# Linear Layouts of Bipartite Planar Graphs

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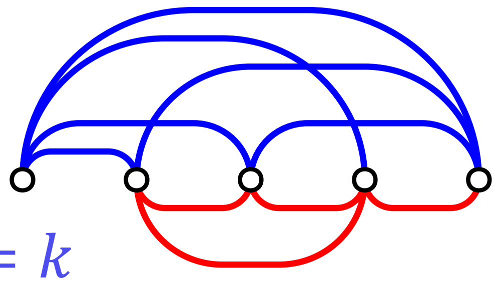
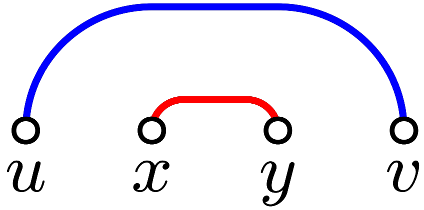
# Queue Layouts

- *Queue Layout* $\Gamma$ of a graph $G = (V, E)$
  - *linear order* $\prec$ of $V$
  - *partition* of $E$ into *queues* $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ such that *no two edges of the same queue nest*
    - $(u, v)$ *nests* $(x, y)$ if $u \prec x \prec y \prec v$






  - *Queue Number* $qn(\Gamma) = k$
- *Queue Number* $qn(G)$ of a graph $G$ = *smallest queue number* of any queue layout of $G$
  - $qn(\mathcal{G})$ of a graph family $\mathcal{G}$ = *largest queue number* of any $G \in \mathcal{G}$ (typically *dependent on the number of vertices* $n$)

# Queue Layouts

- Why are queue layouts *interesting?*
  - *Curious combinatorial* problem
  - *Theorem:* If the *queue number* and the *acyclic chromatic number* are bounded, $G$ has a *3D straight-line drawing in $O(n)$ volume*. [Dujmović, Wood 2004]

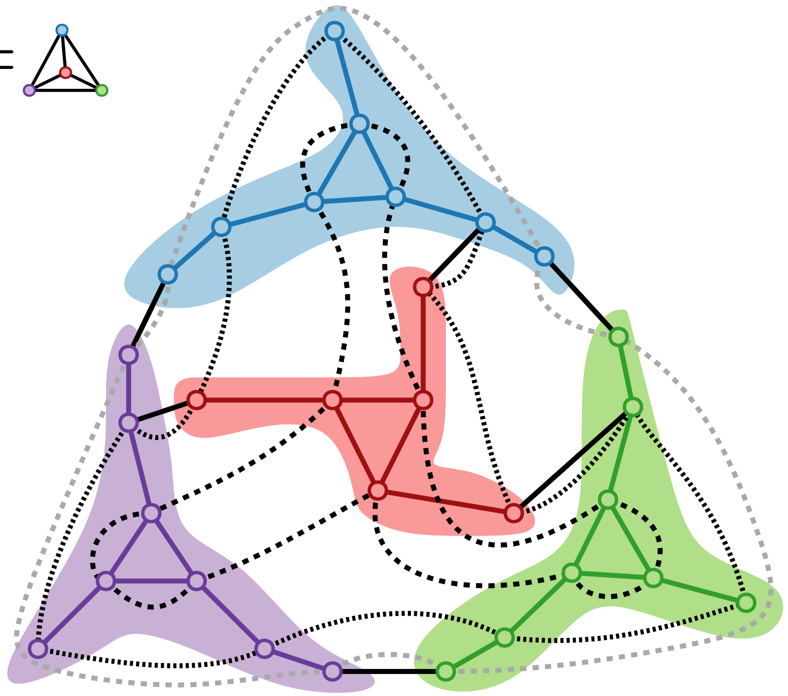
# Queue Number of Important Graph Classes

