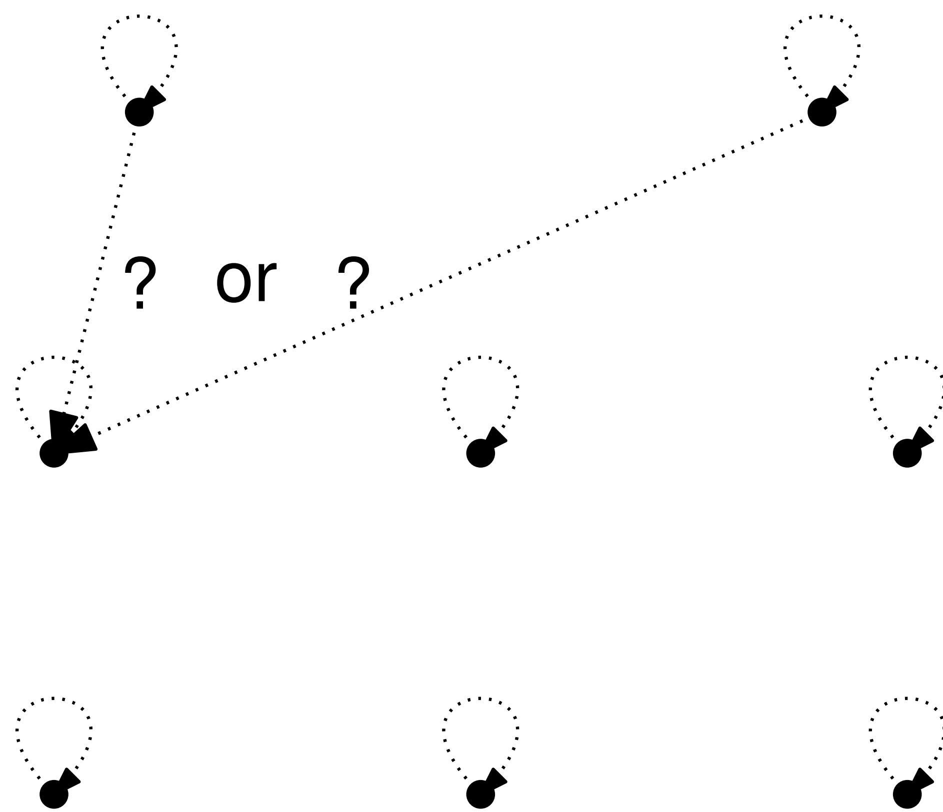
This reveals something about the second-LSB of their in-degree

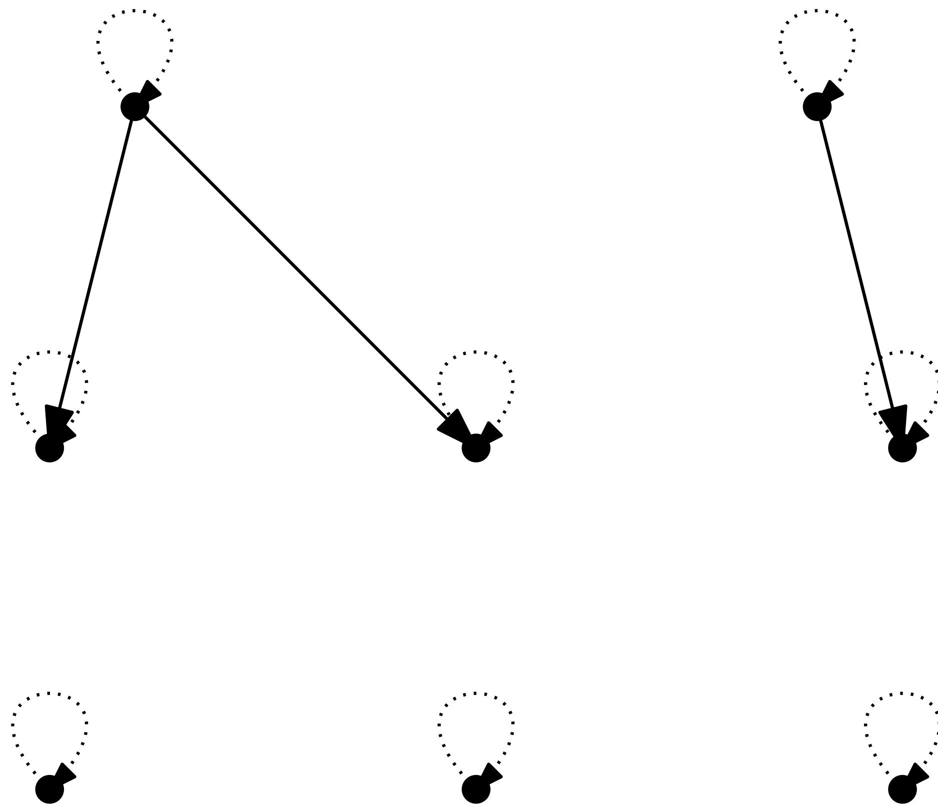
Realizable Matoušek-type USOs: Upper Bound