- *Conjecture:* The queue number of *planar* graphs is *bounded* [Heath, Leighton, Rosenberg 1992]
- *Conjecture:* The queue number of *planar* graphs is *not bounded* [Heath, Rosenberg 2011]
- *Theorem:* The queue number of graphs of *bounded tree width* is *bounded* [Dujmović, Morin, Wood 2005]
- *Theorem:* The queue number of graphs of *bounded degree* is *not bounded.* [Wood 2008]
- *Theorem:* The queue number of *planar* graphs of *bounded degree* is *bounded.* [Bekos, F., Gronemann, Mchedlidze, Montecchiani, Raftopoulou, Ueckerdt 2019]

# Queue Number of Planar Graphs

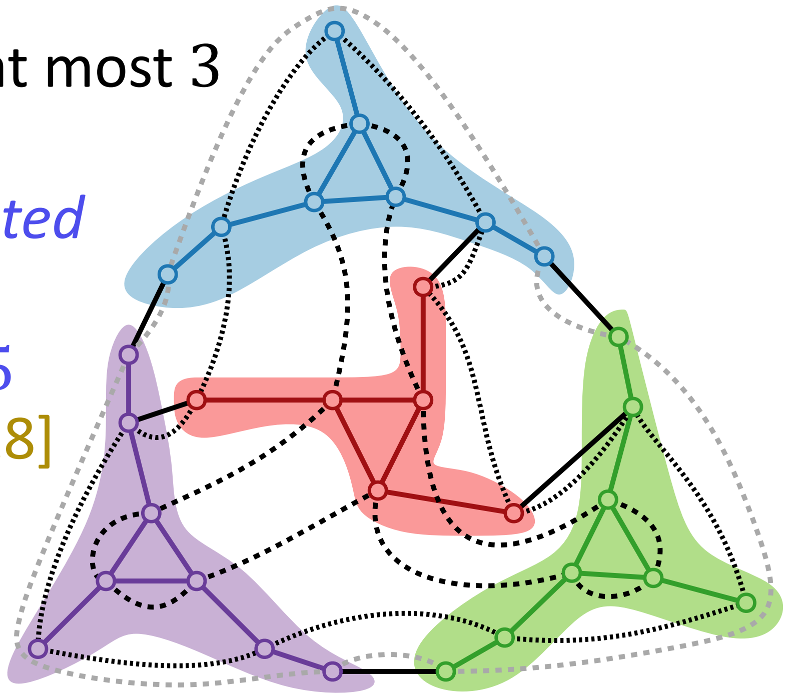
- *Theorem:* The queue number of *planar* graphs is *at most 49.* [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood 2019/2020]
  - *H-Partition* $\mathcal{H} = (H, \{V_x : x \in V(H)\})$ of $G$
    - $\{V_x : x \in V(H)\}$ is partition of $V(G)$
    - For edge $(u, v) \in E(G)$: Either
      - $u, v \in V_x$ *(intra-bag)* or
      - $u \in V_x, v \in V_y, (x, y) \in E(H)$ *(inter-bag)*
  - BFS-Layering $\mathcal{L} = (V_0, V_1, \ldots)$ of $G$
    - For each edge $(u, v) \in E(G)$: Either
      - $u, v \in V_i$ *(intra-layer)* or
      - $u \in V_i$ and $v \in V_{i+1}$ *(inter-layer)*
    - $\mathcal{H}$ has layered width $\ell$ if $|V_x \cap V_i| \leq \ell$

$H =$



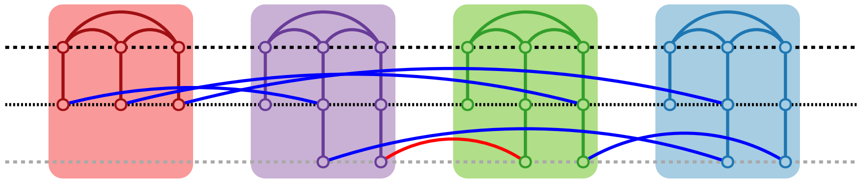
# Queue Number of Planar Graphs

- *Theorem:* The queue number of *planar* graphs is *at most 49.* [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood 2019/2020]
  - For every *planar graph* $G$ there is an *$H$-partition* $\mathcal{H} = (H, \{V_x : x \in V(H)\})$ of $G$ with *layered width* at most 3 and where $H$ is *planar* has *tree width* at most $3$.
  - Such an *$H$-partition* can be computed via a *BFS rooted at any* $r \in V(G)$.
  - $tw(H) \leq 3$ and $H$ being *planar* implies $qn(H) \leq 5$ [Alam, Bekos, Gronemann, Kaufmann, Pupyrev 2018]

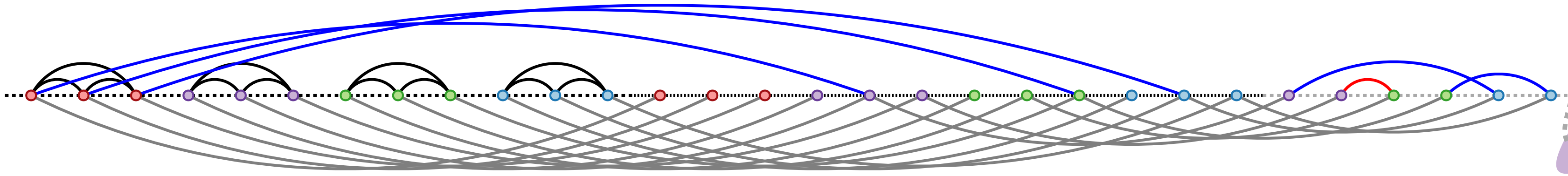


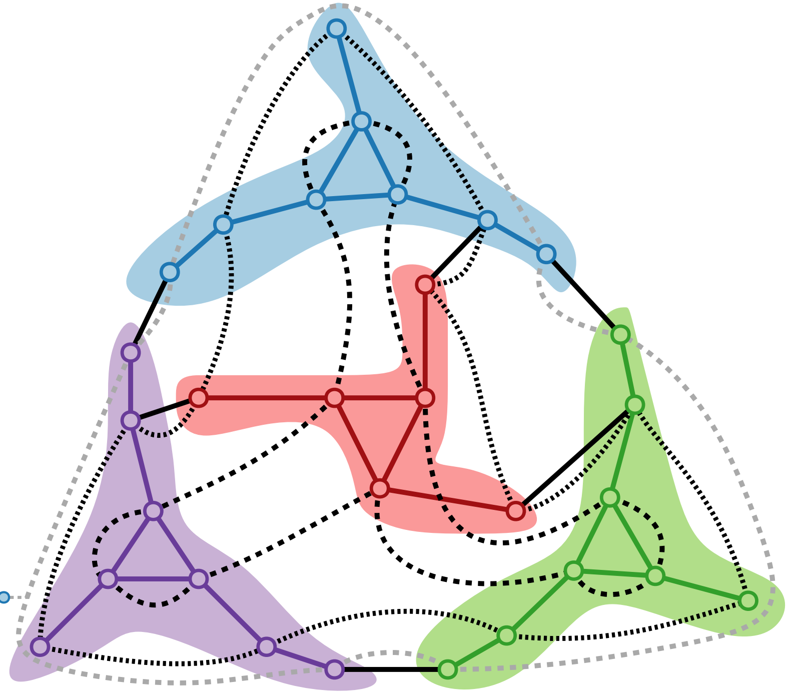
# Queue Number of Planar Graphs

- *Theorem:* The queue number of *planar* graphs is *at most 49.* [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood 2019/2020]
  - *Construct queue layout as follows:*
    - Construct *queue layout* of $H$ with *queue number 5*
    - Replace vertices of $H$ with *layered content of bags*