Levelling: $O(\log n)$



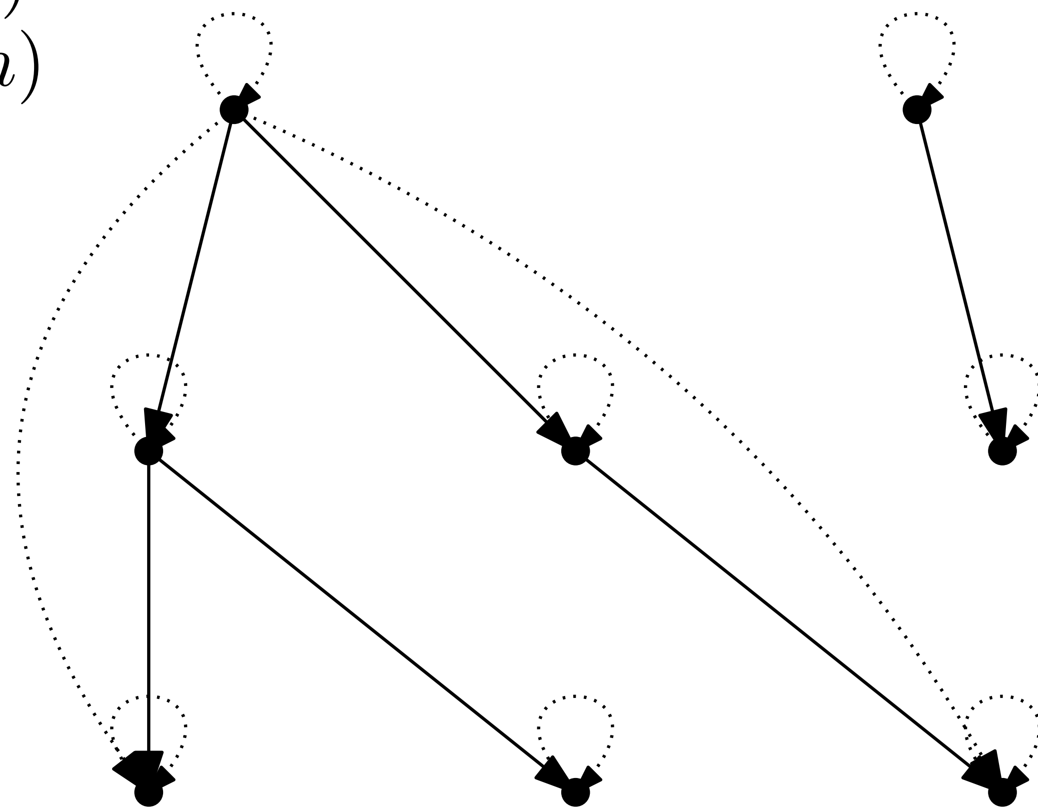
Realizable Matoušek-type USOs: Upper Bound

Levelling: $O(\log n)$
One level: $O(\log n)$



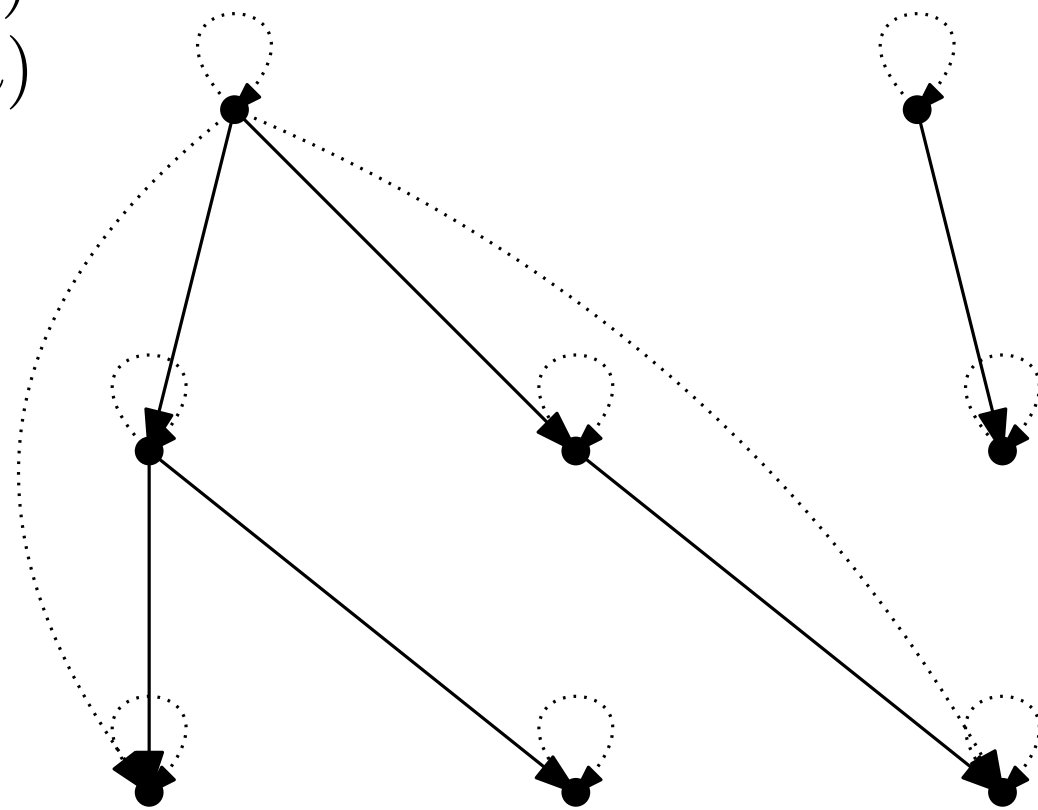
Realizable Matoušek-type USOs: Upper Bound

Levelling: $O(\log n)$
One level: $O(\log n)$
All levels: $O(n \log n)$



Realizable Matoušek-type USOs: Upper Bound

Levelling: $O(\log n)$
One level: $O(\log n)$
All levels: $O(\log^2 n)$



Open Questions

Can we find a complexity gap or algorithms making use of realizability for larger (more relevant) classes of USOs?

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What is the true query complexity of sink-finding on realizable Matoušek-type USOs?

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Does the complexity gap also hold for randomized algorithms?