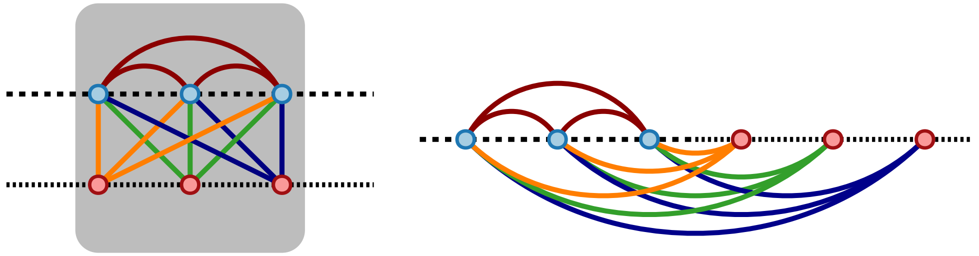
    - Layout *layer by layer* consecutively



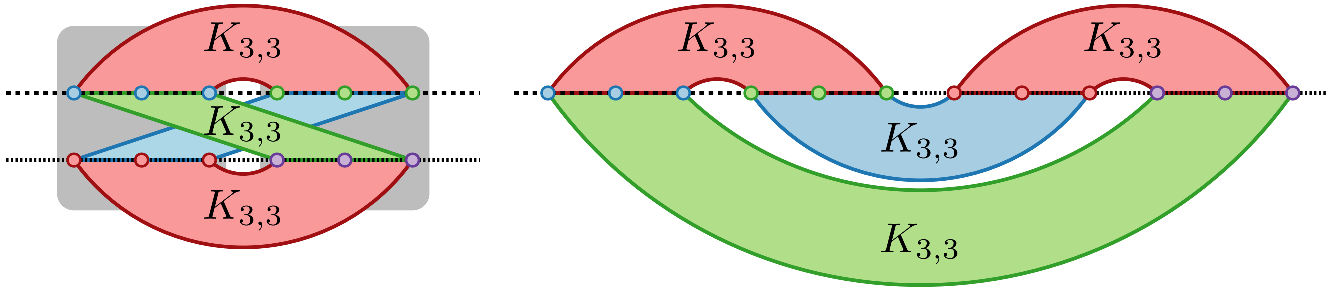


# Queue Number of Planar Graphs

- *Theorem:* The queue number of *planar* graphs is *at most 49.* [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood 2019/2020]
  - *Assignment to queues:*
    - *Intra-Bag Edges:* 1 queue for *intra-layer* und 3 for *inter-layer*



    - *Inter-Bag Edges for each of the 5 queues of $H$:* 3 queues *for intra-layer,* 6 for *inter-layer*



# Queue Number of Bipartite Planar Graphs

- How about *bipartite planar graphs?*
  - *Now:* $G = (A, B, E)$
  - $A$ and $B$ are two *disjoint vertex sets*
  - $E \subseteq A \times B$, i.e., $A$ and $B$ are *independent sets*
- *Our first result:* The queue number of *bipartite planar graphs* is at most 28.
  - *Adjust* the proof of Dujmović et al. to obtain *33-queue layout*

# Queue Number of Bipartite Planar Graphs

*Theorem:* The queue number of *bipartite planar graphs* is at most 33.

- *$H$-Partition* $\mathcal{H} = (H, \{V_x : x \in V(H)\})$ of $G$
  - $\{V_x : x \in V(H)\}$ is partition of $V(G)$
  - For edge $(u, v) \in E(G)$: Either
    - $u, v \in V_x$ *(intra-bag)* or
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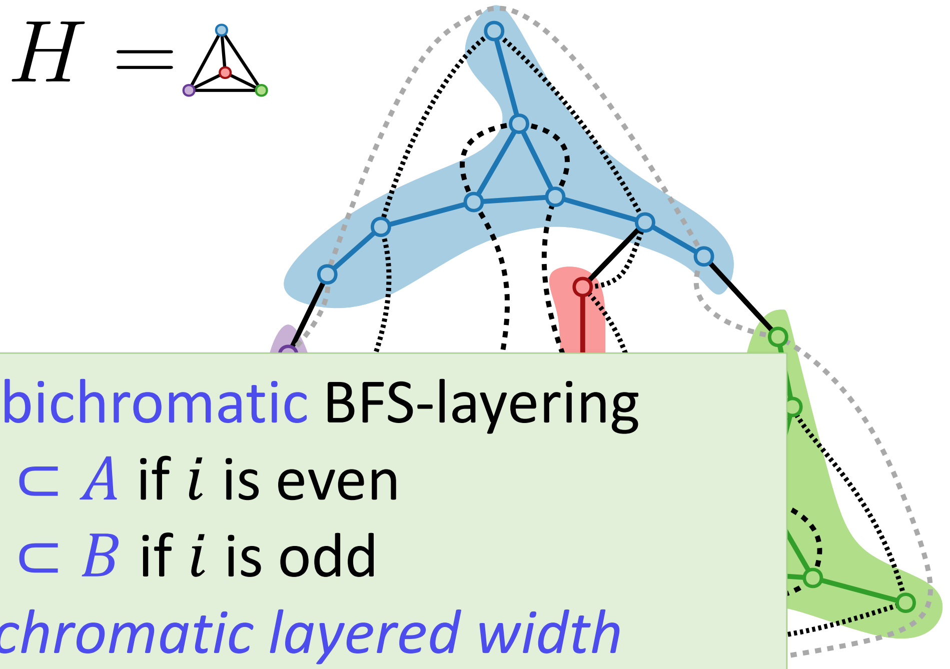


- *Now:* bichromatic BFS-layering
  - $V_i \subset A$ if $i$ is even
  - $V_i \subset B$ if $i$ is odd
  - *bichromatic layered width*

# Queue Number of Bipartite Planar Graphs

*Theorem:* The queue number of *bipartite planar graphs* is at most 33.

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and where $H$ is *planar* has *tree width* at most 3.
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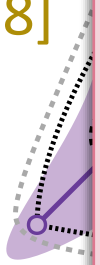
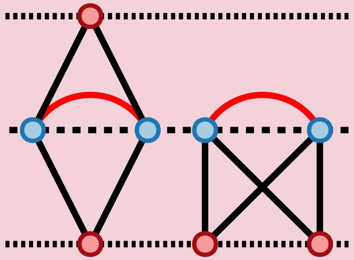


- *Triangulation* is required later
  - Quadrangulate $G$, i.e., add edges until *every face* is a *4-cycle*
  - *Perform BFS* from arbitrary root $r \in V(G)$
  - *Triangulate* quadrangles



# Queue Number of Bipartite Planar Graphs

*Theorem:* The queue number of *bipartite planar graphs* is at most 33.

- *Assignment to queues:*
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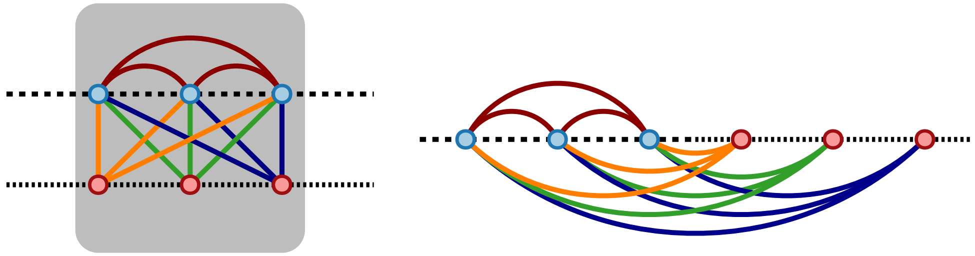
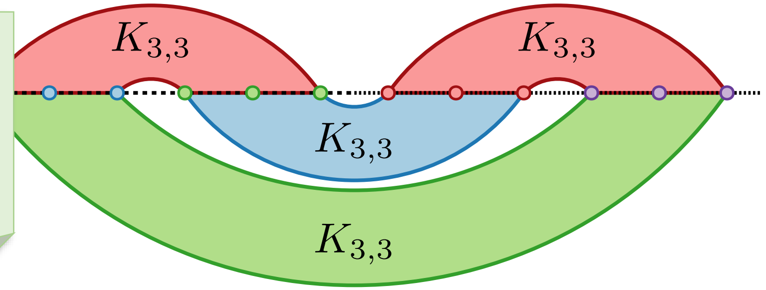


  - *Inter-Bag Edges for each of the 5 queues of $H$:* ~~3 queues *for intra-layer,*~~ 6 for *inter-layer*




- *Now:* bichromatic BFS-layering
  - *No intra-layer edges*

# Queue Number of Bipartite Planar Graphs

- How about *bipartite planar graphs?*
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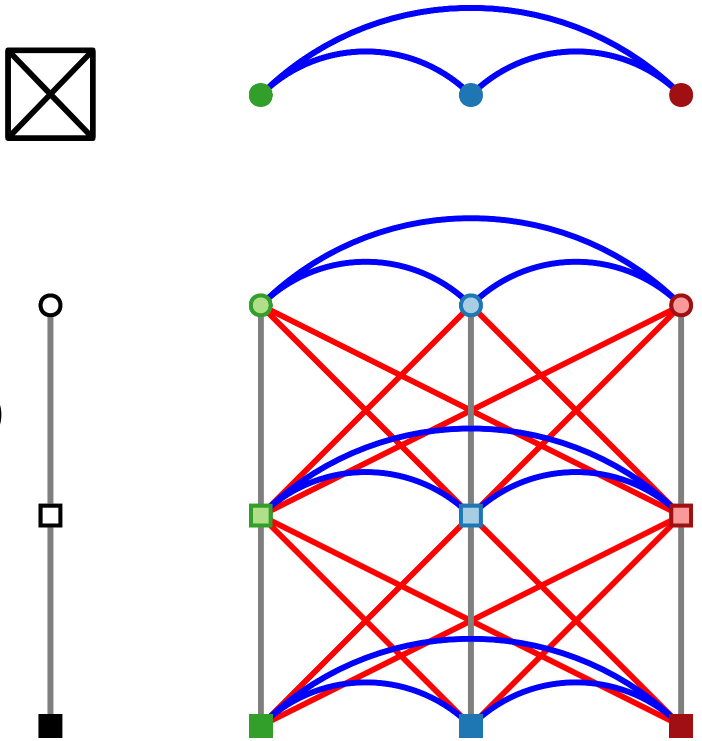
  - $A$ and $B$ are two *disjoint vertex sets*
  - $E \subseteq A \times B$, i.e., $A$ and $B$ are *independent sets*
- *Our first result :* The queue number of *bipartite planar graphs* is at most 28.
  - *Adjust* the proof of Dujmović et al. to obtain *33-queue layout*
  - *Apply improvements* by [Bekos, Gronemann, Raftopoulou 2022] to *reduce number of queues* by 5

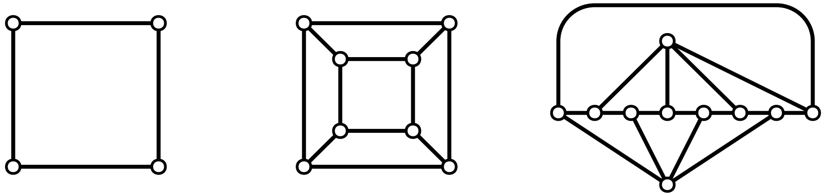
# Graph Product Structure

- *Different interpretation of these results:* If $G$ is *planar*, it is a *subgraph* of $P \boxtimes K_3 \boxtimes H$ where
  - $P$ is a *path*
  - $H$ has *treewidth* at most 3
  - $\boxtimes$ is the *strong graph product*
    - The *product* of $G_1$ and $G_2$ has *vertices* $V(G_1) \times V(G_2)$,
    - an *edge* $((u,x),(v,x))$ if $(u,v) \in E(G_1)$ (similar for $((u,x),(u,y))$)
    - an *edge* $((u,x),(v,y))$ if $(u,v) \in E(G_1)$ and $(x,y) \in E(G_2)$
- *Can we find better bounds for the queue number of bipartite planar graphs?*
- *Or a better graph product?*

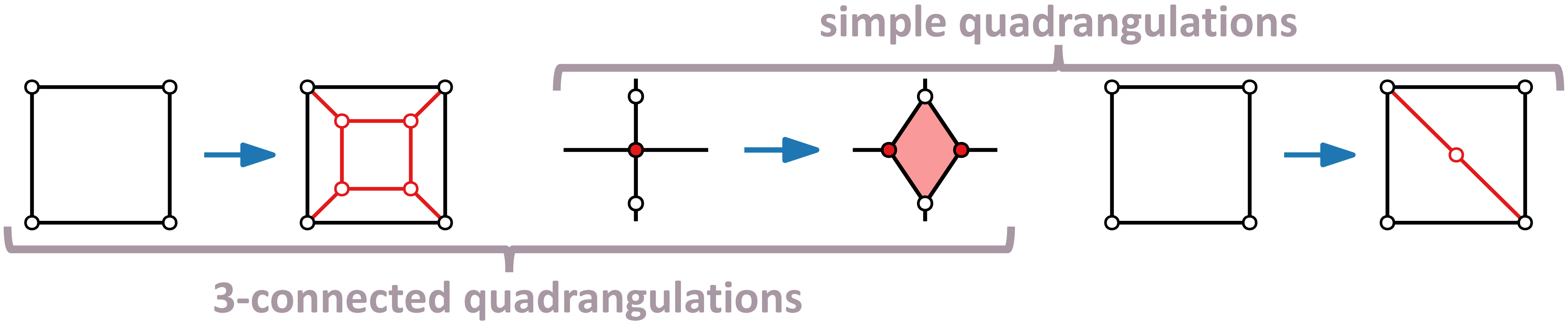


# Recursive Definition of Bipartite Planar Graphs

- *Edge-maximal bipartite planar* graphs are exactly *quadrangulations*
- *Recursive definition for quadrangulations:*
  - *Base graphs:* 4-cycle, cube graph, generalizations of the cube graph

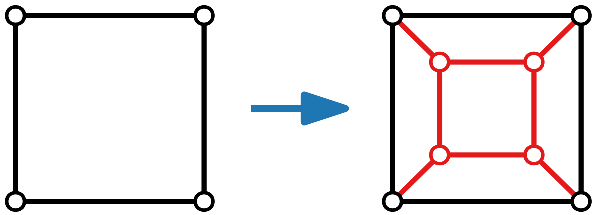


  - *Iterative Steps:* HENNEBERG steps [Felsner, Huemer, Kappes, Orden 2010]

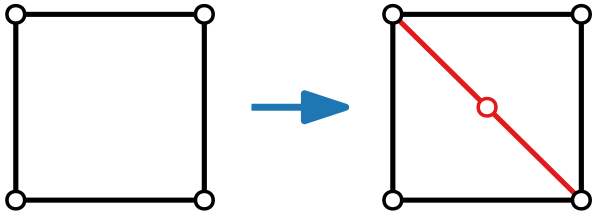


# Results for Special Quadrangulations

- *Stacked Quadrangulations*
  - *Base graph:* 4-cycle
  - *Iterative Step:*



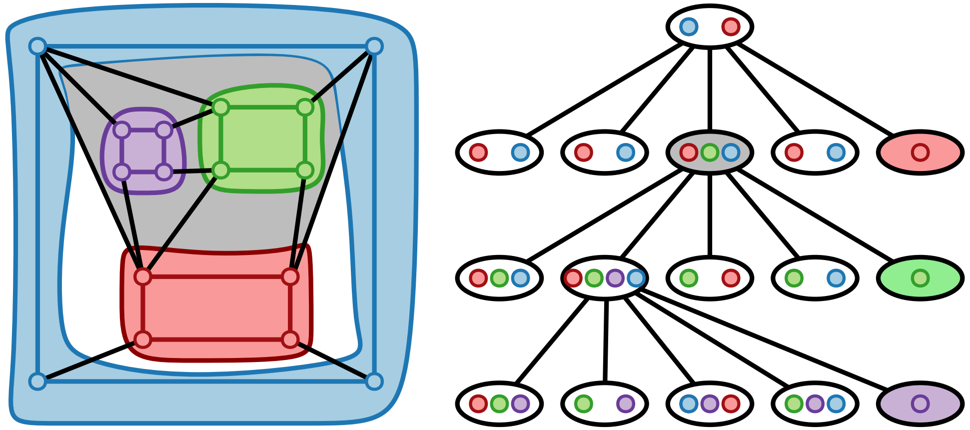
- *2-Degenerate Quadrangulations*
  - *Base graph:* 4-cycle
  - *Iterative Step:*



- **Subgraphs of $C_4 \boxtimes H$ where $C_4$ is a 4-cycle and $H$ is planar and has treewidth at most 3**
- **Corollary: Queue number at most 21**

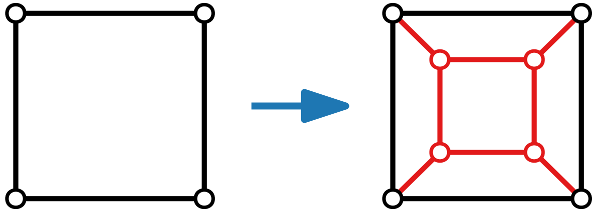
# Stacked Quadrangulations

- *Theorem:* Let $G$ be a *stacked quadrangulation*. Then, $G$ is *subgraph* of $C_4 \boxtimes H$ where $H$ is *planar* and has *treewidth at most 3*.
  - *In other words:* We iteratively construct an *$H$-partition* and its *tree decomposition*.
    - *Tree decomposition* $(T, \{V_x : x \in V(T)\})$
      - *for each edge* $(u, v) \in E(G)$ there is a bag $V_x$ containing *both endvertices,* i.e., $u, v \in V_x$
      - *vertices* may occur in *multiple bags*, and
      - *for each vertex,* the subforest of $T$ *induced by ist bags* is *connected*
      - *Tree width* = # vertices per bag - 1
    - *Tree width 3: Each face* contains an edge belonging to *one bag of $H$* when it is created
      - *Plus one vertex* corresponding to the *cycle* added *inside the face* later

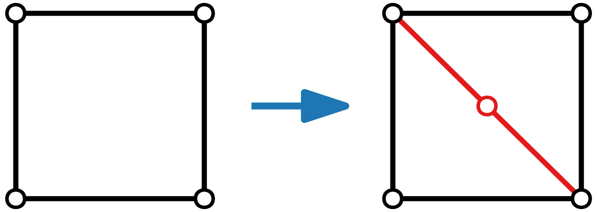


# Results for Special Quadrangulations

- *Stacked Quadrangulations*
  - *Base graph:* 4-cycle
  - *Iterative Step:*



- *2-degenerate Quadrangulations*
  - *Base graph:* 4-cycle
  - *Iterative Step:*

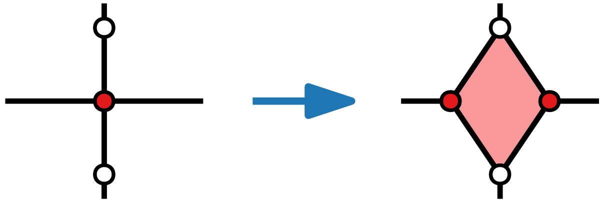


- **Subgraphs of $C_4 \boxtimes H$ where $C_4$ is a 4-cycle and $H$ is planar and has treewidth at most 3**
- **Corollary: Queue number at most 21**

- **Queue number at most 5**
  - Yet another *decomposition technique*
- **Queue number at least 3**
  - 259 vertices
  - Extends to *mixed linear layouts*

# Open Problems

- How does the *third HENNEBERG step* behave w.r.t. *the queue number*?



- *Explicitly construct* queue layouts for *stacked quadrangulations*
- *Close the gap* for *2-degenerate quadrangulations*
- *For non-bipartite planar graphs:*
  - queue number is *at most 42*
  - queue number is *at least 4*
  - *Which bound is closer to the truth?*